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ON CONVERGENCE OF q-CHLODOVSKY-TYPE MKZD OPERATORS

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Abstract. In the present paper, we define a new kind of MKZD operators for functions defined on $[0,b_n]$, named q-Chlodovsky-type MKZD operators, and give some approximation properties.

1. Introduction

For a function defined on the interval $[0,1]$, the Meyer-König and Zeller operators $M_n(f,x)$ [10] are defined as

$$M_n(f;x)=\sum_{k=0}^{\infty} m_{n,k}\,(x)\,f\left(\frac{k}{n+k}\right) \tag{1.1}$$

where $m_{n,k}=\binom{n+k-1}{k}x^k(1-x)^n$. In 1989 Guo [2] introduced the integrated Meyer-König and Zeller operators $\widetilde{M}_n$ by the means of the operators (1.1), to approximate Lebesgue integrable functions on the interval $[0,1]$. Such operators have been defined as

$$\widetilde{M}_n(f;x)=\sum_{k=0}^{\infty} \widetilde{m}_{n,k}\,(x)\int_{I_k} f(t)\,dt \tag{1.2}$$

where $I_k=[\frac{k}{n+k},\frac{k+1}{n+k+1}]$ and $\widetilde{m}_{n,k}\,(x)=(n+1)\binom{n+k+1}{k}x^k(1-x)^n$. Similar results may be also found in the papers [3, 4].

Recently, Karsli [8] defined the following MKZD operators for functions defined on $[0,b_n]$, named Chlodovsky-type MKZD operators as

$$L_n(f;x)=\sum_{k=0}^{\infty}\frac{n+k}{b_n}m_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n} f(t)b_{n,k}\left(\frac{t}{b_n}\right)dt,\ \ 0\le x,\ t\le b_n, \tag{1.3}$$

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where (b_n) is a positive increasing sequence with the properties

$$\lim_{n\to\infty} b_n = \infty \text{ and } \lim_{n\to\infty} \frac{b_n}{n} = 0$$

and $b_{n,k}(t) = n\binom{n+k}{k} t^k (1-t)^{n-1}$. We now deal with the q-analogue of Chlodovsky-type MKZD operators $L_{n,q}$, defined as

$$L_{n,q}(f;x) = \sum_{k=0}^{\infty} \frac{[n+k]_q}{b_n} m_{n,k,q}\left(\frac{x}{b_n}\right) \int_0^{b_n} q^{-k} f(t) b_{n,k,q}\left(\frac{qt}{b_n}\right) d_q t, \ \ 0 \le x \le b_n, \tag{1.4}$$

where

$$m_{k,n,q}(x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1} (1-q^s x)$$

and

$$b_{n,k,q}(t) = [n]_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q t^k \prod_{s=0}^{n-2} (1-q^s t) \ (0 \le t, x \le 1),$$

provided the q-integral and the infinite series on the r.h.s. of (1.4) are well-defined. It can be easily verified that in the case $q = 1$ the operators defined by (1.4) reduce to the Chlodovsky-type MKZD operators defined by (1.3).

Actually the q-analogue of the linear positive operators was started in the last decade when Phillips [11] first introduced q-Bernstein polynomials, and later their Durrmeyer variants were studied and discussed in [5, 6]. Very recently Govil and Gupta [1] studied the approximation properties of q-MKZD operators. Here our aim is to study the q-analogue of summation-integral-type CMKZD operators. We shall prove that the operators $L_{n,q}f$ being defined in (1.4) converge to the limit f.

Before getting onto the main subject, we first give definitions of *q-integer*, *q-binomial* coefficient and *q-integral*, which are required in this paper. For any fixed real number $q > 0$ and non-negative integer r the *q-integer* of the number r is defined by

$$[r]_q = \begin{cases} (1-q^r)/(1-q), & q \ne 1 \\ r, & q = 1. \end{cases}$$

The *q-factorial* is defined by

$$[r]_q! = \begin{cases} [r]_q [r-1]_q \cdots [1]_q, & r = 1,2,3,\ldots \\ 1, & r = 0. \end{cases}$$

and *q-binomial* coefficient is defined as

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q! [n-r]_q!},$$

for integers $n \ge r \ge 0$. The q-integral is defined as (see [9])

$$\int_0^a f(x)\, d_q x = (1-q)\, a \sum_{n=0}^{\infty} f(aq^n)\, q^n$$

provided the sum converges absolutely. Note that the series on the right-hand side is guaranteed to be absolutely convergent as the function f is such that, for some $M > 0$, $\alpha > -1$, $|f(x)| < Mx^{\alpha}$ in a right neighbourhood of $x = 0$.

Definition 1.1. A function f is q-integrable on $[0, \infty)$ if the series

$$\int_0^{\infty} f(x)\, d_q x = (1-q) \sum_{n \in \mathbb{Z}} f(q^n)\, q^n$$

converges absolutely. We use the notation

$$(a-b)_q^n = \prod_{j=0}^{n-1} (a - q^j b).$$

The q-analogue of Beta function (see [7]) is defined as

$$B_q(m,n) = \int_0^1 t^{m-1} (1-qt)_q^{n-1} d_q t, \quad m, n > 0.$$

Also

$$B_q(m,n) = \frac{[m-1]![n-1]!}{[m+n-1]!}.$$

2. Auxiliary results

In this section we give certain results, which are necessary to prove our main theorem.

Lemma 2.1. *For $s \in \mathbb{N}$,*

$$(L_{n,q} t^s)(x) = b_n^s \sum_{k=0}^{\infty} m_{n,k,q}\left(\frac{x}{b_n}\right) \frac{[n+k]_q!}{[k]_q!} \frac{[k+s]_q!}{[k+s+n]_q!}. \tag{2.1}$$

Proof. We have

$$\begin{aligned}(L_{n,q} t^s)(x) &= \sum_{k=0}^{\infty} \frac{[n+k]_q}{b_n} m_{n,k,q}\left(\frac{x}{b_n}\right) \int_0^{b_n} q^{-k} t^s b_{n,k,q}\left(\frac{qt}{b_n}\right) d_q t \\ &= \sum_{k=0}^{\infty} \frac{[n+k]_q}{b_n} m_{n,k,q}\left(\frac{x}{b_n}\right) \int_0^{b_n} t^s \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{t}{b_n}\right)^k \left(1 - \frac{qt}{b_n}\right)_q^{n-1} d_q t.\end{aligned}$$

Setting $u = t/b_n$, we get

$$\begin{aligned}(L_{n,q} t^s)(x) &= \sum_{k=0}^{\infty} \frac{[n+k]_q}{b_n} m_{n,k,q}\left(\frac{x}{b_n}\right) b_n^{s+1} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \int_0^1 u^{k+s} (1-qu)_q^{n-1} d_q u \\ &= \sum_{k=0}^{\infty} \frac{[n+k]_q}{b_n} m_{n,k,q}\left(\frac{x}{b_n}\right) b_n^{s+1} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q B_q(k+s+1, n)\end{aligned}$$

$$= b_n^s \sum_{k=0}^{\infty} [n+k]_q \, m_{n,k,q}\left(\frac{x}{b_n}\right) \frac{[n+k-1]_q!}{[n-1]_q! \; [k]_q!} \frac{\Gamma_q(k+s+1)\,\Gamma_q(n)}{\Gamma_q(k+s+n+1)}$$

$$= b_n^s \sum_{k=0}^{\infty} m_{n,k,q}\left(\frac{x}{b_n}\right) \frac{[n+k]_q!}{[k]_q!} \frac{[k+s]_q!}{[k+s+n]_q!}.$$

For $s = 0, 1$ and 2 in (2.1), we get respectively

$$(L_{n,q}1)(x) = \sum_{k=0}^{\infty} m_{n,k,q}\left(\frac{x}{b_n}\right) = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n-1}\left(1 - q^s \frac{x}{b_n}\right) = 1, \tag{2.2}$$

since

$$\frac{1}{\prod_{s=0}^{n-1}\left(1 - q^s \frac{x}{b_n}\right)} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k.$$

$$(L_{n,q}t)(x) = b_n \sum_{k=0}^{\infty} m_{n,k,q}\left(\frac{x}{b_n}\right) \frac{[n+k]_q!}{[k]_q!} \frac{[k+1]_q!}{[n+k+1]_q!}$$

$$= b_n \prod_{s=0}^{n-1}\left(1 - q^s \frac{x}{b_n}\right) \sum_{k=0}^{\infty} \frac{[n+k-1]_q!}{[n-1]_q!\,[k]_q!} \frac{[k+1]_q}{[n+k+1]_q} \left(\frac{x}{b_n}\right)^k$$

$$= b_n \prod_{s=0}^{n-1}\left(1 - q^s \frac{x}{b_n}\right) \sum_{k=1}^{\infty} \frac{[n+k-2]_q!}{[n-1]_q!\,[k-1]_q!} \left(\frac{x}{b_n}\right)^k \frac{[k+1]_q}{[n+k+1]_q} \frac{[n+k-1]_q}{[k]_q}$$

$$= b_n \prod_{s=0}^{n-1}\left(1 - q^s \frac{x}{b_n}\right) \sum_{k=1}^{\infty} \frac{[n+k-2]_q!}{[n-1]_q!\,[k-1]_q!} \left(\frac{x}{b_n}\right)^k \frac{[k+1]_q}{[k]_q} \frac{[n+k-1]_q}{[n+k+1]_q}$$

$$\geq b_n \prod_{s=0}^{n-1}\left(1 - q^s \frac{x}{b_n}\right) \sum_{k=1}^{\infty} \frac{[n+k-2]_q!}{[n-1]_q!\,[k-1]_q!} \left(\frac{x}{b_n}\right)^k \frac{[k+1]_q}{[k]_q} \frac{[n-1]_q}{[n+1]_q}$$

$$= \frac{[n-1]_q}{[n+1]_q} b_n \prod_{s=0}^{n-1}\left(1 - q^s \frac{x}{b_n}\right) \sum_{k=1}^{\infty} \frac{[n+k-2]_q!}{[n-1]_q!\,[k-1]_q!} \left(\frac{x}{b_n}\right)^k$$

$$= \frac{[n-1]_q}{[n+1]_q} b_n \prod_{s=0}^{n-1}\left(1 - q^s \frac{x}{b_n}\right) \sum_{k=0}^{\infty} \frac{[n+k-1]_q!}{[n-1]_q!\,[k]_q!} \left(\frac{x}{b_n}\right)^{k+1}$$

$$= \frac{[n-1]_q}{[n+1]_q} \frac{x}{b_n} b_n \sum_{k=0}^{\infty} \frac{[n+k-1]_q!}{[n-1]_q!\,[k]_q!} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n-1}\left(1 - q^s \frac{x}{b_n}\right)$$

$$= \frac{[n-1]_q}{[n+1]_q} x \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n-1}\left(1 - q^s \frac{x}{b_n}\right)$$

$$= \frac{[n-1]_q}{[n+1]_q} x, \tag{2.3}$$

and

$$
\begin{aligned}
(L_{n,q}t^2)(x) &= b_n^2 \sum_{k=0}^{\infty} m_{n,k,q}\left(\frac{x}{b_n}\right)\frac{[n+k]_q!}{k!}\frac{[k+2]_q!}{[k+2+n]_q!}\\
&= b_n^2 \prod_{s=0}^{n-1}\left(1-q^s\frac{x}{b_n}\right)\sum_{k=0}^{\infty}\frac{[n+k-1]_q!}{[n-1]_q!\,[k]_q!}\left(\frac{x}{b_n}\right)^k\frac{[k+2]_q\,[k+1]_q}{[k+2+n]_q\,[k+1+n]_q}\\
&= b_n^2 \prod_{s=0}^{n-1}\left(1-q^s\frac{x}{b_n}\right)\sum_{k=0}^{\infty}\frac{[n+k-1]_q!}{[n-1]_q!\,[k]_q!}\left(\frac{x}{b_n}\right)^k\frac{1+q+q\,[k]_q+2q^2\,[k]_q+q^3\,[k]_q^2}{[k+2+n]_q\,[k+1+n]_q}\\
&\le b_n^2 \prod_{s=0}^{n-1}\left(1-q^s\frac{x}{b_n}\right)\frac{1}{[n-1]_q!}\times\\
&\quad\times\sum_{k=0}^{\infty}\frac{[n+k-3]_q!}{[k]_q!}\left(\frac{x}{b_n}\right)^k\left(1+q+q\,[k]_q+2q^2\,[k]_q+q^3\,[k]_q^2\right)\\
&= (1+q)\,b_n^2 \prod_{s=0}^{n-1}\left(1-q^s\frac{x}{b_n}\right)\frac{1}{[n-1]_q\,[n-2]_q}\sum_{k=0}^{\infty}\frac{[n+k-3]_q!}{[n-3]_q!\,[k]_q!}\left(\frac{x}{b_n}\right)^k\\
&\quad+\left(q+2q^2\right)b_n^2 \prod_{s=0}^{n-1}\left(1-q^s\frac{x}{b_n}\right)\frac{1}{[n-1]_q}\sum_{k=0}^{\infty}\frac{[n+k-2]_q!}{[n-2]_q!\,[k]_q!}\left(\frac{x}{b_n}\right)^{k+1}\\
&\quad+q^3b_n^2 \prod_{s=0}^{n-1}\left(1-q^s\frac{x}{b_n}\right)\frac{1}{[n-1]_q!}\sum_{k=1}^{\infty}\frac{[n+k-3]_q!}{[k-1]_q!}\left(\frac{x}{b_n}\right)^k[k]_q\\
&= (1+q)\,b_n^2\frac{1}{[n-1]_q\,[n-2]_q}+\left(q+2q^2\right)b_n^2\frac{x}{[n-1]_q}\\
&\quad+q^3b_n^2 \prod_{s=0}^{n-1}\left(1-q^s\frac{x}{b_n}\right)\frac{1}{[n-1]_q!}\sum_{k=1}^{\infty}\frac{[n+k-3]_q!}{[k-1]_q!}\left(\frac{x}{b_n}\right)^k\left(1+q\,[k-1]_q\right)\\
&= (1+q)\,b_n^2\frac{1}{[n-1]_q\,[n-2]_q}+\left(q+2q^2\right)b_n^2\frac{1}{[n-1]_q}\frac{x}{b_n}\\
&\quad+q^3b_n^2 \prod_{s=0}^{n-1}\left(1-q^s\frac{x}{b_n}\right)\frac{1}{[n-1]_q!}\sum_{k=0}^{\infty}\frac{[n+k-2]_q!}{[k]_q!}\left(\frac{x}{b_n}\right)^{k+1}\\
&\quad+q^4b_n^2 \prod_{s=0}^{n-1}\left(1-q^s\frac{x}{b_n}\right)\frac{1}{[n-1]_q!}\sum_{k=0}^{\infty}\frac{[n+k-1]_q!}{[k]_q!}\left(\frac{x}{b_n}\right)^{k+2}\\
&= \frac{(1+q)\,b_n^2}{[n-1]_q\,[n-2]_q}+\left(q+2q^2+q^3\right)\frac{b_n}{[n-1]_q}x+q^4x^2. \qquad (2.4)
\end{aligned}
$$

From (2.2), (2.3) and (2.4), an easy computation gives

$$
\left(L_{n,q}(t-x)^2\right)(x) \le \frac{(1+q)\,b_n^2}{[n-1]_q\,[n-2]_q}+\frac{\left(q+2q^2+q^3\right)b_n}{[n-1]_q}x
$$

$$+\left[q^4-2\frac{[n-1]_q}{[n+1]_q}+1\right]x^2:=A_{n,q}(x). \qquad (2.5)$$

It is observed here that for $0<q<1$, one has $[n]_q\to\frac{1}{1-q}$ as $n\to\infty$. This implies that $(L_{n,q}t^2)(x)$ and $\left(L_{n,q}(t-x)^2\right)(x)$ does not converge to x^2 and 0 respectively, as $n\to\infty$. To obtain some convergence results for q-CMKZD operators defined in (1.4), we will consider a sequence (q_n) of real numbers such that $0<q_n<1$, $\lim_{n\to\infty}q_n=1$, and

$$\lim_{n\to\infty}\frac{b_n}{[n]_{q_n}}=0. \quad ■ \qquad (2.6)$$

3. Main results

Now we are ready to obtain some convergence results on q-CMKZD operators.

THEOREM 3.1. *Let (q_n) be a sequence of real numbers such that $0<q_n<1$ and $\lim_{n\to\infty}q_n=1$. If $f\in C[0,\infty)$, we have*

$$|(L_{n,q_n}f)(x)-f(x)|\le 2\omega(f,\sqrt{A_{n,q_n}(x)}), \qquad (3.1)$$

where $\omega(f,\cdot)$ is the usual modulus of continuity of f in the space of continuous functions.

Proof. Using (1.4) for $q=q_n$, we have

$$\begin{aligned}
&|(L_{n,q_n}f)(x)-f(x)|\\
&=\left|\sum_{k=0}^{\infty}\frac{[n+k]_{q_n}}{b_n}m_{n,k,q_n}\left(\frac{x}{b_n}\right)\int_0^{b_n}q_n^{-k}f(t)b_{n,k,q_n}\left(\frac{q_nt}{b_n}\right)d_{q_n}t-f(x)\right|\\
&\le\sum_{k=0}^{\infty}\frac{[n+k]_{q_n}}{b_n}m_{n,k,q_n}\left(\frac{x}{b_n}\right)\int_0^{b_n}q_n^{-k}\left|f(t)-f(x)\right|b_{n,k,q_n}\left(\frac{q_nt}{b_n}\right)d_{q_n}t\\
&\le\sum_{k=0}^{\infty}\frac{[n+k]_{q_n}}{b_n}m_{n,k,q_n}\left(\frac{x}{b_n}\right)\int_0^{b_n}q_n^{-k}\left(\frac{|t-x|}{\delta}+1\right)\omega(f,\delta)b_{n,k,q_n}\left(\frac{q_nt}{b_n}\right)d_{q_n}t\\
&=\omega(f,\delta)\sum_{k=0}^{\infty}\frac{[n+k]_{q_n}}{b_n}m_{n,k,q_n}\left(\frac{x}{b_n}\right)\int_0^{b_n}q_n^{-k}b_{n,k,q_n}\left(\frac{q_nt}{b_n}\right)d_{q_n}t\\
&\quad+\frac{\omega(f,\delta)}{\delta}\sum_{k=0}^{\infty}\frac{[n+k]_{q_n}}{b_n}m_{n,k,q_n}\left(\frac{x}{b_n}\right)\int_0^{b_n}q_n^{-k}\left|t-x\right|b_{n,k,q_n}\left(\frac{q_nt}{b_n}\right)d_{q_n}t\\
&\le\omega(f,\delta)+\frac{\omega(f,\delta)}{\delta}\left\{\left(L_{n,q_n}(t-x)^2\right)(x)\right\}^{1/2}\\
&\le\omega(f,\delta)+\frac{\omega(f,\delta)}{\delta}\left\{A_{n,q_n}(x)\right\}^{1/2}
\end{aligned}$$

Now, if we choose $\delta^2=A_{n,q_n}(x)$, we get

$$|(L_{n,q_n}f)(x)-f(x)|\le 2\omega(f,\sqrt{A_{n,q_n}(x)}),$$

and the proof of Theorem 3.1 is thus complete. ■

It is easy to see that, the right-hand side of formula (3.1) can diverge. Indeed, for $x = \frac{b_n}{2}$ we cannot guarantee $\delta \to 0$ as $n \to \infty$.

From Lemma 2.1 and Theorem 3.1, we can immediately give the following Bohman-Korovkin-type theorem.

THEOREM 3.2. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and $\lim_{n\to\infty} q_n = 1$. Then, for $f \in C[0,\infty)$, the sequence $L_{n,q_n}(f,x)$ converges uniformly to $f(x)$ on any closed finite subinterval $[0,A]$, where $A > 0$ being a constant.*

DEFINITION 3.3. For $f \in C[a,b]$ and $t > 0$, the Peetre-K Functional are defined by

$$K(f,\delta) := \inf_{g\in C^2[a,b]} \left\{ \|f-g\|_{C[a,b]} + t\,\|g\|_{C^2[a,b]} \right\}.$$

THEOREM 3.4. *If $g \in C^2[0,A]$, then*

$$|(L_{n,q}g)(x) - g(x)| \le A_{n,q}(x)\,\|g\|_{C^2[0,A]}\,,$$

where $A > 0$ is a constant.

Proof. By Taylor formula with integral reminder term, we write

$$g(t) = g(x) + (t-x)g'(x) + \int_0^{t-x} (t-x-u)^2 g''(x+u)\,du. \tag{3.2}$$

If we apply the operator (1.4) to (3.2), we get

$$\begin{aligned}
&|(L_{n,q}g)(x) - g(x)| \\
&\quad = \left| g'(x)(L_{n,q}(t-x))(x) + \left(L_{n,q}\left(\int_0^{t-x}(t-x-u)^2 g''(x+u)\,du\right)\right)(x)\right| \\
&\quad \le \|g'\|_{C[0,A]} |(L_{n,q}(t-x))(x)| \\
&\qquad + \|g''\|_{C[0,A]} \left|\left(L_{n,q}\left(\int_0^{t-x}(t-x-u)^2\,du\right)\right)(x)\right|.
\end{aligned}$$

Since

$$\int_0^{t-x}(t-x-u)^2\,du = \frac{(t-x)^2}{2},$$

one gets from (2.5)

$$|(L_{n,q}g)(x) - g(x)| \le \|g'\|_{C[0,A]}\,\{A_{n,q}(x)\}^{1/2} + \|g''\|_{C[0,A]}\,A_{n,q}(x).$$

Now noting that

$$\|g\|_{C^2[a,b]} = \|g\|_{C[a,b]} + \|g'\|_{C[a,b]} + \|g''\|_{C[a,b]}\,,$$

we get

$$|(L_{n,q}g)(x) - g(x)| \le A_{n,q}(x)\,\|g\|_{C^2[0,A]}\,,$$

and this completes the proof of Theorem 3.4. ■

Now, we are ready to prove the following theorem.

THEOREM 3.5. *Let* (q_n) *be a sequence of real numbers such that* $0 < q_n < 1$ *and* $\lim_{n\to\infty} q_n = 1$. *If* $f \in C[0,\infty)$, *then*

$$\|(L_{n,q_n}f) - f\|_{C[0,A]} \leq 2K(f, B_{n,q_n}),$$

where B_{n,q_n} *is the maximum value of* $A_{n,q_n}(x)$ *on* $[0,A]$, $A > 0$ *is a constant; namely,*

$$B_{n,q} = \frac{(1+q)\,b_n^2}{[n-1]_q\,[n-2]_q} + \frac{\left(q+2q^2+q^3\right)b_n}{[n-1]_q}A + \left[q^4 - 2\frac{[n-1]_q}{[n+1]_q} + 1\right]A^2.$$

Proof. By the linearity property of (L_{n,q_n}), we get

$$\begin{aligned}
&|(L_{n,q_n}f)(x) - f(x)| \\
&\quad\leq |(L_{n,q_n}f)(x) - (L_{n,q_n}g)(x)| + |(L_{n,q_n}g)(x) - g(x)| + |g(x) - f(x)| \\
&\quad\leq \|f-g\|_{C[0,A]}\,|(L_{n,q_n}1)(x)| + \|f-g\|_{C[0,A]} + |(L_{n,q_n}g)(x) - g(x)|\,.
\end{aligned}$$

From Theorem 3.4, one has

$$|(L_{n,q_n}f)(x) - f(x)| \leq 2\,\|f-g\|_{C[0,A]} + A_{n,q_n}(x)\,\|g\|_{C^2[0,A]}\,,$$

and hence

$$\|(L_{n,q_n}f) - f\|_{C[0,A]} \leq 2\,\|f-g\|_{C[0,A]} + B_{n,q_n}\,\|g\|_{C^2[0,A]}\,. \tag{3.3}$$

If we take the infimum on the right-hand side of (3.3) over all $g \in C^2[0,A]$, we get

$$\|(L_{n,q_n}f) - f\|_{C[0,A]} \leq 2K(f, B_{n,q_n}).$$

This completes the proof. ■

THEOREM 3.6. *Let* (q_n) *be a sequence of real numbers such that* $0 < q_n < 1$ *and* $\lim_{n\to\infty} q_n = 1$. *If* $f \in Lip_M^{\alpha}[0,\infty)$, *then for any* $A > 0$ *and* $x \in [0,A]$ *the inequality*

$$|(L_{n,q_n}f)(x) - f(x)| \leq M\left\{B_{n,q_n}\right\}^{\frac{\alpha}{2}}$$

holds with the constant M, *which is independent of* n *and* B_{n,q_n} *is as defined in Theorem 3.5.*

Proof. For convenience we write $L_{n,q_n}(f;x)$ instead of $(L_{n,q_n}f)(x)$. Note that

$$\begin{aligned}
|L_{n,q_n}(f;x) - f(x)| &\leq L_{n,q_n}(|f(t)-f(x)|\,;x) \\
&= \sum_{k=0}^{\infty}\frac{[n+k]_{q_n}}{b_n}m_{n,k,q_n}\left(\frac{x}{b_n}\right)\int_0^{b_n} q_n^{-k}\,|f(t)-f(x)|\,b_{n,k,q_n}\left(\frac{q_n t}{b_n}\right)d_{q_n}t
\end{aligned}$$

$$\leq M\int_0^{b_n} q_n^{-k}\left|t-x\right|^{\alpha}\sum_{k=0}^{\infty}\frac{[n+k]_{q_n}}{b_n}m_{n,k,q_n}\left(\frac{x}{b_n}\right)b_{n,k,q_n}\left(\frac{q_nt}{b_n}\right)d_{q_n}t.$$

If we choose $p_1=\frac{2}{\alpha}$ and $p_2=\frac{2}{2-\alpha}$, then $\frac{1}{p_1}+\frac{1}{p_2}=1$. Therefore

$$\begin{aligned}&\left|L_{n,q_n}(f;x)-f(x)\right|\\&\leq M\int_0^{b_n}\left\{\left|t-x\right|^2q_n^{-k}\sum_{k=0}^{\infty}\frac{[n+k]_{q_n}}{b_n}m_{n,k,q_n}\left(\frac{x}{b_n}\right)b_{n,k,q_n}\left(\frac{q_nt}{b_n}\right)\right\}^{\frac{1}{p_1}}\times\\&\times\left\{q_n^{-k}\sum_{k=0}^{\infty}\frac{[n+k]_{q_n}}{b_n}m_{n,k,q_n}\left(\frac{x}{b_n}\right)b_{n,k,q_n}\left(\frac{q_nt}{b_n}\right)\right\}^{\frac{1}{p_2}}d_{q_n}t.\end{aligned}$$

By Hölder inequality, we have

$$\begin{aligned}&\left|L_{n,q_n}(f;x)-f(x)\right|\\&\leq M\left\{\int_0^{b_n}q_n^{-k}\left|t-x\right|^2\sum_{k=0}^{\infty}\frac{[n+k]_{q_n}}{b_n}m_{n,k,q_n}\left(\frac{x}{b_n}\right)b_{n,k,q_n}\left(\frac{q_nt}{b_n}\right)d_{q_n}t\right\}^{\frac{1}{p_1}}\times\\&\times\left\{\int_0^{b_n}q_n^{-k}\sum_{k=0}^{\infty}\frac{[n+k]_{q_n}}{b_n}m_{n,k,q_n}\left(\frac{x}{b_n}\right)b_{n,k,q_n}\left(\frac{q_nt}{b_n}\right)d_{q_n}t\right\}^{\frac{1}{p_2}}\\&=M\left\{\int_0^{b_n}q_n^{-k}\left|t-x\right|^2\sum_{k=0}^{\infty}\frac{[n+k]_{q_n}}{b_n}m_{n,k,q_n}\left(\frac{x}{b_n}\right)b_{n,k,q_n}\left(\frac{q_nt}{b_n}\right)d_{q_n}t\right\}^{\frac{\alpha}{2}}.\end{aligned}$$

From (2.5) we obtain

$$\left|L_{n,q_n}(f;x)-f(x)\right|\leq M\left\{A_{n,q_n}(x)\right\}^{\frac{\alpha}{2}}.$$

This implies that for $x\in[0,A]$

$$\left|(L_{n,q_n}f)(x)-f(x)\right|\leq M\left\{B_{n,q_n}\right\}^{\frac{\alpha}{2}}$$

which in view of (2.5) and (2.6) tends to zero as $n\to\infty$. ■

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