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ON $\mathcal{I}_\tau^\mathcal{K}$-CONVERGENCE OF NETS IN LOCALLY SOLID RIESZ SPACES

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Abstract. In this short note we continue our investigation of nets in locally solid Riesz spaces from [P. Das, E. Savas, On $\mathcal{I}$-convergence of nets in locally solid Riesz spaces, Filomat, 27 (1) (2013), 84–89] and introduce the idea of $\mathcal{I}_\tau^\mathcal{K}$-convergence of nets which is more general than $\mathcal{I}_\tau^*$-convergence and obtain some of its basic properties.

1. Introduction

The notion of Riesz space was first introduced by F. Riesz [25] in 1928 and since then it has found several applications in measure theory, operator theory, optimization and also in economics (see [3, 22, 24]). A Riesz space is an ordered vector space which is also a lattice, endowed with a linear topology. Further if it has a base consisting of solid sets at zero then it is known as a locally solid Riesz space. In a very recent development, the idea of statistical convergence of sequences was studied by Pehlivan and Albayrak [1] in locally solid Riesz spaces.

On the other hand the notions of usual convergence and statistical convergence of sequences were further generalized in [11] where the notions of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence of a sequence was introduced by using ideals of the set of positive integers. One can see [6–9, 13, 15, 17–19, 21, 22] for more works in this direction where many more references can be found. The idea of ideal convergence has also been investigated in (ℓ)-groups (a structure more general than Riesz spaces) in [3, 4]. In particular in [21] the notion of $\mathcal{I}^*$-convergence was further extended to $\mathcal{I}^\mathcal{K}$-convergence.

In an interesting development, the notion of usual convergence of nets was extended to ideal convergence of nets in [19] where the basic topological nature of these convergence was established (also continued in [8, 9]). As a natural consequence, in [10], we introduced the idea of ideal-τ convergence of nets in a locally solid Riesz space and studied some of its properties by using the mathematical tools

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of the theory of topological vector spaces. As a continuation, in this short note we continue our investigation of nets in locally solid Riesz spaces from [10] and introduce the idea of $\mathcal{I}_\tau^{\mathcal{K}}$-convergence of nets which is more general than $\mathcal{I}_\tau^*$-convergence and obtain some of its basic properties.

2. Preliminaries

In this section we recall some of the basic concepts of Riesz spaces and ideal convergence of nets and interested readers can look into [1, 3, 10, 19] for details.

Definition 2.1. Let L be a real vector space and let $\leq$ be a partial order on this space. L is said to be an ordered vector space if it satisfies the following properties:

(i) If $x, y \in L$ and $y \leq x$ then $y + z \leq x + z$ for each $z \in L$.

(ii) If $x, y \in L$ and $y \leq x$ then $\lambda y \leq \lambda x$ for each $\lambda \geq 0$.

If in addition L is a lattice with respect to the partial ordering, then L is said to be a Riesz space (or a vector lattice).

For an element x of a Riesz space L the positive part of x is defined by $x^+ = x \vee \theta$, the negative part of x by $x^- = (-x) \vee \theta$ and the absolute value of x by $|x| = x \vee (-x)$, where θ is the element zero of L. A subset S of a Riesz space L is said to be solid if $y \in S$ and $|x| \leq |y|$ imply $x \in S$.

A topology τ on a real vector space L that makes the addition and scalar multiplication continuous is said to be a linear topology, that is when the mappings

$$(x, y) \to x + y \text{ (from } (L \times L, \tau \times \tau) \to (L, \tau))$$
$$(\lambda, x) \to \lambda x \text{ (from } (R \times L, \sigma \times \tau) \to (L, \tau))$$

are continuous where σ is the usual topology on R. In this case the pair (L, τ) is called a topological vector space.

Every linear topology τ on a vector space L has a base $\mathcal{N}$ for the neighborhoods of θ satisfying the following properties:

a) Each $V \in \mathcal{N}$ is a balanced set, that is $\lambda x \in V$ holds for all $x \in V$ and every $\lambda \in R$ with $|\lambda| \leq 1$.

b) Each $V \in \mathcal{N}$ is an absorbing set, that is for every $x \in L$, there exists a $\lambda > 0$ such that $\lambda x \in V$.

c) For each $V \in \mathcal{N}$ there exists some $W \in \mathcal{N}$ with $W + W \subset V$.

Definition 2.2. [3] A linear topology τ on a Riesz space L is said to be locally solid if τ has a base at zero consisting of solid sets. A locally solid Riesz space (L, τ) is a Riesz space L equipped with a locally solid topology τ.

$\aleph_{sol}$ will stand for a base at zero consisting of solid sets and satisfying the properties (a), (b) and (c) in a locally solid topology.

We now recall the following basic facts from [10, 19] (see also [5, 6]).

A family $\mathcal{I}$ of subsets of a non-empty set X is said to be an ideal if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (ii) $A \in \mathcal{I}, B \subset A$ imply $B \in \mathcal{I}$. $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \{\phi\}$ and $X \notin \mathcal{I}$. $\mathcal{I}$ is admissible if it contains all singletons. If $\mathcal{I}$ is a proper non-trivial ideal then the family of sets $F(\mathcal{I}) = \{M \subset X : M^c \in \mathcal{I}\}$ is a filter on X (where c stands for the complement.) It is called the filter associated with the ideal $\mathcal{I}$.

Throughout the paper $(D, \geq)$ will stand for a directed set and $\mathcal{I}$ a non-trivial proper ideal of D. A net is a mapping from D to X and will be denoted by $\{s_\alpha : \alpha \in D\}$. Let for $\alpha \in D$, $D_\alpha = \{\beta \in D : \beta \geq \alpha\}$. Then the collection $F_0 = \{A \subset D : A \supset D_\alpha \text{ for some } \alpha \in D\}$ forms a filter in D. Let $\mathcal{I}_0 = \{A \subset D : A^c \in F_0\}$. Then $\mathcal{I}_0$ is a non-trivial ideal of D.

A nontrivial ideal $\mathcal{I}$ of D will be called D-admissible if $D_\alpha \in F(\mathcal{I})$ $\forall \alpha \in D$.

Definition 2.3. A net $\{s_\alpha : \alpha \in D\}$ in a topological space (X, τ) is said to be $\mathcal{I}$-convergent to $x_0 \in X$ if for any open set U containing x_0, $\{\alpha \in D : s_\alpha \notin U\} \in \mathcal{I}$.

We also recall the following definitions from [10].

Definition 2.4. Let (L, τ) be a locally solid Riesz space and $\{\delta_\alpha : \alpha \in D\}$ be a net in L. $\{\delta_\alpha : \alpha \in D\}$ is said to be ideal τ-convergent ($\mathcal{I}_\tau$-convergent) to $x_0 \in L$ if for any τ-neigbourhood U of zero, $\{\alpha \in D : \delta_\alpha - x_0 \notin U\} \in \mathcal{I}$. In this case we write $\mathcal{I}_\tau\text{-}\lim \delta_\alpha = x_0$.

Definition 2.5. A net $\{\delta_\alpha : \alpha \in D\}$ in a locally solid Riesz space (L, τ) is said to be $\mathcal{I}_\tau$-bounded if for any τ-neigbourhood U of zero there exists some $\lambda > 0$ such that $\{\alpha \in D : \lambda\delta_\alpha - x_0 \notin U\} \in \mathcal{I}$.

Definition 2.6. A net $\{\delta_\alpha : \alpha \in D\}$ in a locally solid Riesz space (L, τ) is said to be $\mathcal{I}_\tau$-Cauchy if for any τ-neigbourhood U of zero there exists some $\beta \in D$ such that $\{\alpha \in D : \delta_\alpha - \delta_\beta \notin U\} \in \mathcal{I}$.

3. $\mathcal{I}^\mathcal{K}$-topological convergence of nets in locally solid Riesz spaces

We first introduce our main definition.

Definition 3.1. Let $\mathcal{K}$ be a non-trivial D-admissible ideal of D. A net $\{s_\alpha : \alpha \in D\}$ in a locally solid Riesz space (L, τ) is said to be $\mathcal{I}^\mathcal{K}$-topologically convergent ($\mathcal{I}_\tau^\mathcal{K}$-convergent in short) to $x_0 \in L$ if there exists a $M \in \mathcal{F}(\mathcal{I})$ such that M itself is a directed set and the net $\{t_\alpha : \alpha \in D\}$ defined by $t_\alpha = s_\alpha$ if $\alpha \in M$ and $t_\alpha = x_0$ if $\alpha \in D \setminus M$ is $\mathcal{K}$-convergent to x_0.

Example 3.1. In the locally solid Riesz space $(\mathbb{R}^2, \|\cdot\|)$ with the Euclidean norm $\|\cdot\|$ and coordinate ordering choose the neighborhood system $\mathcal{N}_{x_0}$ of any point $x_0 \in \mathbb{R}^2$. It is known that $\mathcal{N}_{x_0}$ is itself a directed set D with respect to inclusion. Take two proper nontrivial D-admissible ideals $\mathcal{K}$ and $\mathcal{I}$ of D such that $\mathcal{K}$ contains $\mathcal{I}_0$ properly. Choose $C \in \mathcal{K}/\mathcal{I}_0$. Let $\{s_U : U \in D\}$ be given by

$$s_U \in U \ \forall U \in \mathcal{N}_{x_0} \setminus C$$
$$s_U = y_0 \ \forall U \in C$$

where $x_0 \neq y_0$. Then it is easy to observe that $\{s_U : U \in D\}$ cannot converge to x_0 usually but $\mathcal{I}_\tau^\mathcal{K}$-converges to x_0 as choosing $M = D$ we have $M \in \mathcal{F}(\mathcal{I})$ and

$$\{U \in M : s_U - x_0 \notin U\} = C \in \mathcal{K}$$

for any τ-neighbourhood U of zero which does not contain $y_0 - x_0$ (such neighborhoods exist because of Hausdorffness of $\mathbb{R}^2$).

Note that the above example can be formulated in any Hausdorff locally solid Riesz space (L, τ) with a point x_0 for which $\mathcal{N}_{x_0}$ contains infinitely many members.

Theorem 3.1. *Let (L, τ) be a locally solid Riesz space and $\{s_\alpha : \alpha \in D\}$, $\{t_\alpha : \alpha \in D\}$ be two nets in L. Then*

(i) $\mathcal{I}_\tau^\mathcal{K}$-$\lim s_\alpha = x_0 \Longrightarrow \mathcal{I}_\tau^\mathcal{K}$-$\lim \alpha s_\alpha = \alpha x_0$ *for each* $\alpha \in \mathbb{R}$.

(ii) $\mathcal{I}_\tau^\mathcal{K}$-$\lim s_\alpha = x_0$, $\mathcal{I}_\tau^\mathcal{K}$-$\lim t_\alpha = y_0 \Longrightarrow \mathcal{I}_\tau^\mathcal{K}$-$\lim (s_\alpha + t_\alpha) = x_0 + y_0$.

Proof. (*i*) Let U be a τ-neighbourhood of zero. Choose $V \in \mathcal{N}_{sol}$ such that $V \subset U$. Since $\mathcal{I}_\tau^\mathcal{K}$-$\lim s_\alpha = x_0$, there is a $M \in \mathcal{F}(\mathcal{I})$ such that the net $\{t_\alpha : \alpha \in D\}$ defined by $t_\alpha = s_\alpha$ if $\alpha \in M$ and $t_\alpha = x_0$ if $\alpha \in D \setminus M$ is $\mathcal{K}$-convergent to x_0. Then

$$\{\alpha \in D : t_\alpha - x_0 \notin V\} \in \mathcal{K}.$$

But $\{\alpha \in D : t_\alpha - x_0 \notin V\} = \{\alpha \in M : t_\alpha - x_0 \notin V\}$ (as $\forall \alpha \in D \setminus M$, $t_\alpha - x_0 = x_0 - x_0 \in V$). Thus $\{\alpha \in M : s_\alpha - x_0 \notin V\} \in \mathcal{K}$, i.e., $(D \setminus M) \cup \{\alpha \in M : s_\alpha - x_0 \in V\} \in \mathcal{F}(\mathcal{K})$. Let $|a| \leq 1$. Since V is balanced, $s_\alpha - x_0 \in V$ implies that $a(s_\alpha - x_0) \in V$. Hence we have

$$\{\alpha \in M : s_\alpha - x_0 \in V\} \subset \{\alpha \in M : as_\alpha - ax_0 \in V\} \subset \{\alpha \in M : as_\alpha - ax_0 \in U\}$$

and so $(D \setminus M) \cup \{\alpha \in M : as_\alpha - ax_0 \in U\} \in \mathcal{F}(\mathcal{K})$.

Now let $|a| > 1$ and as usual let $[|a|]$ be the smallest integer greater or equal to $|a|$. Choose a $W \in \mathcal{N}_{sol}$ such that $[|a|] W \subset V$. As before, $(D \setminus M) \cup A \in \mathcal{F}(\mathcal{K})$ where $A = \{\alpha \in M : s_\alpha - x_0 \in W\}$. Then we have

$$|ax_0 - as_\alpha| = |a| \, |x_0 - s_\alpha| \leq [|a|] \, |x_0 - s_\alpha| \in [|a|] \, W \subset V \subset U$$

for each $\alpha \in A$. Since the set V is solid, we have $ax_0 - as_\alpha \in V$ and so $ax_0 - as_\alpha \in U$ for each $\alpha \in A$. So we get

$$B = \{\alpha \in M : as_\alpha - ax_0 \in U\} \supset A$$

and so $(D \setminus M) \cup B \in \mathcal{F}(\mathcal{K})$. Clearly then $\{t'_\alpha : \alpha \in D\}$ defined by $t'_\alpha = s_\alpha$ if $\alpha \in M$ and $t'_\alpha = ax_0$ if $\alpha \in D \setminus M$ is $\mathcal{K}$-convergent to ax_0 and so $\mathcal{I}_\tau^\mathcal{K}$-$\lim \alpha s_\alpha = \alpha x_0$ for each $\alpha \in \mathbb{R}$.

(ii) Let U be an arbitrary τ-neighbourhood of zero. Choose V and $W \in \mathcal{N}_{sol}$ such that $W + W \subset V \subset U$. Since $\mathcal{I}_\tau^\mathcal{K}$-$\lim s_\alpha = x_0$, $\mathcal{I}_\tau^\mathcal{K}$-$\lim t_\alpha = y_0$ so there are M and $M_1 \in \mathcal{F}(\mathcal{I})$ such that $(D \setminus M) \cup A \in \mathcal{F}(\mathcal{K})$ where $A = \{\alpha \in M : s_\alpha - x_0 \in W\}$ and $(D \setminus M_1) \cup B \in \mathcal{F}(\mathcal{K})$ where $B = \{\alpha \in M_1 : t_\alpha - y_0 \in W\}$. Now $M \cap M_1 \in \mathcal{F}(\mathcal{I})$ and clearly

$$(s_\alpha + t_\alpha) - (x_0 + y_0) \in W + W \subset V \subset U$$

for each $\alpha \in A \cap B$. Hence we have

$$\begin{aligned}&(D \setminus (M \cap M_1)) \cup \{\alpha \in M \cap M_1 : (s_\alpha + t_\alpha) - (x_0 + y_0) \in U\}\\ &\quad\supset (D \setminus (M \cap M_1)) \cup (A \cap B)\\ &\quad= ((D \setminus (M \cap M_1)) \cup A) \cap ((D \setminus (M \cap M_1)) \cup B)\\ &\quad\supset ((D \setminus M) \cup A) \cap ((D \setminus M_1) \cup B) \in \mathcal{F}(\mathcal{K})\end{aligned}$$

and so the set on the left hand side also belongs to $\mathcal{F}(\mathcal{K})$. This proves that $\mathcal{I}_\tau^\mathcal{K}$-$\lim (s_\alpha + t_\alpha) = x_0 + y_0$. ∎

THEOREM 3.2. *Let (L, τ) be a locally solid Riesz space and $\{s_\alpha : \alpha \in D\}$, $\{t_\alpha : \alpha \in D\}$, $\{v_\alpha : \alpha \in D\}$ be three nets such that $s_\alpha \le t_\alpha \le v_\alpha$ for each $\alpha \in D$. If $\mathcal{I}_\tau^\mathcal{K}$-$\lim s_\alpha = \mathcal{I}_\tau^\mathcal{K}$-$\lim v_\alpha = x_0$ then $\mathcal{I}_\tau^\mathcal{K}$-$\lim t_\alpha = x_0$.*

Proof. Let U be an arbitrary τ-neighbourhood of zero. Choose V and $W \in \mathcal{N}_{sol}$ such that $W + W \subset V \subset U$. Now by our assumption there are $M, M_1 \in \mathcal{F}(\mathcal{I})$ such that$(D \setminus M) \cup A \in \mathcal{F}(\mathcal{K})$ where $A = \{\alpha \in M : s_\alpha - x_0 \in W\}$ and $(D \setminus M_1) \cup B \in \mathcal{F}(\mathcal{K})$ where $B = \{\alpha \in M_1 : v_\alpha - x_0 \in W\}$. Then $M \cap M_1 \in \mathcal{F}(\mathcal{I})$. For each $\alpha \in A \cap B$,

$$s_\alpha - x_0 \le t_\alpha - x_0 \le v_\alpha - x_0$$

and so

$$|t_\alpha - x_0| \le |s_\alpha - x_0| + |v_\alpha - x_0| \in W + W \subset V.$$

Since V is solid, so $t_\alpha - x_0 \in V \subset U$. Hence

$$\begin{aligned}&(D \setminus (M \cap M_1)) \cup \{\alpha \in M \cap M_1 : t_\alpha - x_0 \in U\}\\ &\qquad\supset (D \setminus (M \cap M_1)) \cup (A \cap B)\\ &\qquad\supset ((D \setminus M) \cup A) \cap ((D \setminus M_1) \cup B) \in \mathcal{F}(\mathcal{K})\end{aligned}$$

and so $(D \setminus (M \cap M_1)) \cup \{\alpha \in M \cap M_1 : t_\alpha - x_0 \in U\} \in \mathcal{F}(\mathcal{K})$. This shows that $\{t_\alpha : \alpha \in D\}$ is $\mathcal{I}_\tau^\mathcal{K}$-convergent x_0. ∎

DEFINITION 3.2. A net $\{s_\alpha : \alpha \in D\}$ is said to be $\mathcal{I}^\mathcal{K}$-topologically bounded ($\mathcal{I}_\tau^\mathcal{K}$-bounded) if there exists a $M \in \mathcal{F}(\mathcal{I})$ and if for any τ-neighbourhood U of zero there exists some $\lambda > 0$ such that $\{\alpha \in M : \lambda s_\alpha - x_0 \notin U\} \in \mathcal{K}$, i.e.

$$(D \setminus M) \cup \{\alpha \in M : \lambda s_\alpha - x_0 \in U\} \in \mathcal{F}(\mathcal{K}).$$

THEOREM 3.3. *If a net $\{s_\alpha : \alpha \in D\}$ in a locally solid Riesz space (L, τ) is $\mathcal{I}_\tau^\mathcal{K}$-convergent then it is $\mathcal{I}_\tau^\mathcal{K}$-bounded.*

Proof. Let $\mathcal{I}_\tau^\mathcal{K}$-$\lim s_\alpha = x_0$. Let U be an arbitrary τ-neighbourhood of zero. Choose V and $W \in \mathcal{N}_{sol}$ such that $W + W \subset V \subset U$. Now there is a $M \in \mathcal{F}(\mathcal{I})$ such that

$$C = \{\alpha \in M : s_\alpha - x_0 \notin W\} \in \mathcal{K}.$$

Since W is absorbing, there exists a $\lambda > 0$ such that $\lambda x_0 \in W$. We can take $\lambda \le 1$ since W is solid. Again since W is balanced, $s_\alpha - x_0 \in W$ implies that $\lambda (s_\alpha - x_0) \in W$. Then we have

$$\lambda s_\alpha = \lambda (s_\alpha - x_0) + \lambda x_0 \in W + W \subset V \subset U$$

for every $\alpha \in M \setminus C$. Hence $\{\alpha \in M : \lambda s_\alpha \notin W\} \in \mathcal{K}$, which shows that $\{s_\alpha : \alpha \in D\}$ is $\mathcal{I}_\tau^{\mathcal{K}}$-bounded. ■

DEFINITION 3.3. A net $\{s_\alpha : \alpha \in D\}$ in a locally solid Riesz space (L, τ) is said to be $\mathcal{I}_\tau^{\mathcal{K}}$-Cauchy if there exists a $M \in \mathcal{F}(\mathcal{I})$ and $\{s_\alpha : \alpha \in M\}$ is $\mathcal{K}|_M$-Cauchy, i.e., If for every τ-neighborhood U of zero there exists a $\beta \in M$ such that $\{\alpha \in M : s_\alpha - s_\beta \notin U\} \in \mathcal{K}$.

THEOREM 3.4. *If a net $\{s_\alpha : \alpha \in D\}$ in a locally solid Riesz space (L, τ) is $\mathcal{I}_\tau^{\mathcal{K}}$-convergent then it is $\mathcal{I}_\tau^{\mathcal{K}}$-Cauchy.*

Proof. Let $\mathcal{I}_\tau^{\mathcal{K}}$-$\lim s_\alpha = x_0$ and Let U be an arbitrary τ-neighbourhood of zero. Choose V and $W \in \mathcal{N}_{sol}$ such that $W + W \subset V \subset U$. Now there is a $M \in \mathcal{F}(\mathcal{I})$ such that $A = \{\alpha \in M : \lambda s_\alpha - x_0 \notin W\} \in \mathcal{K}$. Then for any $\alpha, \beta \in M \setminus A$,

$$s_\alpha - s_\beta = s_\alpha + x_0 - x_0 - s_\beta \in W + W \subset V \subset U.$$

Hence it follows that $\{\alpha \in M : s_\alpha - s_\beta \notin U\} \in \mathcal{K}$. As this is true for every τ-neighbourhood U of zero, $\{s_\alpha : \alpha \in D\}$ is $\mathcal{I}_\tau^{\mathcal{K}}$-Cauchy. ■

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