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EXPONENTIAL DICHOTOMY AND STRONGLY STABLE VECTORS OF HILBERT SPACE CONTRACTION SEMIGROUPS

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Abstract. The paper deals with exponential dichotomy and its relationship with strongly stable vectors associated with Hilbert space semigroups. Contraction semigroups are decomposed by using the strongly stability operator associated with the semigroup. Necessary and sufficient conditions for exponential stability and non-exponential stability are investigated in terms of norm inequalities—instead of a Lyapunov operator equation.

1. Introduction

In this paper $\mathcal{H}$ stands for a complex Hilbert space. Inner product and norm in $\mathcal{H}$ will be denoted by $\langle\cdot\,;\cdot\rangle$ and $\|\cdot\|$. By a subspace of $\mathcal{H}$ we mean a *closed* linear manifold of $\mathcal{H}$. If $\mathcal{S}$ is any subspace of $\mathcal{H}$, then $\mathcal{H}$ admits the orthogonal decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$, where $\oplus$ stand for orthogonal direct sum and $\mathcal{S}^\perp = \mathcal{H} \ominus \mathcal{S}$ is the orthogonal complement of $\mathcal{S}$. Let the Banach algebra of all operators (bounded linear transformations) of $\mathcal{H}$ into itself be denoted by $\mathcal{B}[\mathcal{H}]$; and let the identity operator and the null operator in $\mathcal{B}[\mathcal{H}]$ be denoted by I and O. The norm in $\mathcal{B}[\mathcal{H}]$ is also denoted by $\|\cdot\|$. Throughout the paper $[T(t)] = \{T(t)\,;\ t \geq 0\}$ will stand for a strongly continuous semigroup (C_0-semigroup) of operators $T(t)$ in $\mathcal{B}[\mathcal{H}]$.

A semigroup $[T(t)]$ is *exponentially stable* (*e-stable*) if there exist real constants $M \geq 1$ and $\alpha > 0$ such that

$$\|T(t)x\| \leq Me^{-\alpha t}\,\|x\| \quad \text{for every } t \geq 0 \text{ and every } x \in \mathcal{H}$$

(equivalently, $\|T(t)\| \leq Me^{-\alpha t}$ for every $t \geq 0$). It is *uniformly stable* (*u-stable*) if

$$\|T(t)\| \to 0 \quad \text{as} \quad t \to \infty$$

(i.e., $T(t) \to O$ as $t \to \infty$ in $\mathcal{B}[\mathcal{H}]$). Recall that $[T(t)]$ is uniformly stable if and only if it is exponentially stable [3]. A semigroup $[T(t)]$ *strongly stable* (*s-stable*) if

$$\|T(t)x\| \to 0 \quad \text{as} \quad t \to \infty \quad \text{for every } x \in \mathcal{H}.$$

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We assume that $\mathcal{H}$ is infinite-dimensional; otherwise all the above stability concepts coincide. Since uniform stability (which is equivalent to exponential stability) clearly implies strong stability, it follows that there are two types of s-stability, namely, *s-stability-via-e-stability* (or simply e-stability; that is s-stability because of e-stability), and *s-stability-non-e-stability* (or simply s-stability for non-e-stable semigroups).

An *exponential dichotomy* of a semigroup $[T(t)]$ on $\mathcal{H}$ is an orthogonal decomposition of it, say $\mathcal{H} = \mathcal{R} \oplus \mathcal{R}^\perp$, where $\mathcal{R}$ is a subspace of $\mathcal{H}$ that reduces $T(t)$ for every $t \geq 0$ on which $[T(t)]$ is exponentially stable, while it is non-exponentially stable on the orthogonal complement $\mathcal{R}^\perp$ of $\mathcal{R}$ (if $\mathcal{R}^\perp \neq \{0\}$). It is plain that if $[T(t)]$ is exponentially stable, then the trivial exponential dichotomy of $[T(t)]$ is $\mathcal{H}$ itself, in the sense that $\mathcal{H}$ is naturally identified with $\mathcal{H} \oplus \{0\}$ (which is not dichotomic after all). For nondegenerate exponential dichotomy the trivial cases (either $\mathcal{R} = \{0\}$ or $\mathcal{R}^\perp = \{0\}$) must be excluded.

We consider exponential dichotomy of Hilbert space contraction semigroups using various decompositions of contraction semigroups [1, 6], as well as norm inequalities for exponential stability of semigroups [7].

Exponential dichotomy involves an orthogonal projection (often called exponential dichotomic projection [10], which is such that the semigroup is exponentially stable on its range and non-exponentially stable on its kernel in the sense that all nonzero vectors in it are non-exponentially stable as in the forthcoming Definition 1(c). This exponential dichotomic projection is the orthogonal projection onto the exponential stability subspace, and it is dominated by the orthogonal projection onto the strong stability subspace.

Exponential dichotomy for Hilbert space C_0-semigroups is defined in Section 2. It is shown that if a semigroup is exponentially dichotomic (i.e., if it has an exponential dichotomy), then the reducing subspace on which it is exponentially stable is maximal (in the sense that no exponential stability subspace includes it). This subspace clearly is included in the subspace of strong stability associated with the semigroup. If this inclusion is proper, then we consider the characterization of the two types of s-stable vectors associated with the semigroup. The contraction case is investigated in Section 3. We close the paper in Section 4 with a detour towards strong dichotomy of Hilbert space semigroups.

2. Exponential dichotomy and strongly stable vectors

Recall that we will be dealing with C_0-semigroups $[T(t)]$ on a complex Hilbert space $\mathcal{H}$. We begin by making the following definitions.

Definition 1. Let $[T(t)]$ be a semigroup on $\mathcal{H}$. Take a vector $x \in \mathcal{H}$.

(a) x is *s-stable* if

$$\|T(t)x\| \to 0 \quad \text{as} \quad t \to \infty.$$

(b) x is *e-stable* if there exist real numbers $M(x) \geq 1$ and $\alpha(x) > 0$ such that

$$\|T(t)x\| \leq M(x)\, e^{-\alpha(x)t} \|x\| \quad \text{for every } t \geq 0.$$

Since e-stability obviously implies s-stability, every e-stable vector is s-stable, and so an e-stable vector is also referred to as an *s-stable-via-e-stable* vector. In other words, *x is e-stable if and only if x is s-stable-via-e-stable*.

(c) x is *non-s-stable* if it is not s-stable; that is, if

$$\|T(t)x\| \not\to 0 \quad \text{as} \quad t \to \infty.$$

(d) x is *non-e-stable* if it is not e-stable; that is, if for every $M \geq 1$ and $\alpha > 0$,

$$Me^{-\alpha t}\|x\| < \|T(t)x\| \quad \text{for some } t \geq 0.$$

(e) x is *s-stable-non-e-stable* if it is not e-stable but is s-stable. That is, x is *s-stable-non-e-stable* if $\lim_{t\to\infty} \|T(t)x\| = 0$ and, for every $M \geq 1$ and $\alpha > 0$, there exists $t = t(M, \alpha, x) \geq 0$ such that $Me^{-\alpha t}\|x\| < \|T(t)x\|$.

A subset (in particular, a subspace) of $\mathcal{H}$ consisting entirely of s-stable, or e-stable vectors will be referred to as an s-stable, or e-stable subset (subspace). A subset (subspace) of $\mathcal{H}$ for which all *nonzero* vectors are s-stable-non-e-stable, or non-s-stable or non-e-stable, will be referred to as an s-stable-non-e-stable, or non-s-stable, or non-e-stable subset (subspace), respectively. Observe that an s-stable subset is one that may contains both e-stable (i.e., s-stable-via-e-stable) as well as s-stable-non-e-stable vectors. Also note that if a subspace has a property Π, and if it is maximal (in the sense that it is not included in any subset of $\mathcal{H}$ that has property Π), then its orthogonal complement does not have property Π.

If a subspace $\mathcal{R}$ of $\mathcal{H}$ is $[T(t)]$-invariant, then we say that the semigroup $[T(t)]$ is *e-stable on the subspace* $\mathcal{R}$ if its restriction $[T(t)|_{\mathcal{R}}]$ is e-stable, which means that there exist real constants $M \geq 1$ and $\alpha > 0$ such that

$$\|T(t)u\| \leq Me^{-\alpha t}\,\|u\| \quad \text{for every } t \geq 0 \text{ and every } u \in \mathcal{R}.$$

This means that $\mathcal{R}$ is a *homogeneously e-stable subspace* in the sense that each vector u in $\mathcal{R}$ is e-stable where $M(u)$ and $\alpha(u)$ of Definition 1(b) do not depend on u in $\mathcal{R}$ (i.e., they are constants not only over $t \geq 0$ but also over all $u \in \mathcal{R}$). On the other hand, if a *nonzero* subspace $\mathcal{N}$ of $\mathcal{H}$ is $[T(t)]$-invariant, then we say that a semigroup $[T(t)]$ is *non-e-stable on the subspace* $\mathcal{N}$ if its restriction $[T(t)|_{\mathcal{N}}]$ is not only not e-stable but, more that, if every *nonzero* vector of $\mathcal{N}$ is non-e-stable in the sense of Definition 1(d). That is, for every $0 \neq v \in \mathcal{N}$, $M \geq 1$, and $\alpha > 0$,

$$e^{-\alpha t}\|v\| < \tfrac{1}{M}\|T(t)v\| \quad \text{for some } t \geq 0.$$

For definitions of e-dichotomic projection and e-dichotomy for C_0-semigroups $[T(t)]$ on a complex Hilbert space $\mathcal{H}$ see, for instance, [10, 11] and the references therein. It is worth noticing that there are different versions of e-dichotomy; we follow here the one considered in [11].

Definition 2. Let $[T(t)]$ be a semigroup on $\mathcal{H}$. Take an orthogonal projection $P \in \mathcal{B}[\mathcal{H}]$ so that $\mathcal{R} = \text{range}(P) = P(\mathcal{H})$ and $\mathcal{N} = \text{kernel}(P) = P^{-1}(\{0\})$ are complementary orthogonal subspaces of $\mathcal{H}$; that is, $\mathcal{H} = \mathcal{R} \oplus \mathcal{N}$ with $\mathcal{R}^\perp = \mathcal{N}$. If

(a) P commutes with each $T(t)$ (i.e., $T(t)P = P\,T(t)$ for every $t \geq 0$ or, equivalently, $\mathcal{R}$ and $\mathcal{N}$ are reducing subspaces for $[T(t)]$), and

(b) $[T(t)]$ is e-stable on $\mathcal{R}$ and non-e-stable on $\mathcal{N}$ (in the sense that all nonzero vectors in $\mathcal{N}$ are non-e-stable),

then we say that P is an *e-dichotomic projection* for $[T(t)]$ and the semigroup is said to be *e-dichotomic* (or *P-dichotomic*).

The generator of a C_0-semigroup $[T(t)]$ on $\mathcal{H}$ will be denoted by A, which is a linear (not necessarily bounded) transformation of a dense linear manifold $\mathcal{D}$ of $\mathcal{H}$, the domain of A, into $\mathcal{H}$.

Remark 1. Consider the setup of Definition 2. Let $E \in \mathcal{B}[\mathcal{H}]$ be the complementary projection of P (i.e., $E = I - P$ is the orthogonal projection with kernel(E) = range(P) and range(E) = kernel(P)). Since the subspaces $\mathcal{R}$ = range(P) and $\mathcal{N}$ = kernel(P) reduce each $T(t)$, it follows that $[T(t)|_{\mathcal{R}}]$ and $[T(t)|_{\mathcal{N}}]$ also are C_0-semigroups acting on $\mathcal{R}$ and on $\mathcal{N}$ (i.e., $T(t)|_{\mathcal{R}}\, u = T(t)Px$ and $T(t)|_{\mathcal{N}}\, v = T(t)Ex$ for every $x = (u, v)$ in $\mathcal{H} = \mathcal{R} \oplus \mathcal{N}$ with $u \in \mathcal{R}$ and $v \in \mathcal{N}$). The generators of $[T(t)|_{\mathcal{R}}]$ and $[T(t)|_{\mathcal{N}}]$ are the restrictions, $A|_{\mathcal{R}}$ and $A|_{\mathcal{N}}$, of the generator A of $[T(t)]$ to their respective domains $\mathcal{D}_{\mathcal{R}} = \mathcal{D} \cap \mathcal{R}$ and $\mathcal{D}_{\mathcal{N}} = \mathcal{D} \cap \mathcal{N}$ (i.e., $A|_{\mathcal{R}}\, u = Au$ for every $u \in \mathcal{D} \cap \mathcal{R}$ and $A|_{\mathcal{N}}\, v = Ax$ for every $v \in \mathcal{D} \cap \mathcal{N}$).

Remark 2. Consider again the setup of Definition 2. Observe that $\mathcal{N} \setminus \{0\}$ may be empty. Indeed, there are semigroups (contractive C_0-continuous on infinite-dimensional spaces) for which e-dichotomy degenerates in the following sense: there may be no nonzero e-stable vector (i.e., $\mathcal{R} = \{0\}$) or there may be no nonzero non-e-stable vector (i.e., $\mathcal{N} = \{0\}$). Actually, we noticed in Section 1 that e-stability may be thought of as (degenerate) e-dichotomy on the whole space $\mathcal{H}$ (i.e., $\mathcal{R} = \mathcal{H}$ which means $P = I$ — also see [10]). In the same way, non-e-stability is (degenerate) e-dichotomy on the zero space (i.e., $\mathcal{N} = \mathcal{H}$ which means $P = O$) in the sense that all nonzero vectors are non-e-stable. However, the extra assumption that P is nontrivial (i.e., $O \neq P \neq I$) or, equivalently, that the subspaces $\mathcal{R}$ and $\mathcal{N}$ are nontrivial (i.e., $\{0\} \neq \mathcal{N} \neq \mathcal{H}$ and $\{0\} \neq \mathcal{R} \neq \mathcal{H}$) ensures that e-dichotomy does not degenerate. If e-dichotomy degenerates to $\mathcal{R} = \mathcal{H}$ (i.e., to $\mathcal{N} = \{0\}$), then the semigroup surely has no s-table-non-e-stable vectors; if it degenerates to $\mathcal{N} = \mathcal{H}$ (i.e., to $\mathcal{R} = \{0\}$), then we may still have semigroups with no s-table-non-e-stable. Samples: $T(t) = e^{-\frac{1}{2}t}I$ or $T(t) = I$. Moreover, even if e-dichotomy does not degenerate we may have semigroups with no s-stable-non-e-stable vectors (for instance $T(t) = e^{-\frac{1}{2}t}I \oplus I$ has no s-stable-non-e-stable vectors).

We will characterize s-stable vectors associated with a semigroup $[T(t)]$ on $\mathcal{H}$ by using *exponential dichotomy*.

Let $[T(t)]$ be a semigroup on complex Hilbert space $\mathcal{H}$. Set

$$\mathcal{M} = \big\{x \in \mathcal{H}\colon \lim_{t\to\infty} \|T(t)x\| \to 0\big\}.$$

It is readily verified that $\mathcal{M}$ a subspace of $\mathcal{H}$, which is referred to as the *s-stable subspace of* $[T(t)]$. Thus consider the decomposition

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}.$$

It is clear that $\mathcal{M}$ is $[T(t)]$-invariant (so that $\mathcal{M}^{\perp}$ is $[T^*(t)]$-invariant, where $T^*(t)$ denotes the adjoint of $T(t)$). Moreover, if the invariant subspace $\mathcal{M}$ is reducing, then the restriction of each $T(t)$ to $\mathcal{M}^\perp$ makes the semigroup $[T(t)|_{\mathcal{M}^\perp}]$ non-s-stable in the sense that all nonzero vectors in it are non-s-stable. Note that $\mathcal{M}$ is maximal in the sense that every set of s-stable vectors for $[T(t)]$ is included in $\mathcal{M}$.

Recall that a projection *onto* a subspace means a projection whose range is precisely that subspace.

Theorem 1. *Let $[T(t)]$ be a semigroup on $\mathcal{H}$ and consider its s-stable subspace $\mathcal{M}$. Let Q be the orthogonal projection onto the s-stable subspace $\mathcal{M}$. Suppose $[T(t)]$ is e-dichotomic and consider the e-dichotomic (orthogonal) projection P of* Definition 2 *with range $\mathcal{R}$ and kernel $\mathcal{N}$. Then the following assertions hold true.*

(a) $\mathcal{R} \subseteq \mathcal{M}$. *Equivalently,* $P \le Q$ (i.e., *P is dominated by Q*).

(b) $\mathcal{M} = \mathcal{R} \oplus (\mathcal{M} \cap \mathcal{N})$.

(c) $\mathcal{N} = \mathcal{M}^\perp \oplus (\mathcal{M} \cap \mathcal{N})$.

(d) *The decomposition of* Definition 2 *is refined to* $\mathcal{H} = \mathcal{R} \oplus (\mathcal{M} \cap \mathcal{N}) \oplus \mathcal{M}^\perp$.

(e) *The subspace $\mathcal{R}$ is maximal (in the sense that there is no subspace of $\mathcal{H}$ on which $[T(t)]$ is e-stable).*

(f) *$\mathcal{R} \subset \mathcal{M}$ if and only if $\mathcal{M} \cap \mathcal{N} \ne \{0\}$. In this case $[T(t)]$ is s-stable-non-e-stable on the nonzero subspace $\mathcal{M} \cap \mathcal{N}$.*

Proof. Suppose that $[T(t)]$ is e-dichotomic according to Definition 2. Then there is an orthogonal projection $P \in \mathcal{B}[\mathcal{H}]$ such that $\mathcal{R} = \mathrm{range}(P)$ is $[T(t)]$-reducing and, for some $M \ge 1$ and some $\alpha > 0$,

$$\|T(t)x\| \le Me^{-\alpha t}\|x\| \quad \text{for every } t \ge 0 \text{ and every } x \in \mathcal{R},$$

which implies that $\|T(t)x\| \to 0$ as $t \to \infty$ for every $x \in \mathcal{R}$, and therefore $\mathcal{R} \subseteq \mathcal{M}$, where $\mathcal{M}$ is the s-stable subspace of $[T(t)]$. In other words, $\mathcal{R}$ is e-stable (for $[T(t)]$) and, as it happens with every e-stable subspace, $\mathcal{R}$ is tautologically s-stable-via-e-stable. But the above inclusion is equivalent to following inequality, $P \le Q$, where Q is the orthogonal projection onto $\mathcal{M}$ (i.e., $\mathrm{range}(Q) = Q(\mathcal{H}) = \mathcal{M}$ and $\mathrm{kernel}(Q) = Q^{-1}(\{0\}) = \mathcal{M}^\perp$). Moreover, $\mathcal{R} \subseteq \mathcal{M}$ yields the decomposition

$$\mathcal{M} = \mathcal{R} \oplus (\mathcal{M} \cap \mathcal{N}),$$

where the subspace $\mathcal{M} \cap \mathcal{N}$ is $[T(t)]$-invariant since $\mathcal{M}$ and $\mathcal{N}$ are $[T(t)]$-invariant (in fact $\mathcal{N}$ and $\mathcal{R}$ are both reducing). Since $\mathcal{N} = \mathcal{R}^\perp$, another consequence of the inclusion $\mathcal{R} \subseteq \mathcal{M}$ is $\mathcal{M}^\perp \subseteq \mathcal{N}$, and therefore $\mathcal{N}$ can be decomposed as

$$\mathcal{N} = \mathcal{M}^\perp \oplus (\mathcal{M} \cap \mathcal{N}),$$

and so the decomposition of Definition 2 is refined to

$$\mathcal{H} = \mathcal{R} \oplus (\mathcal{M} \cap \mathcal{N}) \oplus \mathcal{M}^\perp.$$

If $\mathcal{R}$ is not maximal, then there exists a (homogeneously) e-stable subspace $\mathcal{R}'$ for $[T(t)]$ such that $\mathcal{R} \subset \mathcal{R}'$. Then $\mathcal{R}' \cap \mathcal{N} \ne \{0\}$. Take $0 \ne v \in \mathcal{R}' \cap \mathcal{N}$. Since $v \in \mathcal{N}$, for every $M \ge 1$ and every $\alpha > 0$ there exists a $t = t(\alpha, M)$ such that

$$Me^{-\alpha t}\|v\| < \|T(t)v\|.$$

But this contradicts the fact that the vector v is e-stable (which happens because $v \in \mathcal{R}'$). Therefore, $\mathcal{R}$ is maximal. Finally observe that

$$\mathcal{R} \subset \mathcal{M}$$

if and only if $\mathcal{M} \cap \mathcal{N} \neq \{0\}$ (reason: $\mathcal{R} = \mathcal{M}$ if and only if $\mathcal{M} \cap \mathcal{N} = \{0\}$ since $\mathcal{N} = \mathcal{R}^{\perp}$). Thus the proper inclusion $\mathcal{R} \subset \mathcal{M}$ implies that $[T(t)]$ is s-stable-non-e-stable on $\mathcal{M} \cap \mathcal{N}$ because $\mathcal{M} \cap \mathcal{N} \neq \{0\}$, $\mathcal{M}$ is s-stable, and $\mathcal{N}$ is non-e-stable (in the sense that all nonzero vectors in $\mathcal{N}$ are non-e-stable). ■

Remark 3. Theorem 1(f) says that, even though $\mathcal{N}$ is a non-e-stable subspace (in the sense that all nonzero vectors in $\mathcal{N}$ are non-e-stable) it can still contain s-stable (thus s-stable-non-e-stable) vectors whenever the inclusion $\mathcal{R} \subset \mathcal{M}$ is proper. (Note that the proper inclusion clearly implies that the reducing e-stable subspace $\mathcal{R}$ is not the whole space; but it may be zero, and $\mathcal{M}$ may be the whole space). Conversely, if there exist s-stable-non-e-stable vectors, then the inclusion $\mathcal{R} \subset \mathcal{M}$ is proper (since, in this case, the e-stable $\mathcal{R}$ does not contain the s-stable-non-e-stable vectors of $\mathcal{M}$). However, as we have seen in Remark 2, there are semigroups (contractive C_0-continuous e-dichotomic on infinite-dimensional spaces) with no s-stable-non-e-stable vectors for which the inclusion $\mathcal{R} \subseteq \mathcal{M}$ becomes an identity. Examples: $T(t) = I$ is e-dichotomic with $P = O$ and $\mathcal{R} = \mathcal{M} = \{0\}$, and $T(t) = e^{-\frac{1}{2}t} I$ is e-dichotomic with $P = I$ and $\mathcal{R} = \mathcal{M} = \mathcal{H}$. Actually, even if $[T(t)]$ is e-dichotomic with a nontrivial projection P we may still have $\mathcal{R} = \mathcal{M}$. For instance, $T(t) = e^{-\frac{1}{2}t} I \oplus I$ on $\mathcal{H} = \mathcal{R} \oplus \mathcal{N}$ is e-dichotomic with a nontrivial projection P (since $\mathcal{R} \neq \{0\}$ and $\mathcal{N} \neq \{0\}$) such that $\mathcal{R} = \mathcal{M}$.

Corollary 1. *Take a semigroup $[T(t)]$ on $\mathcal{H}$. Let $P \in \mathcal{B}[\mathcal{H}]$ be the e-dichotomic projection of $[T(t)]$ (with range $\mathcal{R}$ and kernel $\mathcal{N}$) and let $Q \in [\mathcal{B}[\mathcal{H}]$ be the orthogonal projection onto the s-stable subspace $\mathcal{M}$ of $[T(t)]$. The following assertions are pairwise equivalent*

(a) $P = Q$.

(b) $\mathcal{M} \cap \mathcal{N} = \{0\}$.

(c) $\mathcal{M} = \mathcal{R}$.

(d) $\mathcal{M} \subseteq \mathcal{R}$.

Proof. The definitions of P and Q ensure that (a) $\Longleftrightarrow$ (c) (since the orthogonal projection onto a subspace is unique). Theorem 1(b) ensures that (c) $\Longleftrightarrow$ (b). Theorem 1(a) ensures that (c) $\Longleftrightarrow$ (b) ■

Observe that assertion (c) in Corollary 1 (and so any of the above equivalent assertions) implies that $\mathcal{M}$ reduces $[T(t)]$ because $\mathcal{R}$ reduces $[T(t)]$ by Definition 2. Moreover, (d) means that the s-stable subspace $\mathcal{M}$ consists only of e-stable vectors (and so $\mathcal{M}$ is an s-stable-via-e-stable subspace if any of the above equivalent assertions holds true).

Also note from Theorem 1(b) that s-stable-via-e-stable vectors (i.e., $\mathcal{R}$) and s-stable-non-e-stable vectors (i.e., $\mathcal{M} \cap \mathcal{N}$) are complementary orthogonal subspaces of the s-stable space $\mathcal{M}$ once the semigroup is e-dichotomic. The s-stable subspace $\mathcal{M}$ plays a central role in e-dichotomy *contraction* semigroups of the next section.

3. The contraction case

A semigroup $[T(t)]$ is *contractive*, or a *contraction* semigroup, if

$$\|T(t)x\| \leq \|x\| \quad \text{for every } \; x \in \mathcal{H} \; \text{ and all } \; t \geq 0$$

or, equivalently, if $\|T(t)\| \leq 1$ for all $t \geq 0$. If $[T(t)]$ is a contraction semigroup on $\mathcal{H}$, then $\lim_{t\to\infty} \|T(t)x\|$ exist in $\mathbb{R}$ for every $x \in \mathcal{H}$ (see e.g., [12, Proposition III.9.1]). Since $\{T^*(t)T(t)\}$ is a bounded monotone family of self-adjoint operators (indeed a nonincreasing family of nonnegative contractions: $\|T^*(t)T(t)\| = \|T(t)\|^2 \leq 1$ and $T^*(t+s)T(t+s) \leq T^*(t)T(t)$ for every $s, t \geq 0$), it follows that it converges strongly as $t \to \infty$. Thus, associated with a contractive semigroup $[T(t)]$ there is an operator $C \in \mathcal{B}[\mathcal{H}]$ which is defined by

$$Cx = \lim_{t\to\infty} T^*(t)T(t)x \quad \text{for every } x \in \mathcal{H},$$

whose kernel coincides with the s-stable space $\mathcal{M}$ for $[T(t)]$, that is,

$$\text{kernel}(C) = \mathcal{M}$$

(since $\|Cx\| = \lim_{t\to\infty} \|T(t)x\|^2$ for every $x \in \mathcal{H}$). The operator C is referred to as the *s-stability operator* (although it not s-stable itself) associated with the contraction semigroup $[T(t)]$. Moreover,

$$\text{kernel}(C - C^2) = \text{kernel}(C) \oplus \text{kernel}(I - C).$$

(see e.g., [6, Proposition 3.3]). We now connect e-dichotomy with the s-stability operator C of a contraction semigroup $[T(t)]$.

Theorem 2. *Let C be the s-stability operator of a contraction semigroup $[T(t)]$ on $\mathcal{H}$. Then $\mathcal{H}$ admits the decomposition*

$$\mathcal{H} = \text{kernel}(C) \oplus \text{kernel}(I - C) \oplus \text{kernel}(C - C^2)^\perp.$$

The subspaces kernel(C)*,* kernel$(I - C)$*, and* kernel$(C - C^2)$ *are $[T(t)]$-invariant, $[T(t)|_{\text{kernel}(C)}]$ is s-stable, $[T(t)|_{\text{kernel}(I-C)}]$ is isometric, and $\|x\| \neq \|T(t)x\| \not\to 0$ as $t \to \infty$ for every nonzero $x \in$* kernel$(C - C^2)^\perp$.

Proof. The decomposition of $\mathcal{H}$ is an immediate consequence of the preceding identities since $\mathcal{H} = \text{kernel}(C - C^2) \oplus \text{kernel}(C - C^2)^\perp$, and so is the s-stability of $[T(t)|_{\text{kernel}(C)}]$ on the invariant subspace kernel(C). It is well known that the subspaces kernel$(I - C)$ and kernel$(C - C^2)$ are $[T(t)]$-invariant, and $T(t)|_{\text{kernel}(I-C)}$ is an isometry for each $t \geq 0$ (see e.g., [1], [6, Chapter 3]). The remaining results are straightforward by the above properties. ∎

Remark 4. Note that if the invariant subspace kernel$(C - C^2)$ reduces $[T(t)]$ (so that kernel$(C - C^2)^\perp$ is also $[T(t)]$-invariant), then the property $\|x\| \neq \|T(t)x\| \not\to 0$ *as $t \to \infty$ for every nonzero $x \in$* kernel$(C - C^2)^\perp$, is equivalent to saying that the semigroup $[T(t)|_{\text{kernel}(C-C^2)^\perp}]$ *is non-s-stable and completely non-isometric*. As we will see in Theorem 3 below, the above decomposition can be further refined if C is a projection. In this case $[T(t)]$ can be decomposed into the direct sum of an s-stable contraction semigroup, a unilateral shift semigroup and a unitary semigroup. Decompositions for the special case where C is a projection, involving both C and C_*, the s-stability operator associated $[T^*(t)]$, see e.g., [9] or [6, Section 5.3].

Corollary 2. *If $[T(t)]$ is contraction semigroup on $\mathcal{H}$, with an s-stability operator C, and is e-dichotomic with respect to an e-dichotomic projection P, then*

$$\mathcal{H} = \operatorname{range}(P) \oplus (\operatorname{kernel}(C) \cap \operatorname{kernel}(P)) \oplus \operatorname{kernel}(I - C) \oplus \operatorname{kernel}(C - C^2)^\perp.$$

$[T(t)]$ is e-stable on the reducing subspace range(P)*, $[T(t)]$ is s-stable-non-e-stable on the invariant subspace* kernel$(C) \cap$ kernel(P)*, $[T(t)]$ is isometric (and so non-s-stable) on the invariant subspace* kernel$(I - C)$*, and $\|x\| \neq \|T(t)x\| \not\to 0$ as $t \to \infty$ for every nonzero $x \in$* kernel$(C - C^2)^\perp$.

Proof. Since $\mathcal{M}$=kernel(C), where $\mathcal{M}$ is the s-stable subspace of $[T(t)]$, the claimed results follow from Theorems 1(d) and 2. ∎

The next theorem applies to the class of contraction semigroups whose s-stability operator C is a projection (cf. Remark 4 above).

Theorem 3. *Let C be the s-stability operator of a contraction semigroup $[T(t)]$ on $\mathcal{H}$. If C is a projection, then $\mathcal{H}$ admits the orthogonal decomposition*

$$\mathcal{H} = \operatorname{range}(P) \oplus (\operatorname{kernel}(C) \cap \operatorname{kernel}(P) \oplus \operatorname{kernel}(I - C))$$

consisting entirely of reducing subspaces for $[T(t)]$. Moreover, for each $t \geq 0$,

$$T(t) = T|_{\operatorname{range}(P)} \oplus T(t)|_{\operatorname{kernel}(C) \cap \operatorname{kernel}(P)} \oplus T(t)|_{\operatorname{kernel}(I-C)},$$

where $[T|_{\operatorname{range}(P)}]$ is e-stable, $[T(t)|_{\operatorname{kernel}(C) \cap \operatorname{kernel}(P)}]$ is s-stable-non-e-stable, and $[T(t)|_{\operatorname{kernel}(I-C)}] = [T(t)|_{\operatorname{range}(C)}]$ is isometric (and so non-s-stable).

Proof. If $C = C^2$, then kernel$(C - C^2)^\perp = \{0\}$. Thus the decomposition in Theorem 2 can be identified with

$$\mathcal{H} = \operatorname{kernel}(C) \oplus \operatorname{kernel}(I - C),$$

and that in Corollary 2 with

$$\mathcal{H} = \operatorname{range}(P) \oplus (\operatorname{kernel}(C) \cap \operatorname{kernel}(P)) \oplus \operatorname{kernel}(I - C).$$

In general, $\mathcal{M}$ = kernel(C) and kernel$(I - C)$ are just $[T(t)]$-invariant and the subspace kernel$(C - C^2)^\perp$ is not necessarily $[T(t)]$-invariant. However, since $\mathcal{R}$ = range(P) and $\mathcal{N}$ = kernel(P) reduce $[T(t)]$ (Definition 2), if the above decompositions hold true, then the remaining subspaces in those decompositions are all orthogonal complements of reducing subspaces, thus reducing themselves. Moreover, the action of the restrictions of $[T(t)]$ on those reducing subspaces follow from Theorem 2 and Corollary 2. Finally, $[T(t)|_{\operatorname{kernel}(I-C)}] = [T(t)|_{\operatorname{range}(C)}]$ because kernel$(I - C)$ = range(C) if $C = C^2$. ∎

It is well known that e-stability of Hilbert space semigroups is characterized by a Lyapunov operator equation [2]. This is also the case of e-dichotomy—see [10, 11] and the references therein. However e-stability as well as non-e-stability of Hilbert space contraction semigroups can also be characterized by norm inequalities, as we will see in the next lemma. These will be applied in the forthcoming Theorem 3 to characterize e-dichotomy.

If $[T(t)]$ is a contraction semigroup, then its generator A is *dissipative*; that is,

$$\mathrm{Re}\,\langle Ax\,;x\rangle \leq 0 \quad \text{for every } \; x \in \mathcal{D},$$

where $\mathcal{D}$ denotes the domain of A (which is a dense linear manifold of $\mathcal{H}$), and A is maximal dissipative in the sense that there is no dissipative extension of it on $\mathcal{H}$ [4, 5, 12]. The generator A is called *strictly dissipative* if

$$\mathrm{Re}\,\langle Ax\,;x\rangle \,<\, 0 \quad \text{for every } \; 0 \neq x \in \mathcal{D}.$$

If A is strictly dissipative, then $(-\mathrm{Re}\,\langle Ax\,;x\rangle)^{\frac{1}{2}}$ defines a norm on $\mathcal{D}$, which is referred to as the *dissipative norm on* $\mathcal{D}$. If $[T(t)]$ is a contraction semigroup, then we say that it is *plain-e-stable* if it is e-stable with $M = 1$; that is, if there exists a constant $\alpha > 0$ such that

$$\|T(t)x\| \leq e^{-\alpha t}\,\|x\| \quad \text{for every } t \geq 0 \text{ and every } x \in \mathcal{H}$$

(equivalently, $\|T(t)\| \leq e^{-\alpha t}$ for every $t \geq 0$). A contraction semigroup $[T(t)]$ is called a *proper contraction* semigroup if

$$\|T(t)x\| < \|x\| \quad \text{for every } \; 0 \neq x \in \mathcal{H} \; \text{ and all } \; t > 0,$$

and it is called a *strict contraction* semigroup if

$$\|T(t)\| < 1 \quad \text{for all } \; t > 0.$$

Definition 3. Let $[T(t)]$ be a contraction semigroup on $\mathcal{H}$. Take an orthogonal projection $P \in \mathcal{B}[\mathcal{H}]$ so that $\mathcal{R} = \mathrm{range}(P) = P(\mathcal{H})$ and $\mathcal{N} = \mathrm{kernel}(P) = P^{-1}(\{0\})$ are complementary orthogonal subspaces of $\mathcal{H}$; that is,

$$\mathcal{H} = \mathcal{R} \oplus \mathcal{N}$$

with $\mathcal{R}^{\perp} = \mathcal{N}$. If

(a) P commutes with each $T(t)$ (i.e., $T(t)P = P\,T(t)$ for every $t \geq 0$ or, equivalently, $\mathcal{R}$ and $\mathcal{N}$ are reducing subspaces for $[T(t)]$), and

(b) $[T(t)]$ is plain-e-stable on $\mathcal{R}$ and non-plain-e-stable on $\mathcal{N}$ (in the sense that all nonzero vectors v in $\mathcal{N}$ are such that for every $\alpha > 0$, there exists $t = t(v, \alpha) \geq 0$ for which $e^{-\alpha t}\|v\| < \|T(t)v\|$),

then we say that P is a *plain-e-dichotomic projection* for $[T(t)]$ and the contraction semigroup is said to be *plain-e-dichotomic* (or *plain-P-dichotomic*).

Lemma 1. *Let $[T(t)]$ contraction semigroup with a strictly dissipative generator A.*

(a) *$[T(t)]$ is plain-e-stable if and only if there exists a constant $\alpha > 0$ such that,*

$$\alpha\,\|x\|^2 \leq -\mathrm{Re}\,\langle Ax\,;x\rangle \quad \textit{for every } \; x \in \mathcal{D}.$$

(b) $[T(t)]$ *is non-plain-e-stable if and only if for each* $\beta > 0$ *there exists a vector* $x_\beta \in \mathcal{D}$ *such that*

$$-\mathrm{Re}\,\langle Ax_\beta\,;x_\beta\rangle < \beta\,\|x_\beta\|^2.$$

(d) $[T(t)]$ *is s-stable if and only if*

$$-\int_0^\infty \mathrm{Re}\,\langle Ax\,;x\rangle\,dt = \|x\|^2 \quad \textit{for every } x \in \mathcal{D}.$$

(e) $[T(t)]$ *is s-stable-non-plain-e-stable if and only if*

$$\|x\|^2 = -\int_0^\infty \mathrm{Re}\,\langle Ax\,;x\rangle\,dt \quad \textit{for every } x \in \mathcal{D},$$

and for every $\beta > 0$ *there is an* $x_\beta \in \mathcal{D}$ *such that*

$$-\tfrac{1}{\beta}\,\mathrm{Re}\,\langle Ax_\beta\,;x_\beta\rangle < \|x_\beta\|^2 = -\int_0^\infty \mathrm{Re}\,\langle Ax_\beta\,;x_\beta\rangle\,dt.$$

Proof. [8, Theorem 2], [7, Remark 1(b), Theorem 2, and Corollary 4]. ∎

Definition 3 and Lemma 1 lead to the following characterization of e-dichotomy for contraction semigroups.

Theorem 4. *Let* $[T(t)]$ *be a contraction semigroup on* $\mathcal{H}$ *with a strictly dissipative generator* A. *An orthogonal projection* $P \in \mathcal{B}[\mathcal{H}]$ *is a plain-e-dichotomic projection for the semigroup* $[T(t)]$ *if and only if there exists a constant* $\alpha > 0$ *such that,*

$$\alpha\,\|x\|^2 \le -\mathrm{Re}\,\langle Ax\,;x\rangle \quad \textit{for every } x \in \mathcal{D}\cap\mathcal{R}$$

with $\mathcal{R} = \mathrm{range}(P)$, *and for each* $\beta > 0$ *there is an* $x_\beta \in \mathcal{D}\cap\mathcal{N}$ *such that*

$$-\mathrm{Re}\,\langle Ax_\beta\,;x_\beta\rangle < \beta\,\|x_\beta\|^2$$

with $\mathcal{N} = \mathrm{kernel}(P)$.

Proof. Definition 3 says that $[T(t)]$ is plain-e-dichotomic if and only if there exists an orthogonal projection $P \in \mathcal{B}[\mathcal{H}]$ such that $\mathcal{R} = \mathrm{range}(P)$ and $\mathcal{N} = \mathrm{kernel}(P)$ reduce $[T(t)]$ and $[T(t)]$ is plain-e-stable on $\mathcal{R}$ and non-plain-e-stable on $\mathcal{N}$. Since $\mathcal{H} = \mathcal{R} \oplus \mathcal{N}$, Lemma 1 ensures the claimed result on the decomposition of $\mathcal{D}$ into the complementary orthogonal linear manifolds $\mathcal{D}\cap\mathcal{R}$ and $\mathcal{D}\cap\mathcal{N}$. ∎

Corollary 3. *Let* $[T(t)]$ *be a contraction semigroup with a strictly dissipative generator* A. *Let* C *be the s-stability operator, and let* $\mathcal{M}$ *be the s-stable subspace, associated with* $[T(t)]$. *Suppose* C *is a projection. Then* $[T(t)]$ *is plain-e-dichotomic with respect to the orthogonal projection* $P = I - C$ *if and only if there exists a constant* $\alpha > 0$ *such that*

$$\alpha\,\|x\|^2 \le -\mathrm{Re}\,\langle Ax\,;x\rangle \quad \textit{for every } x \in \mathcal{D}\cap\mathcal{M},$$

and for each $\beta > 0$ *there is an* $x_\beta \in \mathcal{D}\cap C(\mathcal{H})$ *such that*

$$-\mathrm{Re}\,\langle Ax_\beta\,;x_\beta\rangle < \beta\,\|x_\beta\|^2.$$

Proof. If $C = C^2$ then the projection C is *orthogonal* (see e.g., [6, p. 51]). Thus consider the complementary orthogonal projection $P = I - C$ with $\mathrm{range}(P) = \mathrm{kernel}(C) = \mathcal{M}$ (the s-stable subspace of $[T(t)]$) and $\mathrm{kernel}(P) = \mathrm{range}(C) = C(\mathcal{H})$. Applying Theorem 4 we get the claimed necessary and sufficient condition that P is plain-e-dichotomic for $[T(t)]$. ∎

Under the assumption of Corollary 3 (i.e., if C is a projection and if $I - C$ is plain-e-dichotomic for $[T(t)]$), the s-stable $\mathcal{M}$ and e-stable $\mathcal{R}$ subspaces for $[T(t)]$ coincide (i.e., $\mathcal{M} = \mathcal{R}$ as expected). Also note that it may happen that $\mathcal{D} \cap \mathcal{M}$ or $\mathcal{D} \cap C(\mathcal{H})$ may be zero (even though $\mathcal{D}$ is dense in $\mathcal{H}$, and even under the additional assumption that the projection C is nontrivial—so that $\mathcal{M}$ and $C(\mathcal{H})$ are nontrivial subspaces of $\mathcal{H}$).

4. Concluding remark

Suppose a semigroup $[T(t)]$ is e-dichotomic and consider the *e-stable* subspace $\mathcal{R}$ of the e-dichotomic projection P of Definition 2,

$$\mathrm{range}(P) = \mathcal{R}.$$

Since the orthogonal projection onto a subspace is unique, since P commutes with each $T(t)$ if and only if $\mathrm{range}(P)$ and $\mathrm{kernel}(P)$ are reducing subspaces for each $T(t)$ (and so $[T(t)]$ is e-stable on $\mathcal{R} = \mathrm{range}(P)$ and non-s-stable on $\mathrm{kernel}(P)$), and since $\mathcal{R}$ is maximal, we may restate Definition 2 as follows.

> Let $\mathcal{R}$ be a maximal e-stable subspace for $[T(t)]$. Consider the orthogonal projection P onto $\mathcal{R}$ so that $[T(t)]$ is e-stable on $\mathcal{R}$ and non-s-stable on $\mathrm{kernel}(P)$. If P commutes with each $T(t)$, then P is an *e-dichotomic projection* for $[T(t)]$, which is said to be e-*dichotomic*.

If we replace *exponential stability* with *strong stability* in Definition 2 we come across with *s-dichotomic* semigroups. Precisely, if $[T(t)]$ is a semigroup on $\mathcal{H}$, and if there exists an orthogonal projection in $\mathcal{B}[\mathcal{H}]$ that commutes with each $T(t)$ such that $[T(t)]$ is s-table on its range and non-s-stable on its kernel, then we say that the projection is *s-dichotomic* for $[T(t)]$ and the semigroup is *s-dichotomic*.

Definition 4. Let $[T(t)]$ be a semigroup on $\mathcal{H}$. Take an orthogonal projection $Q \in \mathcal{B}[\mathcal{H}]$ so that $\mathrm{range}(Q) = Q(\mathcal{H})$ and $\mathrm{kernel}(Q) = Q^{-1}(\{0\})$ are complementary orthogonal subspaces of $\mathcal{H}$; that is,

$$\mathcal{H} = \mathrm{range}(Q) \oplus \mathrm{kernel}(Q)$$

with $\mathrm{range}(Q)^\perp = \mathrm{kernel}(Q)$. If

(a) Q commutes with each $T(t)$ (i.e., $T(t)Q = QT(t)$ for every $t \geq 0$ or, equivalently, $\mathrm{range}(Q)$ and $\mathrm{kernel}(Q)$ are reducing subspaces for $[T(t)]$), and

(b) $[T(t)]$ is s-stable on $\mathrm{range}(Q)$ and non-s-stable on $\mathrm{kernel}(Q)$ (in the sense that all nonzero vectors in $\mathrm{kernel}(Q)$ are non-s-stable).

then we say that Q is an *s-dichotomic projection* for $[T(t)]$ and the semigroup is said to be s-*dichotomic*.

Consider the s-stable subspace $\mathcal{M} = \{x \in \mathcal{H}\colon \lim_{t\to\infty} \|T(t)x\| \to 0\}$ of $[T(t)]$. It is clear that $\operatorname{range}(Q) \subseteq \mathcal{M}$. On the other hand, suppose there exists a vector $x = (u, v) \in \mathcal{H}$, with $u \in \operatorname{range}(Q)$ and $v \in \operatorname{kernel}(Q)$, such that x lies in $\mathcal{M} \setminus \operatorname{range}(Q)$. Then $\lim_{t\to\infty} T(t)x = (T(t)u, T(t)v) = (0, 0)$ since $x \in \mathcal{M}$, and so $\lim_{t\to\infty} T(t)v = 0$, which implies that $v = 0$. But $x = (u, 0)$ lies in $\operatorname{range}(Q)$, which is a contradiction. Therefore, $\mathcal{M} \subseteq \operatorname{range}(Q)$ and so

$$\operatorname{range}(Q) = \mathcal{M}.$$

Thus, by uniqueness of the orthogonal projection onto a subspace, Q is in fact the orthogonal projection onto the s-stable subspace $\mathcal{M}$ of $[T(t)]$, which means that this actually is the same orthogonal projection Q of Theorem 1. Since Q commutes with each $T(t)$ if and only if $\operatorname{range}(Q)$ and $\operatorname{kernel}(Q)$ are reducing subspaces for $[T(t)]$ (which implies that $[T(t)]$ must be s-stable on $\mathcal{M} = \operatorname{range}(Q)$ and non-s-stable on $\operatorname{kernel}(Q)$), and since $\mathcal{M}$ is maximal, we may restate Definition 2 as follows.

> Consider the orthogonal projection Q onto $\mathcal{M}$ so that $[T(t)]$ is e-stable on $\mathcal{M}$ and non-s-stable on $\operatorname{kernel}(Q)$. If Q commutes with each $T(t)$, then Q is an *s-dichotomic projection* for $[T(t)]$, which is said to be s-*dichotomic*.

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