GLOBAL EXISTENCE AND BOUNDEDNESS OF SOLUTIONS FOR A GENERAL ACTIVATOR-INHIBITOR MODEL

Said Kouachi

Abstract. The purpose of this paper is to prove global existence in time of solutions for a class of multi-component reaction diffusion systems called multiple Gierer-Meinhardt type. The system describes, following Gierer-Meinhardt's scheme, "m" substances in interaction. The nonlinearities present a difficulty since they are fractions. We prove the global existence by using a series of Lyapunov functionals.

1. Introduction

We consider the reaction-diffusion system

$$\frac{\partial u_i}{\partial t} - d_i \Delta u_i = f_i = \prod_{k=1}^{m} u_k^{p_{ik}}, \quad \text{on } \mathbb{R}^+ \times \Omega, \quad i = 1, \dots, m, \tag{1}$$

where we assume that there is no flux through the boundary, i.e., we impose Neumann boundary conditions

$$\frac{\partial u_i}{\partial \eta} = 0 \quad \text{on } \mathbb{R}^+ \times \partial \Omega, \quad i = 1, \dots, m, \tag{2}$$

the initial data

$$u_i(0, x) = u_i^0(x), \text{ in } \Omega, \ i = 1, \dots, m, \tag{3}$$

are assumed to be nonnegative and in $\mathbb{L}^\infty(\Omega)$. The open domain Ω is bounded and of class $\mathbb{C}^1$, with boundary $\partial\Omega$ and $\partial/\partial\eta$ denotes the outward normal derivative on $\partial\Omega$. The positive constants d_i, $i = 1, \dots, m$, are the diffusion coefficients of the system. We suppose that the reactions are fractions: for some $l = 1, \dots, m$, the exponents p_{ik} satisfy

$$p_{ik} > 0 \text{ with } p_{kk} > 1, \ k = 1, \dots, l \ \text{ and } p_{ik} < 0, \ k = l+1, \dots, m, \ \forall i = 1, \dots, m, \tag{4}$$

---

and that for all $i = 1, \ldots, l$, there exists $j = l + 1, \ldots, m$, such that

$$0 < \frac{p_{ii} - 1}{p_{ji}} < 1, \tag{5}$$

$$\left(\frac{p_{ii} - 1}{p_{ji}} \right) \left(\frac{p_{jj} - 1}{p_{ij}} \right) < 1. \tag{6}$$

If we have more then two equations $(m > 2)$, we suppose

$$\frac{p_{ik}}{p_{ii} - 1} < \frac{p_{jk}}{p_{ji}}, \text{ for all } k = 1, \ldots, m, \quad k \neq i, j. \tag{7}$$

In the case of Coupled Reaction-Diffusion equations $(m = 2)$, problem (1)–(3) describes the pattern formation of spatial tissue structures of hydra in morphogenesis. This mathematical model was proposed by A. Gierer and H. Meinhardt [3] following an idea of A. Turing [14]. The system in the form considered in [5] is the following

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - d_1 \Delta u_1 = \sigma_1(x) - a_1 u_1 + \rho_1(u_1, u_2, x) u_1^{p_{11}} u_2^{p_{12}}, & \text{on } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial u_2}{\partial t} - d_2 \Delta u_2 = \sigma_2(x) - a_2 u_2 + \rho_2(u_1, u_2, x) u_1^{p_{21}} u_2^{p_{22}}, & \text{on } \mathbb{R}^+ \times \Omega, \end{cases} \tag{8}$$

where u_1 and u_2 are the unknowns representing the concentrations of two substances called the activator and the inhibitor and a_1 and a_2 are positive constants. The functions σ_1 and σ_2 are positive and continuously differentiable on Ω and ρ_1 and $\rho_2 \in C^1 \left(\mathbb{R}_+^2 \times \Omega \right) \cap \mathbb{L}^\infty \left(\mathbb{R}_+^2 \times \Omega \right)$. Under the conditions (4)–(6) on the exponents $p_{ij} (i, j = 1, 2)$ the authors in [2] and [5] later, proved global existence of solutions for system (8) with homogenous Neumann boundary conditions and positive bounded initial conditions.

Earlier partial results of global existence were proved when $N = 3$, $p_{11} = 2$, $p_{21} = 2$, $p_{12} = -1$, $p_{22} = 0$ by F. Rothe [13] in 1984. In 1987, K. Masuda and K. Takahashi [9] extended the result to $\frac{p_{11} - 1}{p_{21}} < \frac{2}{N+2}$. On the other hand, when

$$\frac{p_{ii} - 1}{p_{ji}} > 1,$$

it is shown in [10] that there exist initial values such that the solution blows up in finite time. The equality case in the above inequality and (6) seems to be open.

To get a more general form of system (1) and for reasons of biological applications (especially when $m = 2$ and 3), the reaction terms in (1) are perturbed (following K. Masuda and K. Takahashi [9]) by linear terms to become

$$f_i = \sigma_i - a_i u_i + \rho_i \prod_{k=1}^m u_k^{p_{ik}}, \quad i = 1, \ldots, m,$$

where a_i, σ_i, ρ_i are positive constants. As an example of biological applications of system (1) with more then three equations, we cite the extended secretion model

in the case of multiple Gierer-Meinhardt systems described by R. Bauer and al. [1] as follows:

$$\begin{cases} \dfrac{\partial a_i}{\partial t} - D_a \dfrac{\partial^2 a_i}{\partial x^2} = \rho_a - \mu_a a_i + \rho \dfrac{a_i^2}{b_i} - \displaystyle\sum_{j \neq i} r a_j, \\[4mm] \dfrac{\partial b_i}{\partial t} - D_b \dfrac{\partial^2 b_i}{\partial x^2} = \rho_b - \mu_b b_i + \rho a_i^2, \end{cases} \tag{9}$$

where a_i and b_i indicate the activator and inhibitor substance concentrations respectively, $i = 1, \ldots, m$.

In this paper, we generalize these results, concerning global existence of solutions to reaction diffusion systems with m equations by using a series of Lyapunov functionals for coupled components of $(u_1, \ldots, u_m)$ analogous to that considered by the authors in [2] and [5] for (u_1, u_2) (see R. D. Parshad, S. Kouachi and J. B. Gutierrez [11] and S. Kouachi [6,7,8]).

2. Notations and preliminaries

The usual norms in the spaces $\mathbb{L}^p(\Omega)$, $p \geq 1$, $\mathbb{L}^\infty(\Omega)$ and $C\left(\overline{\Omega}\right)$ are respectively denoted by :

$$\|u\|_p^p = \frac{1}{|\Omega|} \int_\Omega |u(x)|^p \, dx,$$

$$\|u\|_\infty = \operatorname*{ess.\,sup}_{x \in \Omega} |u(x)|.$$

In order to show global existence in time of a solution, we start with the following standard local existence and uniqueness result which follows from the basic existence theory for abstract semilinear differential equations (see D. Henry [4]).

PROPOSITION 1. *If the initial conditions are positive and uniformly bounded in Ω, then the problem* (1)–(3) *admits a unique classical and positive solution $u = (u_1, \ldots, u_m)$ on $[0, T_{\max}[\times \Omega$. If $T_{\max} < \infty$ then*

$$\lim_{t \nearrow T_{\max}} \|u(t, .)\|_\infty = \infty.$$

The proof of the local existence of a solution to a multi-component reaction-diffusion system such as (1) with certain regularity requirement is not a trivial issue. The arguments of the proof are classical applied in $\mathbb{L}^2(\Omega)$ (see also A. Pazy [12, Th.6.1.4]), except that we need additional $\mathbb{L}^\infty$ growth estimates and positivity control. Thus we combine the local Lipschitz continuity of f_i which is in $C^1\left(\mathbb{R}_+^m, \mathbb{R}_+\right)$, $i = 1, \ldots, m$, the Gronwall inequality, and the following non-expansiveness property of the semigroup $S_i(t)$ associated to the operator $d_i \Delta$ in $\mathbb{L}^\infty(\Omega)$

$$\forall p \in [1, +\infty] \quad \|S_i(t) \varphi\|_p \leq \|\varphi\|_p \text{ for all } t \in [0, T[\text{ and } \varphi \in \mathbb{L}^\infty(\Omega), \quad i = 1, \ldots, m.$$

The Banach fixed-point theorem gives the following unique local solution (called mild solution) with values in $\mathbb{L}^\infty(\Omega)$:

$$u_i(t, .) = S_i(t) u_i^0 + \int_0^t S_i(t - s) f_i(u_1(s, .), \ldots, u_m(s, .)) \, ds, \ t \in [0, T[, \ i = 1, \ldots, m,$$

for some positive T. We verify by standard arguments that this solution belongs to the space $H_{loc}^1\left((0,T)\,;\mathbb{L}^2(\Omega)\right)\cap C\left([0,T)\,;\mathbb{L}^2(\Omega)\right)$ and satisfies, in the classical sense, the problem (1)–(3). Finally, the continuation principle is used to get the solution on a maximal interval $[0,T_{\max}[$.

Also, we can show by comparison arguments for parabolic equations (D. Henry [4] and F. Rothe [13]) that if the initial data are nonnegative, then the solutions are nonnegative on $[0,T_{\max}[\times\Omega$. Moreover, from similar arguments using the maximum principle, we have

$$u_i(t,x) > \underline{u}_i^0 > 0, \ \text{ on } \ ]0,T_{\max}[\,\times\,\Omega, \ \ i=1,\dots,m, \tag{10}$$

where $\underline{u}_i^0 = \min_{\overline{\Omega}}u_i^0 > 0$. To see this, since the solutions are nonnegative, we have

$$\frac{\partial u_i}{\partial t} - d_i\Delta u_i > \frac{dy_i}{dt}, \ \ \text{ on } \ [0,T_{\max}[\,\times\,\Omega, \ \ i=1,\dots,m,$$

where y_i are the solutions of the following ordinary differential system

$$\frac{dy_i}{dt} = 0, \ \ \text{ on } \ [0,T_{\max}[, \ \ i=1,\dots,m,$$

with initial data

$$y_i\,(0) = \underline{u}_i^0, \ \ i=1,\dots,m,$$

then by comparison arguments using the maximum principle, we have

$$u_i(t,x) > y_i > 0, \ \text{ on } \ ]0,T_{\max}[\,\times\,\Omega, \ \ i=1,\dots,m,$$

and since the solutions of the corresponding ordinary differential system are $y_i \equiv \underline{u}_i^0$, $i=1,\dots,m$, this gives (10).

3. Global existence

It is well known that to prove global existence of solutions to (1)–(3) (see Henry [4]), it suffices, thanks to the L^p-regularity theory for the heat operator, to derive a uniform estimate of $\|f_i\|_p$ on $[0,T_{\max}[$ for some $p > N/2$.

The main result of this paper is the following theorem.

THEOREM. *Let* $u(t,.) = (u_1(t,.),\dots,u_m(t,.))$ *be any solution of the problem* (1)–(3); *then under conditions* (4)–(7) *the functional*

$$t \mapsto L(t) = \int_\Omega \prod_{k=1}^m u_k^{\alpha_k}\,dx, \tag{11}$$

is bounded on $[0,T_{\max}[$ *for all* $\alpha_j > 1$, $j=1,\dots,l$ *and all* $\alpha_j < 0$, $j=l+1,\dots,m$.

For the proof of the Theorem, we need the following Lemma

LEMMA 1. *Let* $\alpha > 0$ *and* $\beta < 0$ *be two numbers; then for all positive numbers* $\underline{u}_1,\dots,\underline{u}_m\,(\underline{u}_i = 0)$ *and all* p_{ik} *and* p_{jk}, $1 \le k \le m$, *satisfying* (4)–(7), *there exist* $\theta\,(\alpha) \in (0,1)$ *and a positive constant* $C_1\,(\alpha)$ *such that*

$$\frac{\alpha}{u_i}\left(u_i^\alpha u_j^\beta\right)\prod_{k=1}^m u_k^{p_{ik}} \le -\frac{\beta}{u_j}\left(u_i^\alpha u_j^\beta\right)\prod_{k=1}^m u_k^{p_{jk}} + C_1\left(u_i^\alpha u_j^\beta\right)^\theta, \tag{12}$$

for all $u_k \ge \underline{u}_k$, $k=1,\dots,m$.

Proof. Relation (12) is equivalent to

$$\frac{\alpha}{u_i} \prod_{k=1}^{m} u_k^{p_{ik}} \leq -\frac{\beta}{u_j} \prod_{k=1}^{m} u_k^{p_{jk}} + C_1 \left(u_i^{\alpha} u_j^{\beta} \right)^{\theta-1}, \quad \text{for all } u_k \geq \underline{u}_k, \ k = 1, \ldots, m. \quad (13)$$

We have

$$\frac{\alpha}{u_i} \prod_{k=1}^{m} u_k^{p_{ik}} = C_2 \left(-\frac{\beta}{u_j} \prod_{k=1}^{m} u_k^{p_{jk}} \right)^{\frac{p_{ii}-1}{p_{ji}}} \left(\prod_{k \neq i,j}^{m} u_k^{p_{ik}-p_{jk}\left(\frac{p_{ii}-1}{p_{ji}}\right)} \right) u_j^{p_{ij}-(p_{jj}-1)\left(\frac{p_{ii}-1}{p_{ji}}\right)}$$

$$= C_2 \left(-\frac{\beta}{u_j} \prod_{k=1}^{m} u_k^{p_{jk}} \right)^{\frac{p_{ii}-1}{p_{ji}}+\epsilon} \left(\prod_{k \neq i,j}^{m} u_k^{p_{ik}-p_{jk}\left(\frac{p_{ii}-1}{p_{ji}}\right)-\epsilon p_{jk}} \right)$$

$$\times \left(u_i^{-\epsilon p_{ji}} u_j^{p_{ij}-(p_{jj}-1)\left(\frac{p_{ii}-1}{p_{ji}}\right)-\epsilon p_{jj}} \right)$$

$$= C_2 \left(-\frac{\beta}{u_j} \prod_{k=1}^{m} u_k^{p_{jk}} \right)^{\frac{p_{ii}-1}{p_{ji}}+\epsilon} \left(u_i^{\alpha} u_j^{\beta} \right)^{-\frac{\epsilon p_{ji}}{\alpha}} \left(\prod_{k \neq i,j}^{m} u_k^{p_{ik}-p_{jk}\left(\frac{p_{ii}-1}{p_{ji}}\right)-\epsilon p_{jk}} \right)$$

$$\times \left(u_j^{p_{ij}\left[1-\left(\frac{p_{jj}-1}{p_{ij}}\right)\left(\frac{p_{ii}-1}{p_{ji}}\right)\right]+\epsilon\left(\frac{\beta}{\alpha}p_{ji}-p_{jj}\right)} \right),$$

where C_2 is a positive constant. As $p_{ij} < 0$ and $u_j \geq \underline{u}_j > 0$, then under condition (6) and for ϵ sufficiently small, we have

$$u_j^{p_{ij}\left[1-\left(\frac{p_{jj}-1}{p_{ij}}\right)\left(\frac{p_{ii}-1}{p_{ji}}\right)\right]+\epsilon\left(\frac{\beta}{\alpha}p_{ji}-p_{jj}\right)} \leq C_3,$$

where C_3 is a positive constant. Since $p_{ii} > 1$ and $u_k \geq \underline{u}_k > 0, \ k = 1, \ldots, m,$ and $k \neq i, j$, then under condition (7) and for ϵ sufficiently small, we have

$$\prod_{k \neq i,j}^{m} u_k^{p_{ik}-p_{jk}\left(\frac{p_{ii}-1}{p_{ji}}\right)-\epsilon p_{jk}} \leq C_4,$$

where C_4 is a positive constant. This implies that

$$\frac{\alpha}{u_i} \prod_{k=1}^{m} u_k^{p_{ik}} \leq C_5 \left(-\frac{\beta}{u_j} \prod_{k=1}^{m} u_k^{p_{jk}} \right)^{\frac{p_{ii}-1}{p_{ji}}+\epsilon} \left(u_i^{\alpha} u_j^{\beta} \right)^{-\frac{\epsilon p_{ji}}{\alpha}}, \quad (14)$$

for all $u_k > \underline{u}_k, \ k = 1, \ldots, m$, where C_5 is a positive constant. Now, under condition (5), Young's inequality in the form

$$ab \leq \epsilon a^p + C(\epsilon) b^q, \ \tfrac{1}{p} + \tfrac{1}{q} = 1,$$

for

$$\frac{1}{p} = \frac{p_{ii}-1}{p_{ji}} + \epsilon,$$

where ϵ is sufficiently small, is applicable to the right-hand side of inequality (14) to deduce (13) with

$$\theta = 1 - \frac{\left(\frac{\epsilon p_{ji}}{\alpha}\right)}{1 - \frac{p_{ii}-1}{p_{ji}} - \epsilon}.$$

This completes the proof of the Lemma. ∎

The proof of the above Lemma can be found in [2].

Proof of the Theorem. Put

$$L_{ij}(t) = \int_\Omega u_i^{\alpha_i} u_j^{\alpha_j} \, dx. \tag{15}$$

Differentiating L_{ij} with respect to t yields

$$\begin{aligned}
L'_{ij}(t) &= \int_\Omega \left(\frac{\alpha_i}{u_i} \frac{\partial u_i}{\partial t} + \frac{\alpha_j}{u_j} \frac{\partial u_j}{\partial t} \right) u_i^{\alpha_i} u_j^{\alpha_j} \, dx \\
&= \int_\Omega \left(d_i \frac{\alpha_i}{u_i} \Delta u_i + d_j \frac{\alpha_j}{u_j} \Delta u_j \right) u_i^{\alpha_i} u_j^{\alpha_j} \, dx \\
&\quad + \int_\Omega \left(\frac{\alpha_i}{u_i} \prod_{k=1}^m u_k^{p_{ik}} + \frac{\alpha_j}{u_j} \prod_{k=1}^m u_k^{p_{jk}} \right) u_i^{\alpha_i} u_j^{\alpha_j} \, dx \\
&= I + J. \tag{16}
\end{aligned}$$

By a simple use of Green's formula, we get

$$\begin{aligned}
I = - \int_\Omega u_i^{\alpha_i} u_j^{\alpha_j} \Bigg(& d_i \left(-\alpha_i + \alpha_i^2 \right) \left| \frac{\nabla u_i}{u_i} \right|^2 \\
& + (d_i + d_j) \alpha_i \alpha_j \frac{\nabla u_i}{u_i} \frac{\nabla u_j}{u_j} + d_j \left(-\alpha_j + \alpha_j^2 \right) \left| \frac{\nabla u_j}{u_j} \right|^2 \Bigg) \, dx.
\end{aligned}$$

Therefore, $I \le 0$, if

$$[(d_i + d_j)\, \alpha_i \alpha_j]^2 - 4 d_i d_j \left(-\alpha_i + \alpha_i^2 \right) \left(-\alpha_j + \alpha_j^2 \right) < 0,$$

that is

$$\left(\frac{\alpha_i - 1}{\alpha_i} \right) \left(\frac{\alpha_j - 1}{\alpha j} \right) > \frac{(d_i + d_j)^2}{4 d_i d_j}.$$

If for all $\alpha_i > 1$, we choose $\alpha_j < 0$ sufficiently close to zero, we get $I \le 0$.

For the integral J given by (16), we use (12) to get

$$J \le C_1 \int_\Omega \left(u_i^{\alpha_i} u_j^{\alpha_j} \right)^\theta \, dx. \tag{17}$$

Since $0 < \theta < 1$, by application of Hölder's inequality, we get

$$J \le C_6 \left(\int_\Omega u_i^{\alpha_i} u_j^{\alpha_j} \, dx \right)^\theta, \tag{18}$$

where C_6 is a positive constant. Since $I \le 0$, we get

$$L'_{ij} \le C_6 L_{ij}^\theta, \quad \text{on } [0, T_{\max}[. \tag{19}$$

If $T_{\max} < +\infty$, a simple integration gives the boundedness of the functional L_{ij} on the interval $[0, T_{\max}[$. As u_j is bounded below, we get the boundedness of the functional L_{ij} on the interval $[0, T_{\max}[$.

Since for each $\alpha_i > 1$, $i = 1, \ldots, l$, there exists an $\alpha_j < 0$, $j = l+1, \ldots, m$, such that the functional L_{ij} is bounded on the interval $[0, T_{\max}[$, we have l functionals $L_{1j_1}, L_{2j_2}, \ldots, L_{lj_l}$ of the form (15) which are bounded on $[0, T_{\max}[$.

By application of Hölder's inequality to the functional L_l defined by

$$L_l(t) = \int_\Omega \prod_{i=1}^l u_i^{\alpha_i} u_{j_i}^{\alpha_{j_i}} \, dx, \tag{20}$$

we get

$$L_l(t) \leq \prod_{i=1}^l \left[\int_\Omega \left(u_i^{\alpha_i} u_{j_i}^{\alpha_{j_i}} \right)^{p_i} \, dx \right]^{\frac{1}{p_i}},$$

where the exponents $p_i > 1$, $i = 1, \ldots, l$, satisfy $\sum_{i=1}^l \frac{1}{p_i} = 1$. As the functionals L_{ij_i}, $i = 1, \ldots, l$ are bounded on $[0, T_{\max}[$, for all $\alpha_i > 1$ and all $\alpha_{j_i} < 0$, $i = 1, \ldots, l$, the functional L_l given by (20) is bounded on $[0, T_{\max}[$. For the remaining non-positive exponents α_{j_i} intervening in the expression of the functional L given by (11) and which are not present in the expression of the functional L_l (if they exist), we use the fact that the corresponding components u_{j_i} of the solution u are uniformly bounded below on $[0, T_{\max}[$ by a positive constant. This gives the boundedness of the functional L and ends the proof of the theorem. ∎

COROLLARY 1. *Under conditions* (5)–(7), *all solutions of problem* (1)–(3) *with positive initial data in* $\mathbb{L}^\infty(\Omega)$ *are global.*

Proof. Using the Theorem, all reaction terms $f_i = \prod_{k=1}^m u_k^{p_{ik}}$, $i = 1, \ldots, m$, are in $\mathbb{L}^\infty(0, T_{\max}; \mathbb{L}^p(\Omega))$ for all $p \geq 1$. We take $p > N/2$ to derive a uniform estimate of $\|f_i\|_p$ on $[0, T_{\max}[$. This gives, from the preliminary observations, that the solution will never blow up in $\mathbb{L}^\infty(\Omega)$ at any finite time $T_{\max}$, hence it exists globally ($T_{\max} = +\infty$). ∎

COROLLARY 2. *The solution of problem* (1)–(3) *remains global when the reaction terms are perturbed by linear terms to get the following form*

$$f_i = \sigma_i(x) - a_i u_i + \rho_i(x, u) \prod_{k=1}^m u_k^{p_{ik}}, \quad i = 1, \ldots, m, \tag{21}$$

where a_i are positive constants, $\sigma_i \in C^1(\overline{\Omega})$, $\sigma_i \geq 0$, $\rho_i \in C^1(\overline{\Omega} \times \mathbb{R}_+^m) \cap \mathbb{L}^\infty(\overline{\Omega} \times \mathbb{R}_+^m)$, with $\rho_i > 0$, $i = 1, \ldots, m$.

it *Proof.* In this case the integral I remains non-positive and

$$J = \int_\Omega \left(\frac{\alpha_i \sigma_i}{u_i} + \frac{\alpha_j \sigma_j}{u_j} \right) u_i^{\alpha_i} u_j^{\alpha_j} \, dx + (a_i + a_j) \int_\Omega u_i^{\alpha_i} u_j^{\alpha_j} \, dx$$

$$+ \int_\Omega \left(\rho_i \frac{\alpha_i}{u_i} \prod_{k=1}^m u_k^{p_{ik}} + \rho_j \frac{\alpha_j}{u_j} \prod_{k=1}^m u_k^{p_{jk}} \right) u_i^{\alpha_i} u_j^{\alpha_j} \, dx. \tag{22}$$

Using the fact that the u_i respectively the σ_i are uniformly bounded below on $[0, T_{\max}[$ by a positive constant, respectively uniformly bounded on Ω, we get for the first integral in (22)

$$\int_\Omega \left(\frac{\alpha_i \sigma_i}{u_i} + \frac{\alpha_j \sigma_j}{u_j} \right) u_i^{\alpha_i} u_j^{\alpha_j} \, dx \le C_7 L_{ij} \text{ on } [0, T_{\max}[,$$

where C_7 is a positive constant. Finally, since ρ_i / ρ_j are uniformly bounded on $\overline{\Omega} \times \overline{\mathbb{R}}_+^m$, using the Lemma, we obtain an analogous differential inequality to (19). Following the same reasoning as in the Theorem, we deduce the global existence of solutions to problem (1)–(3) with the reaction terms (21). ∎

REMARK 1. In the case when $p_{ij} = 0$ for some $i = 1, \ldots, l$ and $j = l+1, \ldots, m$, condition (6) is replaced by

$$(p_{jj} - 1) \left(\frac{p_{ii} - 1}{p_{ji}} \right) > 0.$$

REMARK 2. Condition (6) can be replaced by the following simpler but stronger condition

$$\frac{p_{jj} - 1}{p_{ij}} < 1,$$

which, together with (5), implies the condition (6).

REFERENCES

[1] R. Bauer, F. Zubler, A. Hauri, D. R. Muir and R. J. Douglas, *Developmental Origin of Patchy Axonal Connectivity in the Neocortex: A Computational Model*, Cerebral Cortex Advance Access published November 6: 1–14, 2012 (doi:10.1093/cercor/bhs327).

[2] Ming De Li, Shao Hua Chen and Yu Chun Qin, *Boundedness and blow up for the general activator-inhibitor model*, Acta Math. Appl. Sinica (English Ser.) **11**(1) (1995), 59–68.

[3] A. Gierer and H. Meinhardt, *A theory of biological pattern formation*, Kybernetik **12** (1972), 30–39.

[4] D. Henry, *Geometric Theory of Semi-linear Parabolic Equations*, Lecture Notes in Mathematics **840**, Springer-Verlag, New-York, 1984.

[5] H. Jiang, *Global existence of solutions of an activator-inhibitor model*, Discrete Contin. Dyn. Syst. **14** (2006), 737–751.

[6] S. Kouachi, *Global existence for coupled reaction diffusion systems modelling some reversible chemical reactions*, Dynamics Partial Diff. Equat. **8**(2) (2011), 79–88.

[7] S. Kouachi, *Global existence for reaction diffusion systems without nonlinearities growth condition*, Math. Methods Appl. Sci. **34**(7) (2011), 798–802.

[8] S. Kouachi, *Existence of global solutions to reaction diffusion systems with no homogeneous boundary conditions via a Lyapunov functional*, Electron. J. Diff. Eqns **2002**, 1–13.

[9] K. Masuda and K. Takahashi, *Reaction-diffusion systems in the Gierer-Meinhardt theory of biological pattern formation*, Japan J. Appl. Math. **4**(1) (1987), 47–58.

[**10**] W.-M. Ni, K. Suzuki and I. Takagi, *The dynamics of a kinetic activator-inhibitor system*, J. Diff. Equations **229** (2006), 426–465.

[**11**] R. D. Parshad, S. Kouachi and J. B. Gutierrez, *Global existence and asymptotic behavior of a model for biological control of invasive species via supermale introduction*, Commun. Math. Sci. **11**(4) (2013), 951–972.

[**12**] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sciences **44**, Springer-Verlag, New York, 1983.

[**13**] F. Rothe, *Global solutions of reaction-diffusion systems*, Lecture Notes in Mathematics **1072**, Springer-Verlag, Berlin, 1984.

[**14**] A. M. Turing, *The chemical basis of morphogenesis*, Philosophical Transactions of the Royal Society (B), **237** (1952), 37–72.

Department of Mathematics, College of Science, Qassim University, P.O.Box 6644, Al-Gassim, Buraydah 51452, Saudi Arabia

E-mail: `kouachi@hotmail.com`