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ON CERTAIN LINEAR OPERATOR DEFINED BY BASIC HYPERGEOMETRIC FUNCTIONS

Rabha W. Ibrahim

Abstract. By employing the basic hypergeometric series, we introduce here a linear operator for analytic functions. By means of this linear operator, we define and investigate a class of analytic functions. Also, as an application of Jack's lemma, sufficient conditions for univalence, starlikeness and strong starlikeness of certain analytic functions are obtained.

1. Introduction

The hypergeometric function plays an important role in mathematical analysis and its applications. This function is employed in solving many interesting problems, such as conformal mapping of triangular domains bounded by line segments or circular arcs, various problems of quantum mechanics, etc. Moreover, various special functions such as the Legendre polynomials, the Chebyshev polynomials, the ultraspherical polynomials, the Jacobi polynomials, etc, can be expressed in terms of the hypergeometric function.

The classical theories of hypergeometric function and the q-basic hypergeometric function involve many well known summation and transformation formulae such as the binomial theorem, the Vandermonde summations and their analogues. The q-hypergeometric function are usually called the basic hypergeometric function, where "basic" refers to the base q. The theory of basic hypergeometric function arises in combinatorics and classical analysis, number theory, statistic, physics and the theory of quantum Lie algebra.

For convenience, we recall some standard notations for basic hypergeometric series (cf. [1]). Let q be a complex number such that $0 < |q| < 1$. Define the $q-$shifted factorial for all integers k (including infinity) by

$$(\alpha; q)_k = \prod_{j=1}^{k}(1 - \alpha q^j).$$

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We write

$$ {}_r\phi_{r-1}\left[\begin{matrix}\alpha_1,\alpha_2,\dots,\alpha_r;\\ \beta_1,\beta_2,\dots,\beta_{r-1};\end{matrix}q,z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1,q)_n(\alpha_2,q)_n\dots(\alpha_r,q)_n}{(q,q)_n(\beta_1,q)_n(\beta_2,q)_n\dots(\beta_{r-1},q)_n}\,z^n $$

where $\alpha_1,\dots,\alpha_r$ are called the upper parameters, $\beta_1,\dots,\beta_{r-1}$ the lower parameters, z is the argument, and q the base of the series. The basic hypergeometric ${}_r\phi_{r-1}$ series terminates if one of the upper parameters, say α_r, is of the form q^{-n}, for a nonnegative integer n. If the basic hypergeometric series does not terminate then it converges by the ratio test when $|z|<1$.

2. Preliminaries

Let $\mathcal{H}$ be the class of functions analytic in the unit disk $U=\{z:|z|<1\}$ and for $a\in\mathbb{C}$ (set of complex numbers) and $n\in\mathbb{N}$ (set of natural numbers), let $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\cdots$. Let $\mathcal{A}$ be the class of functions f, analytic in U and normalized by the conditions $f(0)=f'(0)-1=0$. Then a function $f\in\mathcal{A}$ is called convex or starlike if it maps U into a convex or starlike region, respectively. Corresponding classes are denoted by $\mathcal{K}$ and S^*: It is well known that $\mathcal{K}\subset S^*$; that both are subclasses of the class of univalent functions and have the following analytical representations

$$ f\in S^*\Leftrightarrow \Re\frac{zf'(z)}{f(z)}>0,\ z\in U $$

and

$$ f\in\mathcal{K}\Leftrightarrow \Re\left[1+\frac{zf''(z)}{f'(z)}\right]>0,\ z\in U. $$

The Hadamard product for the functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n, g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{A}$ is

$$ f(z)*g(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n,\quad (z\in U). $$

Let f be analytic in U, g analytic and univalent in U and $f(0)=g(0)$. Then, by the symbol $f(z)\prec g(z)$ (f subordinate to g) in U, we shall mean $f(U)\subset g(U)$.

Let $\phi:\mathbb{C}^2\to\mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z)),zp'(z))\prec h(z)$ then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, $p\prec q$. If p and $\phi(p(z)),zp'(z))$ are univalent in U and satisfy the differential superordination $h(z)\prec\phi(p(z)),zp'(z))$ then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q\prec p$.

Corresponding to the function $\Phi(z)$ given by

$$ \begin{aligned}\Phi(z)&=z{}_r\phi_{r-1}\left[\begin{matrix}\alpha_1,\alpha_2,\dots,\alpha_r;\\ \beta_1,\beta_2,\dots,\beta_{r-1};\end{matrix}q,z\right]\\ &=z+\sum_{n=2}^{\infty}\frac{(a_1,q)_{n-1}(a_2,q)_{n-1}\dots(a_r,q)_{n-1}}{(q,q)_{n-1}(b_1,q)_{n-1}(b_2,q)_{n-1}\dots(b_{r-1},q)_{n-1}}z^n,\end{aligned} $$

we introduce here the following linear operator:

$$\mathcal{B}_r[\alpha,\beta] := \mathcal{B}_r\Big(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_{r-1},q;z\Big)f(z) = \Phi(z)*f(z)$$
$$= z + \sum_{n=2}^{\infty} \frac{(\alpha_1,q)_{n-1}(\alpha_2,q)_{n-1}\dots(\alpha_r,q)_{n-1}}{(q,q)_{n-1}(\beta_1,q)_{n-1}(\beta_2,q)_{n-1}\dots(\beta_{r-1},q)_{n-1}} a_n z^n. \quad (1)$$

It is clear that

$$\mathcal{B}_r\Big(q,0,\dots,0;0,\dots,0,q;z\Big)f(z) = f(z).$$

When $q = (-\frac{1}{\lambda})^{1/j}, \lambda \geq 1, \alpha_1 = q, \alpha_2 = 1, \dots, \alpha_r = 1; \beta_1 = n, \dots, \beta_{r-1} = n$, we observe that the operator defined by (1) reduces to the operator

$$\mathcal{B}_r\Big(q,1,\dots,1;n,\dots,n,(-\frac{1}{\lambda})^{1/j};z\Big)f(z) = z + \sum_{n=2}^{\infty}\left[\frac{\lambda+1}{\lambda+n}\right]^m a_n z^n, \quad m \in \mathbb{N}. \quad (2)$$

Operator (2) involves the well known operators defined by Bernardi [2,3], Srivastava and Attiya [4] (see also Răaducanu and Srivastava [5], Liu [6] and Prajapat and Goyal [7]) for integer power. Moreover, when $q = (-\frac{1}{\lambda})^{1/j}, \lambda \geq 1, \alpha_1 = q, \alpha_2 = n, \dots, \alpha_r = n; \beta_1 = 1, \dots, \beta_{r-1} = 1$, we observe that the linear operator defined by (1) reduces to the operator

$$\mathcal{B}_r\Big(q,n,\dots,n;1,\dots,1,(-\frac{1}{\lambda})^{1/j};z\Big)f(z) = z + \sum_{n=2}^{\infty}\left[\frac{\lambda+n}{\lambda+1}\right]^m a_n z^n, \quad m \in \mathbb{N}. \quad (3)$$

Operator (3) involves the operator introduced by Cho-Kim (see [8]). Finally for $q = (-\frac{1}{\lambda})^{1/j}, \lambda \geq 1, \alpha_1 = q, \alpha_2 = n-1, \dots, \alpha_r = n-1; \beta_1 = 0, \dots, \beta_{r-1} = 0$, operator (1) imposes the Al-Oboudi differential operator (see [9])

$$\mathcal{B}_r\Big(q,n-1,\dots,n-1;0,\dots,0,(-\frac{1}{\lambda})^{1/j};z\Big)f(z) = z + \sum_{n=2}^{\infty}[1+(n-1)\lambda]^m a_n z^n, \ m \in \mathbb{N}. \quad (4)$$

Note that the Al-Oboudi differential operator is a generalization for Sălăgean differential operator [10].

In the present paper, our main results reduce to the following subclass: $T(\mu)$ of the analytic function class $\mathcal{A}$ which consists of functions $f \in \mathcal{A}$ satisfying the subordination relation given below:

$$(1-\mu)f'(z) + \mu\Big(1 + \frac{zf''(z)}{f'(z)}\Big) \prec F(z)$$

where $\mu \in \mathbb{R}$, F is the conformal mapping of the unit disk U with $F(0) = 1$. We study the univalence, starlikeness and strong starlikeness of this class. We need the following preliminaries:

Lemma 1. [11] *Let $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then*

$$z_0 w'(z_0) = k w(z_0), \quad (5)$$

where k is a real number and $k \geq 1$.

Lemma 2. [12] *Let $f \in \mathcal{A}$ be such that $f' \prec 1 + az$, $0 < a \le 1$, then*

$$\frac{zf'(z)}{f(z)} \prec \Big(\frac{1+z}{1-z}\Big)^{\delta}, \quad z \in U, 0 < \delta < 1.$$

Lemma 3. [13] *Let $f \in \mathcal{A}$ be such that $f' \prec 1 + az$, $0 < a \le \frac{1}{2}$, then*

$$\frac{zf'(z)}{f(z)} \prec 1 + \Big(\frac{3a}{2-a}\Big)z, \quad z \in U.$$

3. Main results

Theorem 1. *Let P be the class of analytic functions in U of the form $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$, satisfy the condition*

$$(1-\mu)p(z) + \mu\frac{zp'(z)}{p(z)} \prec (1-\mu)(1+\gamma z)^b, \tag{6}$$

where $0 < \mu \le 1$, $0 < \gamma \le \frac{\mu}{2}$, $b \in \mathbb{R}^+$, then

$$p(z) \prec (1 + \frac{2\gamma}{\mu}z)^b. \tag{7}$$

Proof. Let us assume that $p(z) = (1 + \frac{2\gamma}{\mu}w(z))^b$, where $w(z)$ be analytic in U with $w(0) = 0$. Our aim is to show that $|w(z)| < 1$, $z \in U$. If $|w(z)| \nless 1$, by Lemma 1, there exists z_0, $|z_0| < 1$ such that $|w(z_0)| = 1$ and $z_0 w'(z_0) = kw(z_0)$ where $k \ge 1$. When we put $w(z_0) = e^{i\theta}$, we have

$$\begin{aligned}\Big|(1-\mu)p(z_0) + \mu\frac{z_0 p'(z_0)}{p(z_0)}\Big| &= \Big|(1-\mu)(1+\frac{2\gamma}{\mu}w(z_0))^b + \frac{2\gamma b z_0 w'(z_0)}{(1+\frac{2\gamma}{\mu}w(z_0))}\Big| \\ &= \Big|(1-\mu)(1+\frac{2\gamma}{\mu}w(z_0))^b + \frac{2\gamma bkw(z_0)}{(1+\frac{2\gamma}{\mu}w(z_0))}\Big| \\ &= \Big|(1-\mu)(1+\frac{2\gamma}{\mu}e^{i\theta})^b + \frac{2\gamma bke^{i\theta}}{(1+\frac{2\gamma}{\mu}e^{i\theta})}\Big|.\end{aligned}$$

By choosing $e^{i\theta} \to 1$, we observe that $\frac{2\gamma bke^{i\theta}}{(1+\frac{2\gamma}{\mu}e^{i\theta})} > 0$; thus

$$\begin{aligned}\Big|(1-\mu)p(z_0) + \mu\frac{z_0 p'(z_0)}{p(z_0)}\Big| &\ge \Big|(1-\mu)(1+\frac{2\gamma}{\mu})^b\Big| \\ &\ge \Big|(1-\mu)(1+\frac{\gamma}{\mu})^b\Big| \ge \Big|(1-\mu)(1+\gamma)^b\Big|, \quad \frac{1}{\mu} \ge 1,\end{aligned}$$

which is a contradiction with (6). Therefore, we must have $|w(z)| < 1, z \in U$. Hence the assertion (7) holds. ■

By letting $p(z) = \mathcal{B}_r[\alpha, \beta]'$ in Theorem 1, we have the following result

COROLLARY 1. *Let $f \in \mathcal{A}$. If for $0 < \mu \leq 1$, $0 < \gamma \leq \frac{\mu}{2}$, $b \in \mathbb{R}^+$, the relation*

$$(1-\mu)\mathcal{B}_r[\alpha, \beta]' + \mu\Big(1 + \frac{z\mathcal{B}_r[\alpha, \beta]''}{\mathcal{B}_r[\alpha, \beta]'}\Big) \prec (1-\mu)(1+\gamma z)^b$$

holds then

$$\mathcal{B}_r[\alpha, \beta]' \prec (1 + \frac{2\gamma}{\mu}z)^b. \tag{8}$$

By putting $\alpha_1 = q, \alpha = 0, \beta = 0$ in Corollary 1, we have the following result

COROLLARY 2. *Let $f \in \mathcal{A}$. If the condition*

$$(1-\mu)f'(z) + \mu\frac{zf''(z)}{f'(z)} \prec (1-\mu)(1+\gamma z)^b$$

is satisfied then

$$f'(z) \prec (1 + \frac{2\gamma}{\mu}z)^b \tag{9}$$

and therefore f is a bounded turning function (univalent).

By taking $p(z) = \frac{\mathcal{B}_r[\alpha,\beta]}{z}$ in Theorem 1, we have the following result

COROLLARY 3. *Let $f \in \mathcal{A}$. Assume that for $0 < \mu \leq 1$, $0 < \gamma \leq \frac{\mu}{2}$, $b \in \mathbb{R}^+$, the relation*

$$(1-\mu)\frac{\mathcal{B}_r[\alpha, \beta]}{z} + \mu(\frac{z\mathcal{B}_r[\alpha, \beta]'}{\mathcal{B}_r[\alpha, \beta]} - 1) \prec (1-\mu)(1+\gamma z)^b$$

is satisfied then $\frac{\mathcal{B}_r[\alpha, \beta]}{z} \prec (1 + \frac{2\gamma}{\mu}z)^b$.

By assuming $\alpha_1 = q, \alpha = 0, \beta = 0$ in Corollary 3, we have the following result

COROLLARY 4. *Let $f \in \mathcal{A}$. Assume that for $0 < \mu \leq 1$, $0 < \gamma \leq \frac{\mu}{2}$, $b \in \mathbb{R}^+$, the relation*

$$(1-\mu)\frac{f(z)}{z} + \mu(\frac{zf'(z)}{f(z)} - 1) \prec (1-\mu)(1+\gamma z)^b$$

is satisfied then $\frac{f(z)}{z} \prec (1 + \frac{2\gamma}{\mu}z)^b$.

By considering $b = 1, a = \frac{2\gamma}{\mu}$ in Corollary 2 together with Lemma 2, we have the following result

COROLLARY 5. *Let $f \in \mathcal{A}$. If the condition*

$$(1-\mu)f'(z) + \mu\frac{zf''(z)}{f'(z)} \prec (1-\mu)(1+\gamma z)$$

is satisfied then $\frac{zf'(z)}{f(z)} \prec (\frac{1+z}{1-z})^{\delta}$ *and hence* f *is strongly starlike for* $0 < \mu \leq 1$ *in* U.

By considering $b = 1, a = \frac{2\gamma}{\mu}$ in Corollary 2 together with Lemma 3, we have the following result:

COROLLARY 6. *Let* $f \in \mathcal{A}$. *If the condition*

$$(1-\mu)f'(z) + \mu\frac{zf''(z)}{f'(z)} \prec (1-\mu)(1+\gamma z)$$

is satisfied then $\frac{zf'(z)}{f(z)} \prec 1 + (\frac{3a}{2-a})z$ *and thus* f *is starlike for* $0 < \mu \leq 1$ *and* $\gamma \leq \frac{\mu}{4}$ *in* U.

Note that Corollary 1 together with Lemmas 2 and 3, gives the starlikeness and strong starlikeness of the operator $\mathcal{B}_r[\alpha, \beta]$ respectively. Now we consider the class $S^b(\gamma, \mu)$ of analytic functions that satisfy condition (9). Then we obtain the following result

THEOREM 2. *Let* $f \in \mathcal{A}$. *If*

$$\Re\{\frac{zf''(z)}{f'(z)}\} < \frac{2b\gamma}{\mu + 2\gamma}, \quad 0 < \mu \leq 1,\, 0 < \gamma \leq \frac{\mu}{2},\, b \in \mathbb{R}^+ \tag{10}$$

then $f \in S^b(\gamma, \mu)$.

Proof. Our aim is to apply Lemma 1. Let us define $w(z)$ by

$$f'(z) = \Big(1 + \frac{2\gamma}{\mu}w(z)\Big)^b.$$

Then $w(z)$ is analytic in U with $w(0) = 0$. It follows that

$$\Re\{\frac{zf''(z)}{f'(z)}\} = \Re\{\frac{\frac{2b\gamma}{\mu}zw'(z)}{(1 + \frac{2\gamma}{\mu}w(z))}\} < \frac{2b\gamma}{\mu + 2\gamma}.$$

Now, we suppose that there exists a point $z_0 \in U$ such that $\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1$. Then, by Lemma 1, we can write that $w(z_0) = e^{i\theta}$ and $z_0w'(z_0) = kw(z_0) = ke^{i\theta}$. Thus we have that

$$\Re\{\frac{z_0f''(z_0)}{f'(z_0)}\} = \Re\{\frac{\frac{2b\gamma}{\mu}z_0w'(z_0)}{(1 + \frac{2\gamma}{\mu}w(z_0))}\} = \Re\{\frac{\frac{2b\gamma}{\mu}ke^{i\theta}}{1 + \frac{2\gamma}{\mu}e^{i\theta}}\} = \frac{\frac{2b\gamma}{\mu}k\cos\theta}{1 + \frac{2\gamma}{\mu}\cos\theta}$$
$$= \frac{\frac{2b\gamma}{\mu}k}{1 + \frac{2\gamma}{\mu}}, \quad \cos\theta \to 1 \geq \frac{2b\gamma}{\mu + 2\gamma}, \quad k \geq 1,$$

which contradicts (10). Therefore, $f \in S^b(\gamma, \mu)$. This completes the proof of the theorem. ∎

Corollary 7. *If $f \in \mathcal{A}$ satisfies the condition in Theorem 2, then*

$$|(f'(z))^{1/b} - 1| < 1.$$

That is $f(z)$ is strongly close-to-convex of order b.

Proof. Since $f(z) \in S^b(\gamma, \mu)$, there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1\,(z \in U)$ such that $f'(z) = \Big(1 + \dfrac{2\gamma}{\mu} w(z)\Big)^b$. Consequently, we have $w(z) = \dfrac{\mu}{2\gamma}\Big((f'(z))^{1/b} - 1\Big)$; hence

$$\frac{\mu}{2\gamma}\left|\Big(f'(z)\Big)^{1/b} - 1\right| = |w(z)| < 1 \implies \left|\Big(f'(z)\Big)^{1/b} - 1\right| < 1.$$

In the same manner as in Theorem 2 and its corollary, condition (8) gives that the operator $\mathcal{B}_r[\alpha, \beta]$ is close to convex in U. ■

REFERENCES

[**1**] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, 2nd ed., Cambridge University Press, Cambridge, 2004.

[**2**] S.D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc. **135** (1969), 429–446 .

[**3**] S.D. Bernardi, *The radius of umvalence of certain analytic functions*, Proc. Amer. Math. Soc. **24** (1970), 312–318.

[**4**] H.M. Srivastava, A.A. Attiya, *An integral operator associated with the Hurwitz-Lerch zeta function and differential subordination*, Integral Transforms Spec. Funct. **18** (2007), 207–216.

[**5**] D. Răaducanu, H.M. Srivastava, *A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch zeta function*, Integral Transforms Spec. Funct. **18** (2007), 933–943.

[**6**] J.-L. Liu, *Subordinations for certain multivalent analytic functions associated with the generalized Srivastava-Attiya operator*, Integral Transforms Spec. Funct. **19** (2008), 893–901.

[**7**] J.K. Prajapat, S.P. Goyal, *Applications of Srivastava-Attiya operator to the classes of strongly starlike and strongly convex functions*, J. Math. Inequal. **3** (2009), 129–137.

[**8**] N.E. Cho, J.A. Kim, *Inclusion properties of certain subclasses of analytic functions defined by a multiplier transformation*, Comp. Math. Appl. **52** (2006), 323-330.

[**9**] F.M. Al-Oboudi, *On univalent functions defined by a generalized Salagean operator*, Intern. J. Math. Math. Sci. **27** (2004), 1429–1436.

[**10**] S. Sălăgean, *Subclasses of univalent functions*, in: Complex Analysis5th Romanian-Finnish seminar, Part 1 (Bucharest, 1981), vol. 1013 of Lecture Notes in Mathematics, pp. 362–372, Springer, Berlin, Germany, 1983.

[**11**] I.S. Jack, *Functions starlike and convex of order K*, J. London Math. Soc. **3** (1971), 469–474.

[**12**] S. Ponnusamy, V. Singh, *Criteria for strongly starlike functions*, Complex Variables: Theory Appl. **34** (1997), 267–291.

[**13**] S. Ponnusamy, *Polya-Schoenberg conjecture for Caratheodory functions*, J. London Math. Soc. **51** (1995), 93–104.

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Institute of Mathematical Sciences University Malaya, 50603 Kuala Lumpur, Malaysia

E-mail: `rabhaibrahim@yahoo.com`