МАТЕМАТИЧКИ ВЕСНИК
65, 4 (2013), 476–480
December 2013

оригинални научни рад
research paper

CLOSED SUBSETS OF STAR σ-COMPACT SPACES

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Abstract. In this paper, we prove the following statements:

(1) There exists a pseudocompact star σ-compact Tychonoff space having a regular-closed subspace which is not star σ-compact.

(2) Assuming $2^{\aleph_0} = 2^{\aleph_1}$, there exists a star countable (hence star σ-compact) normal space having a regular-closed subspace which is not star σ-compact.

1. Introduction

By a space, we mean a topological space. The purpose of this paper is to construct the two examples stated in the abstract. In the rest of this section, we give definitions of terms which are used in the examples. Let X be a space and $\mathcal{U}$ a collection of subsets of X. For $A \subseteq X$, let $St(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$.

Definition. [1, 6] Let P be a topological property. A space X is said to be *star* P if whenever $\mathcal{U}$ is an open cover of X, there exists a subspace $A \subseteq X$ with property P such that $X = St(A, \mathcal{U})$. The set A will be called a *star kernel* of the cover $\mathcal{U}$.

The term star P was coined in [1, 6] but certain star properties, specifically those corresponding to "$\mathcal{P}$=finite" and "$\mathcal{P}$=countable" were first studied by van Douwen et al. in [2] and later by many other authors. A survey of star covering properties with a comprehensive bibliography can be found in [5]. Here, we use the terminology from [1, 6]. In [5] and earlier in [2] a star countable space is called star Lindelöf and strongly 1-star Lindelöf. In [6], a star σ-compact space is called σ-starcompact. From the above definitions, it is clear that every star countable space is star σ-compact and every star σ-compact space is star Lindelöf. In [1], Alas, Junqueira and Wilson studied the relationships of star P properties for $P \in$ {Lindelöf, σ-compact, countable} with other Lindelöf type properties. The author [7] showed that there exists a σ-compact Tychonoff space having a

2010 AMS Subject Classification: 54D20, 54B10, 54D55

Keywords and phrases: Pseudocompact space; normal space; star-Lindelöf space; star σ-compact space.

regular-closed subspace which is not star σ-compact. However, his space is neither pseudocompact nor normal. It is natural for us to consider the following questions:

QUESTION 1. Does there exist a pseudocompact star σ-compact Tychonoff space having a regular-closed subspace which is not star σ-compact?

QUESTION 2. Does there exist a star σ-compact normal space having a regular-closed subspace which is not star σ-compact?

The purpose of this note is to show the two statements stated in the abstract which give a positive answer to question 1 and a consistent answer to Question 2.

Throughout the paper, the cardinality of a set A is denoted by $|A|$. For a cardinal κ, let κ^+ denote the smallest cardinal greater than κ, and $cf(\kappa)$ denote the cofinality of κ. Let $\mathfrak{c}$ denote the cardinality of the continuum, ω_1 the first uncountable cardinal and ω the first infinite cardinal. For a pair of ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$ and $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$. Other terms and symbols that we do not define will be used as in [3].

2. Some examples on star σ-compact spaces

In this section, we construct the two examples stated in the abstract. Recall from [1, 6] that a space X is *star finite* if for every open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $St(F, \mathcal{U}) = X$. In [4], a star finite space is called star compact. It is well-known that countably compactness is equivalent to star compactness for Hausdorff spaces (see [2, 5]). For a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

EXAMPLE 2.1. There exists a pseudocompact star σ-compact Tychonoff space X having a regular-closed subspace which is not star σ-compact.

Proof. Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of the cardinality $\mathfrak{c}$. Let

$$S_1 = (\beta D \times (\mathfrak{c} + 1)) \setminus ((\beta D \setminus D) \times \{\mathfrak{c}\}) = (\beta D \times \mathfrak{c}) \cup (D \times \{\mathfrak{c}\})$$

be the subspace of the product space of βD and $\mathfrak{c} + 1$. Then S_1 is Tychonoff pseudocompact. In fact, it has a countably compact, dense subspace $\beta D \times \mathfrak{c}$. Now, we show that S_1 is not star σ-compact. For each $\alpha < \mathfrak{c}$, let

$$U_\alpha = \{d_\alpha\} \times (\alpha, \mathfrak{c}].$$

Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{\beta D \times \mathfrak{c}\}$$

of S_1 and let F be a σ-compact subset of S_1. Then F is a Lindelöf subset of S_1. Thus $\Lambda = \{\alpha : \langle d_\alpha, \mathfrak{c}\rangle \in F\}$ is countable, since $\{\langle d_\alpha, \mathfrak{c}\rangle : \alpha < \mathfrak{c}\}$ is discrete and closed in S_1. Let $F' = F \setminus \cup\{U_\alpha : \alpha \in \Lambda\}$. Then, if $F' = \emptyset$, choose $\alpha_0 < \mathfrak{c}$ such that $\alpha_0 \notin \Lambda$, then $F \cap U_{\alpha_0} = \emptyset$, hence $\langle d_{\alpha_0}, \mathfrak{c}\rangle \notin St(F, \mathcal{U})$, since U_{α_0} is the only element of $\mathcal{U}$ containing $\langle d_{\alpha_0}, \mathfrak{c}\rangle$. On the other hand, if $F' \neq \emptyset$, since F' is closed in F, then F' is σ-compact and $F' \subseteq \beta D \times \mathfrak{c}$, hence $\pi(F')$ is a σ-compact subspace of the countably compact space $\mathfrak{c}$, where $\pi : \beta D \times \mathfrak{c} \to \mathfrak{c}$ is the projection. Hence there exists $\alpha_1 < \mathfrak{c}$ such that $\pi(F') \cap (\alpha_1, \mathfrak{c}) = \emptyset$. Choose $\alpha < \mathfrak{c}$ such that $\alpha > \alpha_1$ and $\alpha \notin \Lambda$. Then $\langle d_\alpha, \mathfrak{c}\rangle \notin St(F, \mathcal{U})$, since U_α is the only element of $\mathcal{U}$ containing $\langle d_\alpha, \mathfrak{c}\rangle$ and $U_\alpha \cap F = \emptyset$, which shows that S_1 is not star σ-compact.

Let

$$Y = (\beta D \times (\omega + 1)) \setminus ((\beta D \setminus D) \times \{\omega\}) = (\beta D \times \omega) \cup (D \times \{\omega\})$$

be the subspace of the product space of βD and $\omega + 1$. Then Y is star σ-compact, since $\beta D \times \omega$ is a σ-compact dense subset of Y.

Let

$$S_2 = (\beta Y \times (\mathfrak{c} + 1)) \setminus ((\beta Y \setminus Y) \times \{\mathfrak{c}\}) = (\beta Y \times \mathfrak{c}) \cup (Y \times \{\mathfrak{c}\})$$

be the subspace of the product space of βY and $\mathfrak{c} + 1$. Then S_2 is Tychonoff pseudocompact. In fact, it has a countably compact, dense subspace $\beta Y \times \mathfrak{c}$.

We show that S_2 is star σ-compact. To this end, let $\mathcal{U}$ be an open cover of S_2. Since $\beta Y \times \mathfrak{c}$ is countably compact and every countably compact space is star finite, there exists a finite subset $E \subseteq \beta Y \times \mathfrak{c}$ such that

$$\beta Y \times \mathfrak{c} \subseteq St(E, \mathcal{U}).$$

On the other hand, $Y \times \{\omega_1\}$ is star σ-compact, since it is homeomorphic to Y. Thus $Y \times \{\mathfrak{c}\} \subseteq St(((\beta D \times \omega) \times \{\mathfrak{c}\}), \mathcal{U})$, since $(\beta D \times \omega) \times \{\mathfrak{c}\}$ is a σ-compact dense subset of $Y \times \{\mathfrak{c}\}$. Since $Y \times \{\mathfrak{c}\}$ is closed in S_2, then $(\beta D \times \omega) \times \{\mathfrak{c}\}$ is σ-compact in S_2. If we put $F = E \cup ((\beta D \times \omega) \times \{\mathfrak{c}\})$. Then F is a σ-compact subset of S_2 such that $S_2 = St(F, \mathcal{U})$, which shows that S_2 is star σ-compact.

Let $p : D \times \{\mathfrak{c}\} \to (D \times \{\omega\}) \times \{\mathfrak{c}\}$ be a bijection and let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ by identifying $\langle d_\alpha, \mathfrak{c}\rangle$ of S_1 with $p(\langle d_\alpha, \mathfrak{c}\rangle)$ of S_2 for each $\langle d_\alpha, \mathfrak{c}\rangle$ of $D \times \{\mathfrak{c}\}$. Let $\varphi : S_1 \oplus S_2 \to X$ be the quotient map. Then X is pseudocompact, since S_1 and S_2 are pseudocompact. It is clear that $\varphi(S_1)$ is a regular-close subspace of X by the construction of the topology of X which is not star σ-compact.

We shall show that X is star σ-compact. To this end, let $\mathcal{U}$ be an open cover of X. Since $\varphi(S_2)$ is homeomorphic to S_2, then $\varphi(S_2)$ is star σ-compact, hence there exists a σ-compact subset F_1 of $\varphi(S_2)$ such that

$$\varphi(S_2) \subseteq St(F_1, \mathcal{U}).$$

On the other hand, since $\varphi(\beta D \times \mathfrak{c})$ is homeomorphic to $\beta D \times \mathfrak{c}$, then $\varphi(\beta D \times \mathfrak{c})$ is countably compact, hence there exists a finite subset F_2 of $\varphi(S_2)$ such that

$$\varphi(\beta D \times \mathfrak{c}) \subseteq St(F_2, \mathcal{U}).$$

If we put $F = F_1 \cup F_2$, then $X = St(F, \mathcal{U})$. Since $\varphi(S_2)$ is closed in X, then F_1 is σ-compact in X, hence F is a σ-compact subset of X, which shows that X is star σ-compact. ■

For a normal space, we have the following consistent example.

Example 2.2. Assuming $2^{\aleph_0} = 2^{\aleph_1}$, there exists a star countable(hence star σ-compact) normal space having a regular-closed subspace which is not star σ-compact.

Proof. Let L be a set of cardinality $\aleph_1$ disjoint form ω and let $Y = L \cup \omega$ be a separable normal T_1 space under $2^{\aleph_0} = 2^{\aleph_1}$, where L is closed and discrete and each element of ω is isolated. See Example E [9] for the construction of such a space.

Let

$$S_1 = L \cup (\omega_1 \times \omega)$$

and topologize S_1 as follows: A basic neighborhood of $l \in L$ in S_1 is a set of the form

$$G_{U,\alpha}(l) = (U \cap L) \cup ((\alpha, \omega_1) \times (U \cap \omega))$$

for a neighborhood U of l in Y and $\alpha < \omega_1$, and a basic neighborhood of $\langle \alpha, n \rangle \in \omega_1 \times \omega$ in S_1 is a set of the form

$$G_V(\langle \alpha, n \rangle) = V \times \{n\},$$

where V is a neighborhood of α in ω_1. The author [8] showed that S_1 is normal, but it not star Lindelöf, hence it not star σ-compact, since every star σ-compact space is star Lindelöf.

Let S_2 be the same space Y above. Then S_2 is star σ-compact, since ω is a countable dense subset of S_2.

Let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ by identifying l of S_1 with l of S_2 for any $l \in L$. Let $\varphi : S_1 \oplus S_2 \to X$ be the quotient map. Then X is normal, since S_1 and S_2 are normal, and L is closed in S_1 and S_2. It is clear that $\varphi(S_1)$ is a regular-close subspace of X by the construction of the topology of X which is not star σ-compact.

We show that X is star countable. To this end, let $\mathcal{U}$ be an open cover of X. Since ω is a countable dense subset of S_1 and $\varphi(\omega)$ is homeomorphic to $\varphi(S_1)$, then $\varphi(\omega)$ is a countable dense subset of $\varphi(S_1)$, thus $\varphi(S_1) \subseteq St(\varphi(\omega), \mathcal{U})$. On the other hand, since $\omega_1 \times \{n\}$ is countably compact for each $n \in \omega$, then there exists a finite subset F_n of $\varphi(\omega_1 \times \{n\})$ such that $\varphi(\omega_1 \times \{n\}) \subseteq St(F_n, \mathcal{U})$, since $\varphi(\omega_1 \times \{n\})$ is homomorphic to $\omega_1 \times \{n\}$. If we put $F = \varphi(\omega) \cup \cup\{F_n : n \in \omega\}$, then F is a countable subset of X and $X = St(F, \mathcal{U})$, which shows that X is star countable. ■

Remark. It is obvious that $2^{\aleph_0} = 2^{\aleph_1}$ implies negation of CH. Example 2.2 gives a consistent answer to the question 2 above. The author does not know if there is a ZFC counterexample to the question.

Acknowledgement. The author would like to thank Prof. R. Li for his kind help and valuable suggestions. He would also like to thank the referee for his/her careful reading of the paper and a number of valuable suggestions.

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(received 29.10.2011; in revised form 28.11.2011; available online 10.06.2012)

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