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ALMOST METRIC VERSIONS OF ZHONG'S VARIATIONAL PRINCIPLE

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Abstract. A refinement of Zhong's variational principle [Nonlinear Anal. 29 (1997), 1421–1431] is given, in the realm of almost metric structures. Applications to equilibrium points are also provided.

1. Introduction

Let M be some nonempty set; and the map $(x,y) \mapsto d(x,y)$ from $M \times M$ to $R_+ := [0,\infty[$ be a metric over it. Further, take a function $\varphi\colon M \to R \cup \{\infty\}$ with

(a01) φ is inf-proper ($\mathrm{Dom}(\varphi) \neq \emptyset$ and $\varphi_* := \inf[\varphi(M)] > -\infty$).

The following 1979 statement in Ekeland [8] (referred to as Ekeland's variational principle; in short: EVP) is our starting point. Assume that

(a02) d is complete (each d-Cauchy sequence is d-convergent)

(a03) φ is d-lsc ($\liminf_n \varphi(x_n) \geq \varphi(x)$, whenever $x_n \xrightarrow{d} x$).

Theorem 1. *Let the previous conditions hold. Then,*

I) *for each $u \in \mathrm{Dom}(\varphi)$ there exists $v = v(u) \in \mathrm{Dom}(\varphi)$ with*

$$d(u,v) \leq \varphi(u) - \varphi(v) \quad \textit{(hence } \varphi(u) \geq \varphi(v)\textit{)} \tag{1}$$

$$d(v,x) > \varphi(v) - \varphi(x), \quad \textit{for all } x \in M \setminus \{v\}. \tag{2}$$

II) *if $u \in \mathrm{Dom}(\varphi)$, $\rho > 0$ fulfill $\varphi(u) - \varphi_* \leq \rho$, then (1) gives*

$$(\varphi(u) \geq \varphi(v) \quad \textit{and}) \quad d(u,v) \leq \rho. \tag{3}$$

This principle found some basic applications to control and optimization, generalized differential calculus, critical point theory and global analysis; we refer to

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Hyers, Isac and Rassias [10, Ch. 5] for a survey of these. As a consequence, many extensions of EVP were proposed. For example, the (abstract) order one starts from the fact that, with respect to the quasi-order (reflexive and transitive relation)

(a04) $(x, y \in M)$ $x \le y$ if and only if $\varphi(y) + d(x, y) \le \varphi(x)$

the point $v \in M$ appearing in (2) is maximal; so that, EVP is nothing but a variant of the Zorn maximality principle. The dimensional way of extension refers to the ambient space (R) of $\varphi(M)$ being substituted by a (topological or not) vector space; an account of the results in this area is to be found in the 2003 monograph by Goepfert, Riahi, Tammer and Zălinescu [9, Ch. 3]. Further, the (pseudo) metrical one consists in the conditions imposed to the ambient metric over M being relaxed. The basic result in this direction was obtained by Tataru [21]; subsequent extensions of it may be found in Kada, Suzuki and Takahashi [11]. Finally, we must add the "functional" statement by Zhong [26] (referred to as: Zhong's variational principle; in short: ZVP). Let the function $b : R_+ \to R^0_+ :=]0, \infty[$ be locally Riemann integrable; and $B : R_+ \to R_+$ stand for its primitive: $B(t) = \int_0^t b(\tau)\, d\tau$, $t \in R_+$; we say that (b, B) is a normal couple, provided

(a05) b is decreasing and $B(\infty) = \infty$.

Theorem 2. *Under the general assumptions (a02)–(a03), let the normal couple (b, B) and the points $a \in M$, $u \in \mathrm{Dom}(\varphi)$, $\rho > 0$ be such that*

(a06) $\varphi(u) - \varphi_* \le B(d(a, u) + \rho) - B(d(a, u))$.

Then there exists $v = v(u)$ in $\mathrm{Dom}(\varphi)$ with

III) $d(a, v) \le d(a, u) + \rho$, $\varphi(u) \ge \varphi(v)$;

IV) $b(d(a, v))d(v, x) > \varphi(v) - \varphi(x)$, *for each $x \in M \setminus \{v\}$.*

Clearly, ZVP includes (for $b = 1$ and $a = u$) the local version of EVP based upon (3). The relative form of the same, based upon (1) also holds (but indirectly); cf. Bao and Khanh [2]. Summing up, ZVP includes EVP; but, the argument developed there is rather involved; this is equally true for another proof of the same, proposed by Suzuki [19]. A simplification of this reasoning was given in Turinici [22], by a technique due to Park and Bae [16]; note that, as a consequence of this, ZVP $\Leftrightarrow$ EVP. It is our aim in the following to show that such a conclusion continues to hold—under general completeness conditions—in the almost metric framework; details will be given in Section 3. Basic tools for this are a lot of pseudometric variational principles discussed in Section 2. Finally, in Section 4 and Section 5, some applications of these facts to equilibrium points are considered.

2. Pseudometric ordering principles

(A) Let M be a nonempty set; and $\mathcal{R} \subseteq M \times M$ stand for a (nonempty) relation over it. For each $x \in M$, denote $M(x, \mathcal{R}) = \{y \in M; x\mathcal{R}y\}$. The following "Dependent Choices Principle" (in short: DC) is in effect for us:

Proposition 1. *Suppose that*

(b01) $M(c, \mathcal{R})$ *is nonempty, for each* $c \in M$.

Then, for each $a \in M$ *there exists* $(x_n) \subseteq M$ *with* $x_0 = a$ *and* $x_n \mathcal{R} x_{n+1}$, $\forall n$.

This principle, due to Bernays [3] and Tarski [20], is deductible from AC (= the Axiom of Choice), but not conversely; cf. Wolk [25]. Moreover, the alternate Zermelo-Fraenkel system (ZF−AC+DC) is strong enough so as to include the "usual" mathematics; see, for instance, Moskhovakis [14, Ch. 8].

(B) Let M be a nonempty set. Take a quasi-order $(\leq)$ over it, as well as a function $\varphi : M \to R$. Call $z \in M$, $(\leq, \varphi)$-maximal when: $z \leq w \in M$ implies $\varphi(z) = \varphi(w)$; or, equivalently: φ is constant on $M(z, \leq) := \{x \in M; z \leq x\}$; the set of all these will be denoted as $\max(M; \leq; \varphi)$. A basic result about such points is the 1976 Brezis-Browder ordering principle [5] (in short: BB).

Proposition 2. *Suppose that*

(b02) $(M, \leq)$ *is sequentially inductive: each ascending sequence has an upper bound (modulo* $(\leq)$*);*

(b03) φ *is bounded from below and* $(\leq)$*-decreasing.*

Then, $\max(M; \leq; \varphi)$ *is* $(\leq)$*-cofinal in* M $[\forall u \in M, \exists v \in \max(M; \leq; \varphi)$*:* $u \leq v]$ *and* $(\leq)$*-invariant in* M $[z \in \max(M; \leq; \varphi) \implies M(z, \leq) \subseteq \max(M; \leq; \varphi)]$.

This statement includes EVP (see below); and found some useful applications to convex and non-convex analysis (cf. the above references). So, it is natural asking about its existential status. As we shall see, BB is a logical equivalent of DC. The first half of this (DC ⇒ BB) follows from the argument below (see also Turinici [24] and the references therein).

Proof. Define $\beta : M \to R$ as: $\beta(v) := \inf[\varphi(M(v, \leq))]$, $v \in M$. Clearly, β is increasing, and $[\varphi(v) \geq \beta(v)$, for all $v \in M]$. Further, (b03) gives

$$v \text{ is } (\leq, \varphi)\text{-maximal if and only if } \varphi(v) = \beta(v). \tag{4}$$

Now, assume by contradiction that the conclusion in this statement is false; i.e. (if one takes (4) into account) there must be some $u \in M$ such that:

(b04) for each $v \in M_u := M(u, \leq)$, one has $\varphi(v) > \beta(v)$.

Consequently (for all such v), $\varphi(v) > (1/2)(\varphi(v) + \beta(v)) > \beta(v)$; hence

$$v \leq w \text{ and } (1/2)(\varphi(v) + \beta(v)) > \varphi(w), \tag{5}$$

for at least one w (belonging to M_u). The relation $\mathcal{R}$ over M_u introduced via (5) fulfills $M_u(v, \mathcal{R}) \neq \emptyset$, for all $v \in M_u$. So, by (DC), there must be a sequence (u_n) in M_u with $u_0 = u$ and

$$u_n \leq u_{n+1},\ (1/2)(\varphi(u_n) + \beta(u_n)) > \varphi(u_{n+1}), \text{ for all } n. \tag{6}$$

We have thus constructed an ascending sequence (u_n) in M_u for which $(\varphi(u_n))$ is strictly descending and bounded below; hence $\lambda := \lim_n \varphi(u_n)$ exists in R. Taking (b02) into account, there must be some $v \in M$ such that $u_n \le v$, for all n. From (b03), $\varphi(u_n) \ge \varphi(v)$, $\forall n$; whence, $v \in M_u$; moreover (by the properties of β) $\varphi(v) \ge \beta(v) \ge \beta(u_n)$, $\forall n$. The former of these relations gives (by a limit process) $\lambda \ge \varphi(v)$. And the latter of these relations yields (via (6)) $(1/2)(\varphi(u_n)+\beta(v)) > \varphi(u_{n+1})$, for all $n \in N$. Passing to limit as $n \to \infty$, one gets $(\varphi(v) \ge)\beta(v) \ge \lambda$; so, combining with the preceding relation, $\varphi(v) = \beta(v)(= \lambda)$, in contradiction with (b04). ■

(C) A basic application of this result is to pseudometric variational statements. Let M be a nonempty set. By a pseudometric over M we shall mean any map $e : M \times M \to R_+$. Fix such an object; which in addition is reflexive $[e(x,x) = 0, \forall x \in M]$ and triangular $[e(x,z) \le e(x,y) + e(y,z), \forall x,y,z \in M]$; we shall say that $e(\cdot,\cdot)$ is an rt-pseudometric, and (X,e) is a *rt-pseudometric space.*

Define an e-convergence structure on X as: $x_n \xrightarrow{e} x$ if and only if $e(x_n,x) \to 0$ as $n \to \infty$; referred to as: x is an e-limit of (x_n). The set of all these will be denoted $\lim_n(x_n)$; when it is nonempty, we call (x_n), e-convergent. Further, call the sequence (x_n),

(b05) strongly e-asymptotic (in short: e-strasy) if $\sum_n e(x_n,x_{n+1})$ converges;

(b06) e-Cauchy when $[\forall \delta > 0, \exists n(\delta)\text{: } n(\delta) \le p \le q \implies e(x_p,x_q) \le \delta]$.

By the triangular property of e, we have

$$\text{(for each sequence): } e\text{-strasy} \implies e\text{-Cauchy;}$$

but the converse is not true in general. Note that, by the lack of symmetry, an e-convergent sequence in X need not be e-Cauchy.

Finally, let $\varphi : M \to R \cup \{\infty\}$ be some inf-proper function (cf. (a01)). We consider the regularity condition

(b07) (e,φ) is weakly descending complete: for each e-strasy sequence $(x_n) \subseteq \mathrm{Dom}(\varphi)$ with $(\varphi(x_n))$ descending there exists $x \in M$ with $x_n \xrightarrow{e} x$ and $\lim_n \varphi(x_n) \ge \varphi(x)$.

By the generic property above, it is implied by its (stronger) counterpart

(b08) (e,φ) is descending complete: for each e-Cauchy sequence (x_n) in $\mathrm{Dom}(\varphi)$ with $(\varphi(x_n))$ descending there exists $x \in M$ with $x_n \xrightarrow{e} x$ and $\lim_n \varphi(x_n) \ge \varphi(x)$.

A remarkable fact to be added is that the reciprocal inclusion also holds, in the reduced Zermelo-Fraenkel system (ZF-AC):

Lemma 1. *We have, in (ZF-AC),*

$$\textit{(b07)} \implies \textit{(b08); hence (b07)} \iff \textit{(b08).} \tag{7}$$

Proof. Assume that (b07) holds; and let (x_n) be an e-Cauchy sequence in $\mathrm{Dom}(\varphi)$ with $(\varphi(x_n))$, descending. The imposed property upon our sequence assures us (with $\varepsilon = 2^{-m}$) that, for each $m \ge 0$,

$$C(m) := \{n \geq 0; d(x_p, x_q) < 2^{-m}, \quad \text{for } n \leq p \leq q\}$$

is nonempty and $(\leq)$-invariant ($s \geq r \in C(m) \implies s \in C(m)$). In addition, $n \mapsto C(n)$ is $(\subseteq)$-decreasing; hence $n \mapsto g(n) := \min[C(n)]$ is $(\leq)$-increasing. (Note that, such a construction is valid without any form of DC). This finally tells us that $n \mapsto h(n) := n + g(n)$ is strictly $(\leq)$-increasing; wherefrom $(y_n = x_{h(n)}; n \geq 0)$ is a subsequence of (x_n) with

$$(\forall n \geq 0): \quad h(n) \leq p \leq q \implies e(x_p, x_q) < 2^{-n}. \tag{8}$$

In particular, (y_n) is an e-strasy subsequence of (x_n), with $(\varphi(y_n))$, descending. Combining with (b07), yields an $y \in M$ with $y_n \xrightarrow{e} y$ and $\lim_n \varphi(y_n) \geq \varphi(y)$. It is now clear, via (8), that y has all desired in (b08) properties. ■

The following variational principle is our starting point.

Proposition 3. *Let the rt-pseudometric space (X, e) be such that (b07)/(b08) holds. Then, for each $u \in \mathrm{Dom}(\varphi)$, there exists $v = v(u) \in \mathrm{Dom}(\varphi)$ satisfying*

i) $e(u, v) \leq \varphi(u) - \varphi(v)$ *(hence $\varphi(u) \geq \varphi(v)$);*

ii) $[x \in M, e(v, x) \leq \varphi(v) - \varphi(x)] \implies [\varphi(v) = \varphi(x), e(v, x) = 0]$*;*

iii) $e(v, x) > \varphi(v) - \varphi(x)$*, for each $x \in M$ with $e(v, x) > 0$;*

iv) $e(v, x) \geq \varphi(v) - \varphi(x)$*, for all $x \in M$.*

Proof. Let $(\leq)$ stand for the quasi-order (a04) (with e in place of d). Further, denote $M_u = \{x \in M; \varphi(x) \leq \varphi(u)\}$; clearly, $\emptyset \neq M_u \subseteq \mathrm{Dom}(\varphi)$. We claim that the couple $(\leq, \varphi)$ fulfills conditions of BB over M_u; i.e., that (b02) holds. Let (x_n) be an ascending (modulo $(\leq)$) sequence in M_u:

(b09) $e(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m)$, whenever $n \leq m$.

The sequence $(\varphi(x_n))$ is descending bounded; hence a Cauchy one; and, by (b09), (x_n) is e-Cauchy (in M_u). Putting these together, it follows, via (b08), that

$$x_n \xrightarrow{e} y \text{ and } \lim_n \varphi(x_n) \geq \varphi(y), \text{ for some } y \in M. \tag{9}$$

This gives $\varphi(y) \leq \varphi(u)$; wherefrom $y \in M_u$ (because $(x_n) \subseteq M_u$). Moreover, fix some rank n. From (b09) and the triangular property of $e(\cdot, \cdot)$,

$$e(x_n, y) \leq e(x_n, x_m) + e(x_m, y) \leq \varphi(x_n) - \varphi(x_m) + e(x_m, y), \forall m \geq n.$$

This, along with (9), yields by a limit process (relative to m)

$$e(x_n, y) \leq \varphi(x_n) - \lim_m \varphi(x_m) \leq \varphi(x_n) - \varphi(y) \quad \text{(i.e.: } x_n \leq y\text{)}.$$

As n was arbitrarily chosen, y is an upper bound in M_u of (x_n); hence the claim. From BB it follows that, for the starting $u \in M_u$ there exists $v \in M_u$ with

j) $u \leq v$, jj) v is $(\leq, \varphi)$-maximal in M_u ($v \leq x \in M_u \implies \varphi(v) = \varphi(x)$).

The former of these is just i) And the latter one gives ii); because this may be written as: $[x \in M_u,\ e(v, x) \leq \varphi(v) - \varphi(x)] \implies [\varphi(v) = \varphi(x),\ e(v, x) = 0]$. Now, evidently, iii) follows from ii). The only point to be clarified is iv). Assume this would be false: $e(v,x) < \varphi(v) - \varphi(x)$, for some $x \in M$ (hence $x \in M_u$). From ii), one gets $\varphi(v) = \varphi(x)$; so that (by the above) $0 \le e(v,x) < 0$, contradiction. ■

In particular, condition (b07) is retainable under

(b10) (e,φ) is weakly complete: for each e-strasy sequence (x_n) in $\mathrm{Dom}(\varphi)$ there exists $x \in M$ with $x_n \xrightarrow{e} x$ and $\liminf_n \varphi(x_n) \ge \varphi(x)$.

As a consequence, Proposition 3 incorporates the variational principle in Tataru [21]; see also Kang and Park [12].

Call the rt-pseudometric $e : M \times M \to R_+$, an almost metric provided it is in addition sufficient $[e(x,y) = 0 \implies x = y]$; we then say that (X,e) is an almost metric space. A direct application of Proposition 3 to such structures yields:

Theorem 3. *Let the almost metric space e and the inf-proper function φ be as in* (b08). *Then,*

I) *for each $u \in \mathrm{Dom}(\varphi)$ there exists $v = v(u) \in \mathrm{Dom}(\varphi)$ with*

$$e(u,v) \le \varphi(u) - \varphi(v) \text{ (hence } \varphi(u) \ge \varphi(v)\text{)} \tag{10}$$

$$e(v,x) > \varphi(v) - \varphi(x), \text{ for all } x \in M \setminus \{v\} \tag{11}$$

II) *if $u \in \mathrm{Dom}(\varphi)$, $\rho > 0$ fulfill $\varphi(u) - \varphi_* \le \rho$, then* (10) *gives*

$$(\varphi(u) \ge \varphi(v) \text{ and}) \quad e(u,v) \le \rho. \tag{12}$$

Now, evidently, (b08) is retainable whenever

(b11) (e,φ) is complete: for each e-Cauchy sequence (x_n) in $\mathrm{Dom}(\varphi)$ there exists $x \in M$ with $x_n \xrightarrow{e} x$ and $\liminf_n \varphi(x_n) \ge \varphi(x)$.

If e is in addition symmetric $[e(x,y) = e(y,x), \forall x,y \in M]$ (hence, a metric over M), (b11) holds under (a02)+(a03) (modulo e). This tells us that Theorem 3 includes EVP; it will be referred to as the almost metric version of EVP (in short: EVPa).

(D) With these preliminaries, we may now return to the second half (BB $\implies$ DC) of the logical equivalence we just announced. By the developments above, one has the implications: (DC) $\implies$ (BB) $\implies$ (EVPa) $\implies$ (EVP). So, it is natural to ask whether these may be reversed. The setting of this problem is the reduced Zermelo-Fraenkel system (ZF-AC).

Let X be a nonempty set; and $(\le)$ be an order on it. We say that $(\le)$ has the inf-lattice property, provided: $x \wedge y := \inf(x,y)$ exists, for all $x,y \in X$. Further, we say that $z \in X$ is a $(\le)$-maximal element if $X(z,\le) = \{z\}$; the class of all these points will be denoted as $\max(X,\le)$. In this case, $(\le)$ is called a Zorn order when $\max(X,\le)$ is nonempty and cofinal in X [for each $u \in X$ there exists a $(\le)$-maximal $v \in X$ with $u \le v$]. Further aspects are to be described in a metric setting. Let $d : X \times X \to R_+$ be a metric over X; and $\varphi : X \to R_+$ be some function. Then, the natural choice for $(\le)$ above is

$x \le_{(d,\varphi)} y$ if and only if $d(x,y) \le \varphi(x) - \varphi(y)$;

referred to as the Brøndsted order [6] attached to (d, φ). Denote $X(x, \rho) = \{u \in X; d(x, u) < \rho\}$, $x \in X$, $\rho > 0$ [the open sphere with center x and radius ρ]. Call the ambient metric space (X, d), *discrete* when for each $x \in X$ there exists $\rho = \rho(x) > 0$ such that $X(x, \rho) = \{x\}$. Note that, in this case, any function $\psi : X \to R$ is d-continuous over X. However, the d-Lipschitz property ($|\psi(x) - \psi(y)| \leq Ld(x, y)$, $x, y \in X$, for some $L > 0$) cannot be assured, in general.

Now, the statement below is a particular case of EVP:

PROPOSITION 4. *Let the metric space (X, d) and the function $\varphi : X \to R_+$ satisfy*

(b12) *(X, d) is discrete bounded and complete;*

(b13) *$(\leq_{(d,\varphi)})$ has the inf-lattice property;*

(b14) *φ is d-nonexpansive and $\varphi(X)$ is countable.*

Then, $(\leq_{(d,\varphi)})$ is a Zorn order.

We shall refer to it as: the discrete Lipschitz countable version of EVP (in short: (EVPdLc)). Clearly, (EVP) $\implies$ (EVPdLc). The remarkable fact to be added is that this last principle yields (DC); so, it completes the circle between all these.

PROPOSITION 5. *We have (in the reduced Zermelo-Fraenkel system) (EVPdLc) $\implies$ (DC). So (by the above), the maximal/variational principles (BB), (EVPa) and (EVP) are all equivalent with (DC); hence, mutually equivalent.*

For a detailed proof, see Turinici [24]. In particular, when the specific assumptions (b13) and (b14) are ignored, this last result reduces to the one in Brunner [7]. Further aspects may be found in Schechter [18, Ch. 19, Sect. 19.53].

3. Zhong variational statements

(A) Let M be some nonempty set. Take a couple (d, e) of almost metrics over M; we say that e is *d-compatible* provided

(c01) each e-Cauchy sequence is d-Cauchy, too;

(c02) $y \mapsto e(x, y)$ is d-lsc, for each $x \in M$.

Note that both these properties hold when $e = d$. In fact, (c01) is trivial; and (c02) results from the triangular property of d (see Proposition 7 for details). Further, let $\varphi : M \to R \cup \{\infty\}$ be an inf-proper function. The following fact will be useful.

LEMMA 2. *Suppose that e is d-compatible. Then,*

$$[(d, \varphi) = \textit{descending complete}] \implies [(e, \varphi) = \textit{descending complete}].$$

Proof. Let (x_n) be some e-Cauchy sequence in $\mathrm{Dom}(\varphi)$ with $(\varphi(x_n))$ descending. From (c01), (x_n) is d-Cauchy too; so, as (d, φ) is descending complete, there exists $y \in X$ with $x_n \xrightarrow{d} y$ and $\lim_n \varphi(x_n) \geq \varphi(y)$. We claim that this is our desired point. Let $\gamma > 0$ be arbitrary fixed. By the initial choice of (x_n), there exists $k = k(\gamma)$ so that: $e(x_p, x_m) \leq \gamma$, for each $p \geq k$ and each $m \geq p$. Passing to limit upon m one gets (via (c02)) $e(x_p, y) \leq \gamma$, for each $p \geq k$; and since $\gamma > 0$ was arbitrarily chosen, $x_n \xrightarrow{e} y$. This gives the conclusion we want. ■

Now, by simply combining this with Theorem 3, one gets the following "relative" type variational statement (involving these data):

Theorem 4. *Let the couple (d, e) of almost metrics over M and the inf-proper function $\varphi : M \to R \cup \{\infty\}$ be such that (d, φ) be descending complete and e is d-compatible. Then, the following conclusions hold:*

I) *for each $u \in \mathrm{Dom}(\varphi)$ there exists $v = v(u) \in \mathrm{Dom}(\varphi)$ with*

$$e(u, v) \leq \varphi(u) - \varphi(v) \text{ (hence } \varphi(u) \geq \varphi(v)) \tag{13}$$

$$e(v, x) > \varphi(v) - \varphi(x), \text{ for all } x \in M \setminus \{v\} \tag{14}$$

II) *if $u \in \mathrm{Dom}(\varphi)$, $\rho > 0$ fulfill $\varphi(u) - \varphi_* \leq \rho$, then (13) gives*

$$(\varphi(u) \geq \varphi(v) \text{ and}) \quad e(u, v) \leq \rho. \tag{15}$$

For the moment, Theorem 3 $\Longrightarrow$ Theorem 4. The reciprocal is also true; for (see above) $e = d$ is allowed here; so, Theorem 3 $\Longleftrightarrow$ Theorem 4.

This "relative" variational statement may be viewed as an "abstract" version of ZVP. To explain our claim, we need some constructions and auxiliary facts.

(B) Let the locally Riemann integrable function $b : R_+ \to R_+^0$ and its primitive $B : R_+ \to R_+$ be such that (b, B) is normal (cf. Section 1). In particular, we have

$$\int_p^q b(\xi)\, d\xi = (q - p) \int_0^1 b(p + \tau(q - p))\, d\tau, \quad \text{when} \quad 0 \leq p < q < \infty. \tag{16}$$

Some basic facts involving this couple are collected in

Lemma 3. *The following are valid*

i) *B is a continuous order isomorphism of R_+; hence, so is B^{-1};*

ii) *$b(s) \leq (B(s) - B(t))/(s - t) \leq b(t), \forall t, s \in R_+, t < s$;*

iii) *B is almost concave: $t \mapsto [B(t + s) - B(t)]$ is decreasing on R_+, $\forall s \in R_+$;*

iv) *B is concave: $B(t + \lambda(s - t)) \geq B(t) + \lambda(B(s) - B(t))$, for all $t, s \in R_+$ with $t < s$ and all $\lambda \in [0, 1]$;*

v) *B is sub-additive (hence B^{-1} is super-additive).*

The proof is immediate, by (16) above; hence, we do not give details. Note that the properties in iii) and iv) are equivalent to each other, under i). This follows at once from the (non-differential) mean value theorem in Bantaş and Turinici [1].

(C) Now, let M be some nonempty set; and $d : M \times M \to R_+$, an almost metric over it. Further, let $\Gamma : M \to R_+$ be chosen as

(c03) Γ is almost d-nonexpansive ($\Gamma(x) - \Gamma(y) + d(x,y) \geq 0, \forall x, y \in M$).

Define a pseudometric $e = e(B : \Gamma; d)$ over M as

(c04) $e(x,y) = B(\Gamma(x) + d(x,y)) - B(\Gamma(x)), \quad x, y \in M.$

This may be viewed as an "explicit" formula; the "implicit" version of it is

(c05) $d(x,y) = B^{-1}(B(\Gamma(x)) + e(x,y)) - \Gamma(x), \quad x, y \in M.$

We shall establish some properties of this map, useful in the sequel.

First, the "metrical" nature of $(x,y) \mapsto e(x,y)$ is of interest.

Proposition 6. *The pseudometric $e(\cdot,\cdot)$ is an almost metric over M.*

Proof. The reflexivity and sufficiency are clear, by Lemma 3, i); so, it remains to establish the triangular property. Let $x, y, z \in M$ be arbitrary fixed. The triangular property of $d(\cdot,\cdot)$ yields [via Lemma 3, i)]

$$e(x,z) \leq B(\Gamma(x) + d(x,y) + d(y,z)) - B(\Gamma(x) + d(x,y)) + e(x,y).$$

On the other hand, the almost d-nonexpansiveness of Γ gives $\Gamma(x) + d(x,y) \geq \Gamma(y)$; so [by Lemma 3, iii)]

$$B(\Gamma(x) + d(x,y) + d(y,z)) - B(\Gamma(x) + d(x,y)) \leq e(y,z).$$

Combining with the previous relation yields our desired conclusion. ■

By definition, e will be called the Zhong metric attached to d and the couple (B, Γ). The following properties of (d,e) are immediate (via Lemma 3):

Lemma 4. *Under the prescribed conventions,*

vi) $b(\Gamma(x) + d(x,y))d(x,y) \leq e(x,y) \leq b(\Gamma(x))d(x,y)$, *for all* $x, y \in M$;

vii) $e(x,y) \leq B(d(x,y)), \forall x, y \in M$; *hence* $x_n \xrightarrow{d} x$ *implies* $x_n \xrightarrow{e} x$.

A basic property of $e(\cdot,\cdot)$ to be checked is d-compatibility.

Proposition 7. *The Zhong metric $e(\cdot,\cdot)$ is d-compatible (cf. (c01)+(c02)).*

Proof. We firstly check (c02); which may be written as

$$[e(x,y_n) \leq \lambda, \forall n] \text{ and } y_n \xrightarrow{d} y \quad \text{imply } e(x,y) \leq \lambda.$$

So, let x, (y_n), λ and y be as in the premise of this relation. By Lemma 4, we have $y_n \xrightarrow{e} y$ as $n \to \infty$. Moreover (as e is triangular) $e(x,y) \leq e(x,y_n) + e(y_n,y) \leq \lambda + e(y_n,y)$, for all n. It will suffice passing to limit as $n \to \infty$ to get the desired conclusion. Further, we claim that (c01) holds too, in the sense: [(for each sequence) d-Cauchy $\iff$ e-Cauchy]. The left to right implication is clear, via Lemma 4. For the right to left one, assume that (x_n) is an e-Cauchy sequence in M. In particular (by the triangular property) $e(x_i,x_j) \leq \mu$, for all (i,j) with $i \leq j$, and some $\mu \geq 0$. This, along with (c05), yields $d(x_0,x_i) =$

$B^{-1}(B(\Gamma(x_0))+e(x_0,x_i))-\Gamma(x_0) \leq B^{-1}(B(\Gamma(x_0))+\mu)-\Gamma(x_0)$, $\forall i \geq 0$; wherefrom (cf. (c03)) $\Gamma(x_i) \leq \Gamma(x_0) + d(x_0,x_i) \leq B^{-1}(B(\Gamma(x_0)) + \mu)$ [hence $B(\Gamma(x_i)) \leq B(\Gamma(x_0)) + \mu$], for all $i \geq 0$. Putting these facts together yields (again via (c05)) $\Gamma(x_i) + d(x_i,x_j) = B^{-1}(B(\Gamma(x_i)) + e(x_i,x_j)) \leq \nu := B^{-1}(B(\Gamma(x_0)) + 2\mu)$, for all (i,j) with $i \leq j$. And this, via Lemma 4, gives (for the same pairs (i,j)) $e(x_i,x_j) \geq b(\Gamma(x_i) + d(x_i,x_j))d(x_i,x_j) \geq b(\nu)d(x_i,x_j)$. But then, the d-Cauchy property of (x_n) is clear; and the proof is complete. ∎

(D) We are now in position to make precise our initial claim. Let the almost metric d and the inf-proper function φ be such that

(d,φ) is descending complete (according to (b08)).

Further, take a normal couple (b,B); as well as an almost d-nonexpansive map $\Gamma : M \to R_+$. Finally, put $e = e(B;\Gamma;d)$ (the Zhong metric introduced by (c04)/(c05)).

Theorem 5. *Let the conditions above be admitted. Then*

III) *For each $u \in \mathrm{Dom}(\varphi)$ there exists $v = v(u) \in \mathrm{Dom}(\varphi)$ with*

$$b(\Gamma(u) + d(u,v))d(u,v) \leq e(u,v) \leq \varphi(u) - \varphi(v) \tag{17}$$

$$b(\Gamma(v))d(v,x) \geq e(v,x) > \varphi(v) - \varphi(x), \quad \forall x \in M \setminus \{v\} \tag{18}$$

IV) *For each $u \in \mathrm{Dom}(\varphi)$, $\rho > 0$ with $\varphi(u) - \varphi_* \leq B(\Gamma(u) + \rho) - B(\Gamma(u))$, the above evaluation* (17) *gives*

$$d(u,v) \leq \rho; \quad \textit{hence } \Gamma(v) \leq \Gamma(u) + \rho \tag{19}$$

$$b(\Gamma(u) + \rho)d(u,v) \leq \varphi(u) - \varphi(v) \quad \textit{(hence } \varphi(u) \geq \varphi(v)\textit{)}. \tag{20}$$

Proof. By Proposition 6, $e(\cdot,\cdot)$ is an almost metric over M; and, by Proposition 7, it is d-compatible. Hence, Theorem 4 applies to such data. In this case, (17)+(18) are clear via Lemma 4. Moreover, if $u \in \mathrm{Dom}(\varphi)$ is taken as in the premise of IV), then (cf. (17) $e(u,v) \leq \varphi(u) - \varphi(v) \leq \varphi(u) - \varphi_*$; wherefrom (by (c05)) $d(u,v) \leq B^{-1}(B(\Gamma(u)) + \varphi(u) - \varphi_*) - \Gamma(u) \leq \rho$; and (19)+(20) follow as well. ∎

So far, Theorem 3 $\Longrightarrow$ Theorem 4 $\Longrightarrow$ Theorem 5. In addition, Theorem 5 $\Longrightarrow$ Theorem 3; just take $b = 1$ (hence $B =$ identity, $e = d$). Summing up, these three variational principles are mutually equivalent. On the other hand, Theorem 5 may be also viewed as an extended (modulo Γ) version of ZVP. For, if d is symmetric (hence a (standard) metric), (c03) becomes

(c06) $|\Gamma(x) - \Gamma(y)| \leq d(x,y)$, for all $x,y \in M$ (Γ is d-nonexpansive).

In addition, the choice

(c07) $\Gamma(x) = d(a,x)$, $x \in M$, for some $a \in M$

is in agreement with it; hence the claim. For this reason, Theorem 5 will be referred to as the almost metric version of ZVP (in short: ZVPa). This inclusion is technically strict; because the conclusions involving the middle terms in (17)+(18) cannot be obtained in the way described by Zhong [26]. Some related aspects were delineated in Ray and Walker [17]; see also Turinici [23].

4. Application (equilibrium points)

Let M be some nonempty set. Any (extended) function $G : M \times M \to R \cup \{-\infty, \infty\}$ will be referred to as a relative generalized pseudometric on M. Given such an object, we say that $v \in M$ is an equilibrium point of it, when $G(v, x) \geq 0$, $\forall x \in M$. A basic particular case to be considered here is

$$G(x, y) = e(x, y) + F(x, y), \quad x, y \in M \quad \text{(i.e.: } G = e + F\text{)},$$

with $e : M \times M \to R_+$, an almost metric over M and $F : M \times M \to R \cup \{-\infty, \infty\}$, a relative generalized pseudometric on M; when the above definition becomes $e(v, x) \geq -F(v, x)$, $\forall x \in M$. Note that, under the choice (for some $\varphi : M \to R \cup \{\infty\}$)

(d01) $F(x, y) = \varphi(y) - \varphi(x), \quad x, y \in M \quad$ (where $\infty - \infty = 0$),

the above mentioned variational property of v is "close" to the one in Theorem 3. So, existence of such points is deductible from the quoted result; to do this, one may proceed as follows. Assume that the relative generalized pseudometric F is reflexive [$F(x, x) = 0$, $\forall x \in M$] and triangular [$F(x, z) \leq F(x, y) + F(y, z)$, whenever the right member exists]. Define the (extended) function

$$\mu : M \to \mathbf{R}_+ \cup \{\infty\}: \ \mu(x) = \sup\{-F(x, y); y \in M\}, x \in M.$$

The alternative $\mu(M) = \{\infty\}$ cannot be excluded; to avoid this, assume

(d02) μ is proper ($\mathrm{Dom}(\mu) := \{x \in M; \mu(x) < \infty\} \neq \emptyset$).

For the arbitrary fixed $u \in \mathrm{Dom}(\mu)$ put $F_u(\cdot) = F(u, \cdot)$. We have by definition

$$F_u(u) = 0; \ F_u^* := \inf\{F_u(x); x \in M\} = -\mu(u) > -\infty; \tag{21}$$

so that,

$$\begin{gathered} F_u \text{ is inf-proper, for each } u \in \mathrm{Dom}(\mu) \\ \text{(referred to as: } F \text{ is semi inf-proper).} \end{gathered} \tag{22}$$

Further, let d be an almost metric on M with

(d03) (d, F_u) is descending complete, for each $u \in \mathrm{Dom}(\mu)$ (referred to as: (d, F) is semi descending complete).

Theorem 6. *Let* (d02)+(d03) *hold; and let e be d-compatible. Then, for each $u \in \mathrm{Dom}(\mu)$ there exists $v = v(u)$ in M such that*

I) *$e(u, v) \leq -F(u, v) \leq \mu(u) (< \infty)$;*

II) *$e(v, x) > -F(v, x)$, for all $x \in M \setminus \{v\}$.*

Hence, in particular, v is an equilibrium point for $G := e + F$.

Proof. From Theorem 4 it follows that, for the starting $u \in \mathrm{Dom}(\mu)$ (hence $u \in \mathrm{Dom}(F_u)$) there must be another point $v \in \mathrm{Dom}(F_u)$ with the properties

i) $e(u, v) \leq F_u(u) - F_u(v)$; ii) $e(v, x) > F_u(v) - F_u(x), \forall x \in M \setminus \{v\}$.

The former of these is just I), by the reflexivity of F. And the latter yields II); for (by the triangular property) $F(u, v) - F(u, x) \geq F(u, v) - (F(u, v) + F(v, x)) = -F(v, x)$; hence the conclusion. ∎

Now, a basic particular choice of $e(\cdot,\cdot)$ is related to the constructions in Section 3. Precisely, let (b, B) stand for a normal couple; and $\Gamma : M \to R_+$ be almost d-nonexpansive. Let $e = e(B : \Gamma; d)$ stand for the Zhong metric given by (c04)/(c05). By Theorem 5, we then have

THEOREM 7. *Let* (d02)+(d03) *hold. Then, for each* $u \in \mathrm{Dom}(\mu)$ *there exists* $v = v(u)$ *in* M *such that*

III) $b(\Gamma(u) + d(u, v))d(u, v) \le e(u, v) \le -F(u, v) \le \mu(u)$;

IV) $b(\Gamma(v))d(v, x) \ge e(v, x) > -F(v, x), \forall x \in M \setminus \{v\}$.

Hence, in particular, v *is an equilibrium point for* $G(x, y) = F(x, y)+b(\Gamma(x))d(x, y)$, $x, y \in M$. *Moreover,* $u \in \mathrm{Dom}(\mu)$ *whenever*

(d04) $\mu(u) \le B(\Gamma(u) + \rho) - B(\Gamma(u))$, *for some* $\rho > 0$;

and then (as $F_u(u) - F^*_u = \mu(u)$*),* III) *gives* (19) *and*

V) $b(\Gamma(u) + \rho)d(u, v) \le -F(u, v)$ *(hence* $F(u, v) \le 0$*).*

Some remarks are in order. Let $\varphi : M \to R\cup\{\infty\}$ be some inf-proper function. The relative (generalized) pseudometric F over M given as in (d01) is reflexive, triangular and fulfills (d02); because $\mu(.) = \varphi(.) - \varphi_*$ (hence $\mathrm{Dom}(\mu) = \mathrm{Dom}(\varphi)$). In addition, as $F_u(\cdot) = \varphi(\cdot) - \varphi(u)$, $u \in \mathrm{Dom}(\varphi)$, (d03) is identical with (b08) (modulo d). Putting these together, it follows that Theorems 6 and 7 include Theorems 4 and 5 respectively. The reciprocal inclusions are also true, by the very argument above; so that Theorem 6 $\iff$ Theorem 4 and Theorem 7 $\iff$ Theorem 5. In particular, when Γ is taken as in (c07), Theorem 7 yields the main result in Zhu, Zhong and Cho [27]; see also Bao and Khanh [2].

5. The BKP approach

Let (M, d) be a complete metric space. By a relative pseudometric over M we mean any map $g : M \times M \to R$. Given such an object, remember that $v \in M$ is an equilibrium point of it when $g(v, x) \ge 0$, $\forall x \in M$. A basic particular case to be considered here is

$$g(x, y) = d(x, y) + f(x, y), \quad x, y \in M \quad \text{(i.e.: } g = d + f\text{)},$$

with $d : M \times M \to R_+$, taken as before and $f : M \times M \to R$, a relative generalized pseudometric on M; when the above definition becomes $d(v, x) \ge -f(v, x)$, $\forall x \in M$. Note that, under the choice (d01) of f (for some $\varphi : M \to R$) the variational property of v is "close" to the one in EVP. The following 2005 result in the area due to Bianchi, Kassay and Pini [4] (in short: BKP) is available.

THEOREM 8. *Suppose that* f *is reflexive, triangular, and*

(e01) $f(a, .)$ *is bounded from below and lsc, for each* $a \in M$.

Then, for each $u \in M$*, there exists* $v = v(u) \in M$ *such that*

I) $d(u, v) \le -f(u, v)$;

II) $d(v, x) > -f(v, x)$*, for all* $x \in M \setminus \{v\}$.

Hence, in particular, v *is an equilibrium point for* $g := d + f$.

Note that this result is obtainable from Theorem 6 by simply taking $e = d$ and $F = f$. On the other hand, under the same choice (d01) for f, (e01) becomes

(e02) φ is bounded from below and lsc;

and Theorem 8 is just EVP. So, we may ask whether this extension is effective. The answer is negative; i.e., Theorem 8 is deductible from (hence equivalent with) EVP. This will follow from the following

Proof. Define a function $h : M \to R$ as $h(x) = f(u, x)$, $x \in M$. From (e01), EVP is applicable to (M, d) and h; wherefrom, for the starting $u \in M$ there exists $v \in M$ with

i) $d(u, v) \leq h(u) - h(v)$, ii) $d(v, x) > h(v) - h(x)$, $\forall x \in M \setminus \{v\}$.

The former of these gives I), in view of $h(u) = 0$. And the latter one gives II); because (from the triangular property) $h(v) - h(x) \geq -f(v, x)$, for all such x. ∎

This argument (taken from the 2003 paper due to Bao and Khanh [2]) tells us that Theorem 8 is just a formal extension of EVP. This is also true for the 1993 statement in the area due to Oettli and Thera [15]. In fact, the whole reasoning developed in [4] for proving Theorem 8 is, practically, identical with the one of this last paper. Further aspects may be found in [3].

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