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ON SLANT SUBMANIFOLDS OF $N(k)$-CONTACT METRIC MANIFOLDS

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Abstract. The object of the present paper is to study slant submanifolds of an $N(k)$-contact metric manifold. We study the parallelism of Q, and find out necessary and sufficient conditions for the existence of proper slant submanifolds of $N(k)$-contact metric manifolds.

1. Introduction

Slant submanifolds of complex and contact manifolds is an active area of research since slant immersions in complex geometry were defined by B.Y. Chen [6] as a natural generalization of both holomorphic immersions and totally real immersions. Lotta [10] has introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold and he has proved some properties of such immersions. He has also studied [9] the intrinsic geometry of 3-dimensional non-anti-invariant slant submanifolds of K-contact manifolds. Recently, Cabrerizo et al. [4] studied slant submanifolds of Sasakian manifolds and considered the parallelism of Q. Khan et al. [7, 8] studied slant submanifolds of LP-contact and Lorentzian β-Kenmotsu manifolds. Shukla [11] studied slant immersions into quasi-Sasakian manifolds.

Motivated by these works, we have considered proper slant submanifolds of $N(k)$-contact metric manifolds. Cabrerizo et al. [4] cited several examples of slant submanifolds of a Sasakian manifold. For $k = 1$, an $N(k)$-contact metric manifold reduces to a Sasakian manifold. So each example of [4] will be an example of slant submanifolds of an $N(k)$-contact metric manifold. In the present paper We have found necessary and sufficient conditions for the existence of proper slant submanifolds of an $N(k)$-contact metric manifold with parallel endomorphism Q, and also characterised the same using Blair's result [1].

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2. Preliminaries

An $2n+1$-dimensional manifold $\bar{M}^{2n+1}$ is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1,$$
$$\phi\xi = 0, \qquad \eta(\phi X) = 0.$$

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $\bar{M}^{2n+1} \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable, where X is tangent to $\bar{M}$, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $\bar{M}^{2n+1} \times \mathbb{R}$. Let g be the compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{1}$$

Then $\bar{M}^{2n+1}$ becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (1) it can be easily seen that

$$g(X, \phi Y) = -g(\phi X, Y), \qquad g(X, \xi) = \eta(X), \tag{2}$$

for any vector fields X, Y on the manifold. An almost contact metric structure becomes a contact metric structure if $g(X, \phi Y) = d\eta(X, Y)$, for all vector fields X, Y tangent to $\bar{M}$. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$, where R is the curvature tensor [1]. On the other hand, on a manifold $\bar{M}^{2n+1}$ equipped with a Sasakian structure (ϕ, ξ, η, g), one has

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \qquad X, Y \in TM.$$

As a generalization of both $R(X, Y)\xi = 0$ and the Sasakian case, Blair, Koufogiorgos and Papantoniou [3] introduced the case of contact metric manifolds with contact metric structure (ϕ, ξ, η, g) which satisfy

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

for all $X, Y \in TM$.

The (k, μ)-nullity condition on a contact metric manifold is defined by [3].

$$N(k, \mu) = \{W \in T\bar{M} | R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y)\},$$

for all $X, Y \in T\bar{M}$, where $(k, \mu) \in \mathbb{R}^2$. $T\bar{M}$ denotes the tangent space of the manifold $\bar{M}$. A contact metric manifold with $\xi \in N(k, \mu)$ is called a (k, μ)-contact metric manifold. If $\mu = 0$, the (k, μ)-nullity distribution reduces to k-nullity distribution [12]. The k-nullity distribution $N(k)$ of a Riemannian manifold is defined by [12]

$$N(k) = \{Z \in T\bar{M} | R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\},$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call the contact metric manifold an $N(k)$-contact metric manifold.

Let $f\colon (M,g) \to (\bar{M}, g)$ be an isometric immersion. We denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of M and $\bar{M}$ respectively, and by $T^{\perp}(M)$ its normal bundle. Then for vector fields X, Y which are tangent to M, the second fundamental form B is given by the formula

$$B(X,Y) = \overline{\nabla}_X Y - \nabla_X Y. \tag{3}$$

Furthermore, for $N \in T^{\perp}M$,

$$A_N X = \nabla^{\perp}_X N - \overline{\nabla}_X N,$$

where $\nabla^{\perp}$ denotes the normal connection of M. The second fundamental form B and A_N are related by $g(B(X,Y),N) = g(A_N X, Y)$.

However, for an $N(k)$-contact metric manifold M^n of dimension n we have [3]

$$(\overline{\nabla}_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{4}$$

where $h = \frac{1}{2}\mathcal{L}_\xi \phi$. We also have

$$h\xi = 0, \quad h\phi = -\phi h, \quad \ker h = \langle \xi \rangle.$$

In equation (4) putting ξ in place of Y, we obtain

$$\overline{\nabla}_X \xi = -\phi X - \phi h(X) \tag{5}$$

Now we state the following:

Lemma 2.1. [1] *A contact metric manifold M^{2n+1} satisfying $R(X,Y)\xi = 0$ is locally isometric to $E^{n+1} \times S^n(4)$, for $n > 1$ and flat for $n = 1$.*

3. Slant submanifolds

A. Lotta [10] has introduced the following notion of slant immersion in almost contact metric manifolds. Let M be an almost contact metric manifold with structure (ϕ, ξ, η, g). By a slant submanifold of M, we mean an immersed submanifold N of M such that for any $x \in N$ and any $X \in T_x N$ linearly independent on ξ , the angle between ϕX and $T_x N$ is a constant $\theta \in [0, \frac{\pi}{2}]$, called the slant angle of N in M and denoted by $sla(N)$. Invariant and anti-invariant subamnifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively. A slant submanifold which is neither invariant nor anti-invariant is called a proper slant submanifold.

Lotta's definition includes both the cases $\xi \in TM$ and $\xi \in T^{\perp}M$. He proves the following theorem, which generalizes a well-known result of Yano and Kon [13]:

Theorem 3.1. *Let M be a submanifold of a contact metric manifold $\overline{M}$. If ξ is orthogonal to M, then M is anti-invariant.*

Since we are interested to study proper slant submanifold, we will take $\xi \in TM$ and we write, $TM = D \oplus \langle \xi \rangle$, where D is the orthogonal distribution to $\langle \xi \rangle$ in TM.

In this connection a simple result can be proved as follows:

Theorem 3.2. *In a proper slant submanifold of $N(k)$-contact metric manifolds, D remains invariant under h, provided TM is invariant under h.*

Proof. Let $X \in D$. Then $hX \in TM$, since TM is invariant under h. Now,

$$g(hX, \xi) = g(X, h\xi) = 0.$$

Hence the theorem is proved. ■

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{6}$$

where PX is the tangential component of ϕX, and FX is the normal component of ϕX.

Similarly for any $W \in T^{\perp}M$ we have,

$$\phi W = pW + fW,$$

where pW, fW are the tangential and normal components of ϕW, respectively. The submanifold M is called invariant if F is the zero mapping. On the other hand, M will be an anti-invariant submanifold if P is a zero mapping. Now, from (2) and (6), by simple calculation we have

$$g(PX, Y) = -g(X, PY),$$

for any $X, Y \in TM$. So,

$$g(P^2X, Y) = -g(PX, PY) = g(X, P^2Y).$$

Hence if we denote P^2 by Q, we can say, Q is a self-adjoint endomorphism from TM to TM.

Now, let $X \in D$. Then $g(PX, \xi) = -g(X, P\xi) = 0$, since $\phi\xi = 0$. So, $PD \subset D$. Again, $P^2D = P(PD) \subset PD$, since $PD \subset D$. Repeating the steps we have the following lemma:

Lemma 3.1. *Let M be a proper slant submanifold of an $N(k)$-contact metric manifolds $\overline{M}$, then $P^{n+1}D \subset P^nD$, for all positive integer n.*

We define the terms $\nabla P, \nabla F, \nabla Q$ as follows:

(i) $(\nabla_X P)Y = \nabla_X(PY) - P(\nabla_X Y)$,

(ii) $(\nabla_X F)Y = \nabla^{\perp}_X(FY) - F(\nabla_X Y)$,

(iii) $(\nabla_X Q)Y = \nabla_X(QY) - Q(\nabla_X Y)$.

In [4], Cabrerizo et al. have proved the following:

Theorem 3.3. *Let M be a submanifold of an almost contact metric manifold $\overline{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that:*

$$Q = -\lambda\{I - \eta \otimes \xi\}$$

Furthermore, in such case, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

We will make use of this theorem in later sections.

4. Parallelism of Q in slant submanifolds of an $N(k)$-contact metric manifold

Theorem 4.1. *Let M be a slant submanifold of an $N(k)$-contact metric manifold $\overline{M}$, then Q is parallel if and only if M is anti-invariant, provided -1 is not an eigenvalue of h.*

Proof. Let θ be the slant angle of M in $\overline{M}$, then for any $X, Y \in TM$, we have from Theorem 3.3,

$$Q(\nabla_X Y) = \cos^2\theta(-\nabla_X Y + \eta(\nabla_X Y)\xi), \tag{7}$$

and

$$\nabla_X QY = -\cos^2\theta\{\nabla_X Y - \eta(\nabla_X Y)\xi - g(Y, \nabla_X\xi)\xi - \eta(Y)\nabla_X\xi\}, \tag{8}$$

since, $X(\eta(Y)) = \eta(\nabla_X Y) + g(Y, \nabla_X\xi)$.

Hence, $\nabla Q = 0$ if and only if $\cos^2\theta\{g(Y, \nabla_X\xi)\xi + \eta(Y)\nabla_X\xi\} = 0$. So, either $\cos\theta = 0$, which implies M is anti-invariant, or

$$g(Y, \nabla_X\xi)\xi + \eta(Y)\nabla_X\xi = 0. \tag{9}$$

Also, $g(\nabla_X\xi, \xi) = -g(\xi, \nabla_X\xi) = -g(\nabla_X\xi, \xi)$. Hence, $g(\nabla_X\xi, \xi) = 0$, which implies $\nabla_X\xi \in D$.

Now, suppose $\nabla_X\xi \neq 0$. Then (9) implies $\eta(Y) = 0$, that is, $Y \in D$. But then (9) implies, $\nabla_X\xi \in D^\perp = \langle\xi\rangle$, which is impossible.

Hence, $\nabla_X\xi = 0$. Using (5) and (3) we obtain $\nabla_X\xi = -PX - PhX$. So, if -1 is not an eigenvalue of h, then $PX = 0$. This completes the proof. ∎

Thus we have the following:

Corollary 4.1. *Let M be a proper slant submanifold of an $N(k)$-contact metric manifold $\overline{M}$, then Q is parallel if and only if $\nabla_X\xi = 0$, provided -1 is not an eigenvalue of h.*

Now, from (7), (8) and using $\nabla_X\xi = -PX - PhX$, we see that if M is a slant submanifold of an $N(k)$- contact metric manifold $\overline{M}$, then

$$(\nabla_X Q)Y = \cos^2\theta\{g(X + hX, PY)\xi - \eta(Y)(PX + hPX)\},$$

for any $X, Y \in TM$, where θ denotes the slant angle of M.

Lemma 4.1. [10] *Let M be a slant submanifold of an almost contact metric manifold $\overline{M}$. Denote by θ the slant angle of M. Then, at each point x of M, $Q|_D$ has only one eigenvalue $\lambda_1 = -\cos^2\theta$.*

We prove that the converse is also true.

Let $\lambda_1(x)$ be the only eigenvalue of $Q|_D$ at $x \in M$. Let $Y \in D$ be an unit eigenvector associated to λ_1, i.e., $QY = \lambda_1 Y$. Then, we have

$$X(\lambda_1)Y + \lambda_1\nabla_X Y = \nabla_X(QY) = Q(\nabla_X Y) + \lambda g(X + hX, PY)\xi, \tag{10}$$

for any $X \in TM$, since $Y \in D$. Now, since Y is a unit vector we obtain,

$$g(Y, \nabla_X Y) = -g(\nabla_X Y, Y) = -g(Y, \nabla_X Y)$$

and,

$$g(Q(\nabla_X Y), Y) = g(\nabla_X Y, QY) = \lambda_1 g(\nabla_X Y, Y) = 0.$$

Hence, both $\nabla_X Y$ and $Q(\nabla_X Y)$ are orthogonal to Y. Hence, from (10) we conclude that λ_1 is constant on M. Now, let $X \in TM$. Then, $X = \overline{X} + \eta(X)\xi$, where $\overline{X} \in D$. So,

$$QX = Q\overline{X} = \lambda_1\overline{X} = \lambda_1(X - \eta(X)\xi).$$

Hence, by virtue of Theorem 3.3 we conclude that M is slant, and $\lambda_1 = -\cos^2\theta$.

Thus, we obtain:

Theorem 4.2. *Let M be a submanifold of an $N(k)$-contact metric manifold $\overline{M}$ such that $\xi \in TM$. Then, M is slant if and only if*

(i) *$Q|_D$ has only one eigenvalue at each point of M.*

(ii) *There exists $\lambda : M \to [0, 1]$ such that*

$$(\nabla_X Q)Y = \lambda(g(X + hX, PY)\xi - \eta(Y)(PX + PhX)),$$

for $X, Y \in TM$ and $\lambda = \cos^2\theta$, where θ is the slant angle of M.

Theorem 4.3. *Let M be a proper slant submanifold of an $N(k)$-contact metric manifold $\overline{M}$. Q is parallel if and only if D is autoparallel.*

Proof. Suppose Q is parallel. Let $X, Y \in D$. Then we have, $g(\nabla_X Y, \xi) = -g(Y, \nabla_X\xi) = 0$. which implies, $\nabla_X Y \in D$. Hence, D is autoparallel.

Conversely, suppose D is autoparallel. So, for $X, Y \in D, \nabla_X Y \in D$. So, $(\nabla_X Q)Y = \nabla_X(QY) - Q(\nabla_X Y) \in D$, by Lemma 3.1. Since, M is slant, from Theorem 4.2 we have

$$(\nabla_X Q)Y = \cos^2\theta\{g(X + hX, PY)\xi - \eta(Y)(PX + PhX)\} \in \langle\xi\rangle,$$

since $\eta(Y) = 0$ for all $Y \in D$. Combining, we conclude that $(\nabla_X Q)Y = 0$, that is, Q is parallel. ∎

Theorem 4.4. *In a proper slant submanifold of an $N(k)$-contact metric manifold if the endomorphism Q is parallel, then $R(X, Y)\xi = 0$, for all $X, Y \in TM$.*

Proof. The proof follows directly from Corollary 4.1, since

$$R(X,Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi. \quad \blacksquare$$

Thus by virtue of Lemma 2.1 we obtain the result:

PROPOSITION 4.1. *In a proper slant submanifold of an $N(k)$-contact metric manifold if the endomorphism Q is parallel, then M will be locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive curvature 4, that is, $E^{n+1} \times S^n(4)$.*

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