

FABER POLYNOMIAL COEFFICIENT ESTIMATES FOR ANALYTIC BI-BAZILEVIČ FUNCTIONS

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Abstract. A function is said to be bi-univalent in the open unit disk \mathbb{D} if both the function and its inverse are univalent in \mathbb{D} . By the same token, a function is said to be bi-Bazilevič in \mathbb{D} if both the function and its inverse are Bazilevič there. The behavior of these types of functions are unpredictable and not much is known about their coefficients. In this paper we use the Faber polynomial expansions to find upper bounds for the coefficients of classes of bi-Bazilevič functions. The coefficients bounds presented in this paper are better than those so far appeared in the literature. The technique used in this paper is also new and we hope that this will trigger further interest in applying our approach to other related problems.

1. Introduction

Let S denote the set of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

that are analytic and univalent in the open unit disk $\mathbb{D} := \{z : |z| < 1\}$.

For $0 \leq \alpha < 1$ and $0 \leq \beta < 1$, a function $f \in S$ is said to be Bazilevič [6] of order α and type β , denoted by $\mathcal{B}(\alpha; \beta)$, if

$$\operatorname{Re} \left(\left(\frac{z}{f(z)} \right)^{1-\beta} f'(z) \right) > \alpha; \quad z \in \mathbb{D}.$$

If $g = f^{-1}$ is the inverse of the function $f \in S$, then g has a Maclaurin series expansion in some disk about the origin [11]. In 1923, Löwner [24] proved that the inverse of the Koebe function $f(z) = z/(1-z)^2$ provides the best upper bounds for the coefficients of the inverses of the functions $f \in S$. Sharp bounds for the coefficients of the inverses of univalent functions have been obtained in a surprisingly straightforward way, whereas the case for the subclasses of univalent functions turned out to be a challenge. In 1979, Krzyz et al. [18] obtained sharp upper bounds for the first two coefficients of inverses of the functions starlike of order α ; $0 \leq \alpha < 1$. In

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1982, Libera and Zlotkiewicz [20] found the bounds for the first seven coefficients of the inverse of convex functions. Later, in [21] they obtained the bounds for the first six coefficients of the inverse of f provided $\operatorname{Re}f'(z) > 0$; $z \in \mathbb{D}$. In a follow up paper [20] they considered the odd functions $f(z) = z + a_3z^3 + a_5z^5 + \dots$ and showed that if $\operatorname{Re}f'(z) > 0$; $z \in \mathbb{D}$ then $[-z + \log((1+z)/(1-z))]^{-1}$ is the extremal function for the inverse of f . In 1986, Juneja and Rajasekaran [17] obtained coefficient estimates for inverses of α -spiral functions. In 1989, Silverman [28] proved that if $f \in \mathcal{S}$ is such that $\sum_{n=2}^{\infty} n|a_n| \leq 1$ then the n -th coefficient of the inverse of f is bounded above by $\frac{1}{n} \binom{2n-3}{n-2} \frac{1}{2^{n-2}}$. In 1992, Libera and Zlotkiewicz [23] proved that the n -th coefficients of the inverse of starlike functions are bounded above by $[(2n)!/n!(n+1)!]$. Chou [10] in 1994, proved that if $f \in \mathcal{S}$ and $\operatorname{Re}f'(z) > 0$; $z \in \mathbb{D}$ then $-z + 2\log(1+z)$; $z \in \mathbb{D}$ is the extremal function for the inverse of f . Estimates for the first two coefficients of the inverses of subclasses of starlike functions were also obtained in [11] and [30].

Finding coefficient estimates for the inverses of univalent function becomes even more involved when the bi-univalence condition is imposed on these functions. A function $f \in \mathcal{S}$ is said to be *bi-univalent* in \mathbb{D} if its inverse map $g = f^{-1}$ is also univalent in \mathbb{D} . The class of bi-univalent analytic functions was first introduced and studied by Lewin [19] where it was proved that $|a_2| < 1.51$. Brannan and Clunie [7] improved Lewin's result to $|a_2| \leq \sqrt{2}$ and later Netanyahu [25] proved that $|a_2| \leq 4/3$. Brannan and Taha [8] and Taha [31] also investigated certain subclasses of bi-univalent functions and found estimates for their initial coefficients. Recently, Srivastava et al. [29], Frasin and Aouf [13], and Ali et al. [5] found estimates for the first two coefficients of certain subclasses of bi-univalent functions. The bi-univalence requirement makes the behavior of the coefficients of the function f and its inverse $g = f^{-1}$ unpredictable. Not much is known about the higher coefficients of bi-univalent functions as Ali et al. [5] also remarked that finding the bounds for the n -th, ($n \geq 4$) coefficients of classes of bi-univalent functions is an open problem. Hamidi et al. [15,16] used Faber polynomial expansions to find coefficient estimates for classes of meromorphic bi-univalent functions. In this paper we use the Faber polynomial expansions to find upper bounds for the n -th, ($n \geq 3$) coefficients of classes of analytic bi-Bazilevič functions. A function f is said to be bi-Bazilevič of order α and type β in \mathbb{D} if both f and its inverse $g = f^{-1}$ are Bazilevič of order α and type β in \mathbb{D} . We conclude our paper with an examination of the unexpected behavior of the first two coefficients of bi-Bazilevič functions. The coefficient estimates presented in this paper are the best yet appeared in the literature. We hope that the new technique presented in this article triggers further interest in applying our approach to other related problems.

2. Main results

Consider the function $f \in \mathcal{S}$ of the form (1.1). Then $g = f^{-1}$, the inverse map of f , may be represented by the Faber polynomial expansion (see [2,4,32]),

$$g(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots, a_n) w^n, \quad (2.1)$$

where $K_{n-1}^{-n} = K_{n-1}^{-n}(a_2, a_3, \dots, a_n)$, $n - 1 \geq 1$, $-n \in \mathbb{Z}$ and in general, for any real number p , an expansion of $K_n^p = K_n^p(a_2, a_3, \dots, a_n)$ (e.g. see [1–3]) is

$$K_{n-1}^p = pa_n + \frac{p(p-1)}{2}D_{n-1}^2 + \frac{p!}{(p-3)!3!}D_{n-1}^3 + \dots + \frac{p!}{(p-n+1)!(n-1)!}D_{n-1}^{n-1}, \tag{2.2}$$

where $D_{n-1}^p = D_{n-1}^p(a_2, a_3, \dots, a_n)$, are homogeneous polynomials explicated in [2],

$$D_{n-1}^p(a_2, \dots, a_n) = \sum_{n=2}^{\infty} \frac{p!(a_2)^{\mu_1} \dots (a_n)^{\mu_{n-1}}}{\mu_1! \dots \mu_{n-1}!} \quad \text{for } p \leq n - 1,$$

and the sum is taken over all nonnegative integers μ_1, \dots, μ_{n-1} satisfying

$$\begin{cases} \mu_1 + \mu_2 + \dots + \mu_{n-1} = p, \\ \mu_1 + 2\mu_2 + \dots + (n-1)\mu_{n-1} = n - 1. \end{cases}$$

Evidently: $D_{n-1}^{n-1}(a_2, \dots, a_n) = a_2^{n-1}$ (see [1,2,32]).

Examples of the first three terms of $K_{n-1}^{-n} = K_{n-1}^{-n}(a_2, a_3, \dots, a_n)$ are $K_1^{-2} = -2a_2$, $K_2^{-3} = 3(2a_2^2 - a_3)$ and $K_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4)$.

The Faber polynomials introduced by Faber [12] (see also Schur [27]) play an important role in various areas of mathematical sciences, especially in geometric function theory (Gong [14] Chapter III, Schiffer [26], and Todorov [32]). The recent interest in the calculus of Faber polynomials, especially when it involves f^{-1} , the inverse map of f (see [1–4]) beautifully fits our case for the bi-Bazilevič functions. As a result, we are able to state and prove the following

THEOREM 2.1. *For $0 \leq \alpha < 1$ and $0 \leq \beta < 1$ let $f \in \mathcal{B}(\alpha; \beta)$ be bi-Bazilevič in \mathbb{D} . If $a_k = 0$ for $2 \leq k \leq n - 1$, then*

$$|a_n| \leq \frac{2(1-\alpha)}{(n-1)+\beta}; \quad n \geq 3.$$

Proof. For $f \in \mathcal{B}(\alpha; \beta)$ and for its inverse function $g = f^{-1} \in \mathcal{B}(\alpha; \beta)$, there exist positive real part functions $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n$ in \mathbb{D} so that

$$\left(\frac{z}{f(z)}\right)^{1-\beta} f'(z) = \alpha + (1-\alpha)p(z) = 1 + (1-\alpha) \sum_{n=1}^{\infty} c_n z^n, \tag{2.3}$$

and

$$\left(\frac{w}{g(w)}\right)^{1-\beta} g'(w) = \alpha + (1-\alpha)q(w) = 1 + (1-\alpha) \sum_{n=1}^{\infty} d_n w^n. \tag{2.4}$$

Throughout the rest of this article we shall use the inequalities $|c_n| \leq 2$ and $|d_n| \leq 2$ which are known as the Caratheodory Lemma (e.g. [11]).

Using the Faber polynomial expansions given by [4, Eqs. (1.6) and (1.7)], we have

$$\begin{aligned} \left(\frac{f(z)}{z}\right)^p \frac{zf'(z)}{f(z)} &= 1 - \sum_{n=2}^{\infty} F_{n-1}^{n+p-1}(a_2, a_3, \dots, a_n)z^{n-1} \\ &= 1 + \sum_{n=2}^{\infty} \left(1 + \frac{n-1}{p}\right) K_{n-1}^p(a_2, a_3, \dots, a_n)z^{n-1}, \end{aligned}$$

where K_n^p is defined by (2.2). Therefore, the left hand sides of the equations (2.3) and (2.4) can be expressed by

$$\begin{aligned} \left(\frac{z}{f(z)}\right)^{1-\beta} f'(z) &= \left(\frac{f(z)}{z}\right)^\beta \left(\frac{zf'(z)}{f(z)}\right) \\ &= 1 + \sum_{n=2}^{\infty} \left(1 + \frac{n-1}{\beta}\right) K_{n-1}^\beta(a_2, a_3, \dots, a_n) z^{n-1}, \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} \left(\frac{w}{g(w)}\right)^{1-\beta} g'(w) &= \left(\frac{g(w)}{w}\right)^\beta \left(\frac{wg'(w)}{g(w)}\right) \\ &= 1 + \sum_{n=2}^{\infty} \left(1 + \frac{n-1}{\beta}\right) K_{n-1}^\beta(A_2, A_3, \dots, A_n) w^{n-1}, \end{aligned} \quad (2.6)$$

where by (2.1),

$$A_n = \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots, a_n), \quad n = 2, 3, \dots.$$

Comparing the corresponding coefficients of (2.3) and (2.5), we obtain

$$\left(1 + \frac{n-1}{\beta}\right) K_{n-1}^\beta(a_2, a_3, \dots, a_n) = (1-\alpha)c_{n-1}. \quad (2.7)$$

Similarly, from (2.4) and (2.6) we obtain

$$\left(1 + \frac{n-1}{\beta}\right) K_{n-1}^\beta(A_2, A_3, \dots, A_n) = (1-\alpha)d_{n-1}. \quad (2.8)$$

Since $a_k = 0$ for $2 \leq k \leq n-1$, the equations (2.7) and (2.8), respectively, imply

$$(\beta + (n-1))a_n = (1-\alpha)c_{n-1} \quad (2.9)$$

and

$$-(\beta + (n-1))a_n = (1-\alpha)d_{n-1}. \quad (2.10)$$

Now by solving either of the equations (2.9) or (2.10) for a_n and taking the absolute values we obtain

$$|a_n| \leq \frac{2(1-\alpha)}{(n-1) + \beta}. \quad \blacksquare$$

Relaxing the coefficient restrictions imposed on Theorem 2.1, we experience the unpredictable behavior of the coefficients of bi-Bazilevič functions.

THEOREM 2.2. For $0 \leq \alpha < 1$ and $0 \leq \beta < 1$ let $f \in \mathcal{B}(\alpha; \beta)$ be bi-Bazilevič in \mathbb{D} . Then

$$\begin{aligned} \text{(i)} \quad |a_2| &\leq \begin{cases} \sqrt{\frac{4(1-\alpha)}{(1+\beta)(2+\beta)}}, & 0 \leq \alpha < \frac{1}{2+\beta}; \\ \frac{2(1-\alpha)}{1+\beta}, & \frac{1}{2+\beta} \leq \alpha < 1. \end{cases} \\ \text{(ii)} \quad |a_3 - a_2^2| &\leq \frac{2(1-\alpha)}{2+\beta}. \end{aligned}$$

Proof. The equation (2.7) for $n = 2$ and $n = 3$, respectively, implies

$$(\beta + 1)a_2 = (1 - \alpha)c_1, \quad (2.11)$$

and

$$\frac{(\beta - 1)(\beta + 2)}{2}a_2^2 + (\beta + 2)a_3 = (1 - \alpha)c_2. \quad (2.12)$$

Similarly, the equation (2.8) yields $(\beta + 1)A_2 = (1 - \alpha)d_1$, and

$$\frac{(\beta - 1)(\beta + 2)}{2}A_2^2 + (\beta + 2)A_3 = (1 - \alpha)d_2,$$

and for suitable values of A_2 and A_3 we deduce

$$-(\beta + 1)a_2 = (1 - \alpha)d_1, \quad (2.13)$$

and

$$\frac{(\beta - 1)(\beta + 2)}{2}a_2^2 + (\beta + 2)(2a_2^2 - a_3) = (1 - \alpha)d_2. \quad (2.14)$$

Solve either of the equations (2.11) or (2.13) for a_2 and take the absolute values to obtain

$$|a_2| \leq \frac{2(1 - \alpha)}{1 + \beta}.$$

On the other hand, by adding the equations (2.12) and (2.14) we obtain

$$(1 + \beta)(2 + \beta)(a_2)^2 = (1 - \alpha)(c_2 + d_2).$$

Solving the above equation for a_2 and taking the absolute values we obtain

$$|a_2| \leq \sqrt{\frac{4(1 - \alpha)}{(1 + \beta)(2 + \beta)}}.$$

Now the bounds given for $|a_2|$ can be justified upon noting that

$$\sqrt{\frac{4(1 - \alpha)}{(\beta + 1)(\beta + 2)}} < \frac{2(1 - \alpha)}{1 + \beta} \quad \text{if} \quad 0 \leq \alpha < \frac{1}{2 + \beta}.$$

Subtracting (2.14) from (2.12) we obtain

$$2(2 + \beta)(a_3 - a_2^2) = (1 - \alpha)(c_2 - d_2).$$

Dividing by $2(2 + \beta)$ and taking the absolute values yield

$$|a_3 - a_2^2| \leq \frac{1 - \alpha}{2(2 + \beta)} (|c_2| + |d_2|) \leq \frac{2(1 - \alpha)}{2 + \beta}. \quad \blacksquare$$

Letting $\beta = 0$ in Theorem 2.2 we obtain the following corollary for analytic bi-starlike functions of order α ; $0 \leq \alpha < 1$.

COROLLARY 2.1. *Let $f \in \mathcal{B}(\alpha; 0)$ be bi-starlike of order α in \mathbb{D} . Then*

- (i) $|a_2| \leq \begin{cases} \sqrt{2(1 - \alpha)}, & 0 \leq \alpha < \frac{1}{2}; \\ 2(1 - \alpha), & \frac{1}{2} \leq \alpha < 1. \end{cases}$
- (ii) $|a_3 - a_2^2| \leq 1 - \alpha$ for $0 \leq \alpha < 1$.

In the following corollary we show that the bounds given in Theorems 2.1 and 2.2 are better than those given by Srivastava, Mishra and Gochhayat [29, p. 1191, Theorem 2] and Frasin and Aouf [13, p. 1572, Theorem 3.2].

COROLLARY 2.2. *For $0 \leq \alpha < 1$ let $f \in \mathcal{B}(\alpha; 1)$ be bi-Bazilevič in \mathbb{D} . Then*

$$(i) |a_2| \leq \begin{cases} \sqrt{\frac{2(1-\alpha)}{3}}, & 0 \leq \alpha < \frac{1}{3}; \\ 1 - \alpha, & \frac{1}{3} \leq \alpha < 1. \end{cases}$$

$$(ii) |a_3| \leq \frac{2(1-\alpha)}{3}.$$

Proof. Part (i) follows by letting $\beta = 1$ in Theorem 2.2.(i). For part (ii) substitute $\beta = 1$ in the equation (2.12) to obtain $3a_3 = (1 - \alpha)c_2$. Now dividing by 3 and taking the absolute values of both sides yield $|a_3| \leq 2(1 - \alpha)/3$. ■

REMARK. For different values of α and β , Theorem 2.2 demonstrates the fluctuation of the early coefficients of the bi-Bazilevič functions. Determination of extremal functions for bi-univalent functions (in general) and for bi-Bazilevič functions (in particular) remain a challenge.

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