МАТЕМАТИЧКИ ВЕСНИК
64, 4 (2012), 336–346
December 2012

оригинални научни рад
research paper

OSTROWSKI INEQUALITIES FOR COSINE AND SINE OPERATOR FUNCTIONS

George A. Anastassiou

Abstract. Here we present Ostrowski type inequalities on Cosine and Sine Operator Functions for various norms. At the end we give some applications.

1. Introduction

The main motivation here is the famous Ostrowski inequality from 1938, see [1, 2, 10], which follows:

THEOREM 1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[C + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty, \tag{1}$$

for any $x \in [a,b]$, where C is a positive constant. The constant $C = \frac{1}{4}$ is the best possible.

We generalize here (1) to Cosine and Sine operator functions and we expand it in various directions. At the end we give some applications.

2. Background

For notions and results of this section see [4, 6, 8, 9, 11].

Let $(X, \|\cdot\|)$ be a real or complex Banach space. By definition, a cosine operator function is a family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators from X into itself, satisfying

2010 AMS Subject Classification: 26D99, 47D09, 47D99.

Keywords and phrases: Ostrowski inequality; Cosine operator function; Sine operator function.

(i) $C(0) = I$, I the identity operator;

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$, for all $t, s \in \mathbb{R}$ (the last product is composition);

(iii) $C(\cdot) f$ is continuous on $\mathbb{R}$, for all $f \in X$.

Notice that $C(t) = C(-t)$, for all $t \in \mathbb{R}$.

The associated sine operator function $S(\cdot)$ is defined by

$$S(t) f := \int_0^t C(s) f \, ds, \text{ for all } t \in \mathbb{R} \text{ and for all } f \in X.$$

Notice that $S(t) f \in X$ and it is continuous in $t \in \mathbb{R}$.

The cosine operator function $C(\cdot)$ is such that $\|C(t)\| \leq Me^{\omega t}$, for some $M \geq 1$, $\omega \geq 0$, for all $t \in \mathbb{R}_+$; here $\|\cdot\|$ is the norm of the operator.

The infinitesimal generator A of $C(\cdot)$ is the operator from X into itself defined as

$$Af := \lim_{t \to 0+} \frac{2}{t^2} (C(t) - I) f$$

with domain $D(A)$. The operator A is closed and $D(A)$ is dense in X, i.e., $\overline{D(A)} = X$, and satisfies

$$\int_0^t S(s) f \, ds \in D(A) \text{ and } A \int_0^t S(s) f ds = C(t) f - f, \text{ for all } f \in X.$$

Also, $A = C''(0)$ holds, and $D(A)$ is the set of $f \in X$ such that $C(t) f$ is twice differentiable at $t = 0$; equivalently,

$$D(A) = \left\{ f \in X : C(\cdot) f \in C^2(\mathbb{R}, X) \right\}.$$

If $f \in D(A)$, then $C(t) f \in D(A)$, and $C''(t) f = C(t) Af = AC(t) f$, for all $t \in \mathbb{R}$; $C'(0) f = 0$, see [5, 12].

We define $A^0 = I$, $A^2 = A \circ A$, ... , $A^n = A \circ A^{n-1}$, $n \in \mathbb{N}$. Let $f \in D(A^n)$; then $C(t) f \in C^{2n}(\mathbb{R}, X)$, and $C^{(2n)}(t) f = C(t) A^n f = A^n C(t) f$, for all $t \in \mathbb{R}$, and $C^{(2k-1)}(0) f = 0$, $1 \leq k \leq n$, see [8].

For $f \in D(A^n)$, $t \in \mathbb{R}$, we have the cosine operator function's Taylor formula [8, 9] saying that

$$T_n(t) f := C(t) f - \sum_{k=0}^{n-1} \frac{t^{2k}}{(2k)!} A^k f = \int_0^t \frac{(t-s)^{2n-1}}{(2n-1)!} C(s) A^n f \, ds. \tag{2}$$

By integrating (2) we obtain the sine operator function's Taylor formula

$$\begin{aligned} M_n(t) f :&= S(t) f - ft - \frac{t^3}{3!} Af - \cdots - \frac{t^{2n-1}}{(2n-1)!} A^{n-1} f \\ &= \int_0^t \frac{(t-s)^{2n}}{(2n)!} C(s) A^n f ds, \quad \text{for all } t \in \mathbb{R}, \end{aligned} \tag{3}$$

and all $f \in D(A^n)$.

Integrals in (2) and (3) are vector valued Riemann integrals, see [3, 7]. Here $f \in D(A^n)$, $n \in \mathbb{N}$.

Let $a > 0$ and $F \in C([0,a], X)$; then F is vector-Riemann integrable, see [11]. Clearly here $\int_0^a F(t)\,dt \in X$.

3. Ostrowski type inequalities

We first present results on the natural interval; here $[0,a]$, $a > 0$.

THEOREM 2. *Denote*

$$\|\|C(\cdot) A^n f\|\|_{\infty,[0,a]} := \sup_{t\in[0,a]} \|C(t) A^n f\|.$$

Here $t_0 \in [0,a]$. *Then*

$$(i) \quad \left\| \frac{1}{a} \int_0^a T_n(t) f\,dt - T_n(t_0) f \right\| \leq \frac{\|\|C(\cdot) A^n f\|\|_{\infty,[0,a]}}{a(2n)!} \cdot \left[\left(\frac{4nt_0^{2n+1} + a^{2n+1}}{2n+1} \right) - at_0^{2n} \right], \quad (4)$$

$$(ii) \quad \left\| \frac{1}{a} \int_0^a M_n(t) f\,dt - M_n(t_0) f \right\| \leq \frac{\|\|C(\cdot) A^n f\|\|_{\infty,[0,a]}}{a(2n+1)!} \cdot \left[\left(\frac{2(2n+1)t_0^{2(n+1)} + a^{2(n+1)}}{2(n+1)} \right) - at_0^{2n+1} \right]. \quad (5)$$

Proof. By (2) we have $(m := 2n-1)$

$$T_n(t) f = \int_0^t \frac{(t-s)^m}{m!} C(s) A^n f\,ds,$$

and

$$T_n(t_0) f = \int_0^{t_0} \frac{(t_0-s)^m}{m!} C(s) A^n f\,ds,$$

for any $t, t_0 \in [0,a]$. We estimate $E_n(t) f := T_n(t) f - T_n(t_0) f$.

Case of $t \geq t_0$: we have

$$\|E_n(t) f\| = \left\| \int_0^{t_0} \frac{(t-s)^m}{m!} C(s) A^n f\,ds + \int_{t_0}^t \frac{(t-s)^m}{m!} C(s) A^n f\,ds - \int_0^{t_0} \frac{(t_0-s)^m}{m!} C(s) A^n f\,ds \right\|$$

$$= \left\| \frac{1}{m!} \int_0^{t_0} ((t-s)^m - (t_0-s)^m) C(s) A^n f\,ds + \frac{1}{m!} \int_{t_0}^t (t-s)^m C(s) A^n f\,ds \right\|$$

$$\leq \frac{1}{m!} \left[\int_0^{t_0} ((t-s)^m - (t_0-s)^m) \|C(s) A^n f\|\,ds + \int_{t_0}^t (t-s)^m \|C(s) A^n f\|\,ds \right]$$

$$\leq \frac{\|\|C(\cdot)A^n f\|\|_{\infty,[0,a]}}{m!}\left[\int_0^{t_0}\left((t-s)^m-(t_0-s)^m\right)ds+\int_{t_0}^{t}(t-s)^m\,ds\right]$$
$$=\frac{\|\|C(\cdot)A^n f\|\|_{\infty,[0,a]}}{(m+1)!}\left[t^{m+1}-t_0^{m+1}\right].$$

That is

$$\|E_n(t)f\|\leq \frac{\|\|C(\cdot)A^n f\|\|_{\infty,[0,a]}}{(m+1)!}\left[t^{m+1}-t_0^{m+1}\right],$$

for $t\geq t_0$, $t,t_0\in[0,a]$.

Case of $t<t_0$: we similarly find that

$$\|E_n(t)f\|=\left\|\int_0^t\frac{(t-s)^m}{m!}C(s)A^n f\,ds-\int_0^{t_0}\frac{(t_0-s)^m}{m!}C(s)A^n f\,ds\right\|$$
$$\leq \frac{\|\|C(\cdot)A^n f\|\|_{\infty,[0,a]}}{(m+1)!}\left[t_0^{m+1}-t^{m+1}\right],$$

for $t<t_0$, $t,t_0\in[0,a]$. Therefore we get

$$\|E_n(t)f\|\leq \frac{\|\|C(\cdot)A^n f\|\|_{\infty,[0,a]}}{(m+1)!}\left|t^{m+1}-t_0^{m+1}\right|,$$

for all $t,t_0\in[0,a]$.

Next we observe

$$\left\|\frac{1}{a}\int_0^a T_n(t)f\,dt-T_n(t_0)f\right\|=\frac{1}{a}\left\|\int_0^a\left(T_n(t)f-T_n(t_0)f\right)dt\right\|$$
$$\leq \frac{1}{a}\int_0^a\|T_n(t)f-T_n(t_0)f\|\,dt=\frac{1}{a}\int_0^a\|E_n(t)f\|\,dt$$
$$\leq \frac{\|\|C(\cdot)A^n f\|\|_{\infty,[0,a]}}{(m+1)!a}\int_0^a\left|t^{m+1}-t_0^{m+1}\right|dt$$
$$=\frac{\|\|C(\cdot)A^n f\|\|_{\infty,[0,a]}}{(m+1)!a}\left[\left(\frac{2(m+1)t_0^{m+2}+a^{m+2}}{(m+2)}\right)-at_0^{m+1}\right].$$

So that

$$\left\|\frac{1}{a}\int_0^a T_n(t)f\,dt-T_n(t_0)f\right\|$$
$$\leq \frac{\|\|C(\cdot)A^n f\|\|_{\infty,[0,a]}}{(m+1)!a}\left[\left(\frac{2(m+1)t_0^{m+2}+a^{m+2}}{(m+2)}\right)-at_0^{m+1}\right]$$
$$=\frac{\|\|C(\cdot)A^n f\|\|_{\infty,[0,a]}}{a(2n)!}\left[\left(\frac{4nt_0^{2n+1}+a^{2n+1}}{2n+1}\right)-at_0^{2n}\right],$$

proving (4).

Letting $m:=2n$, we similarly obtain

$$\left\|\frac{1}{a}\int_0^a M_n(t)f\,dt-M_n(t_0)f\right\|$$

$$\leq \frac{\||C(\cdot)A^n f\||_{\infty,[0,a]}}{(m+1)!a}\left[\left(\frac{2(m+1)t_0^{m+2}+a^{m+2}}{(m+2)}\right)-at_0^{m+1}\right]$$
$$=\frac{\||C(\cdot)A^n f\||_{\infty,[0,a]}}{(2n+1)!a}\left[\left(\frac{2(2n+1)t_0^{2(n+1)}+a^{2(n+1)}}{2(n+1)}\right)-at_0^{2n+1}\right],$$

proving (5). ∎

When $n=1$ we get

Proposition 3. *Let* $f\in D(A)$, $t_0\in[0,a]$, $a>0$. *We have that*

$$(i)\quad \left\|\frac{1}{a}\int_0^a C(t)f\,dt-C(t_0)f\right\|\leq\frac{\||C(\cdot)Af\||_{\infty,[0,a]}}{2a}\left[\left(\frac{4t_0^3+a^3}{3}\right)-at_0^2\right],$$

and

$$(ii)\quad \left\|\frac{1}{a}\int_0^a S(t)f\,dt-S(t_0)f+f\left(t_0-\frac{a}{2}\right)\right\|$$
$$\leq\frac{\||C(\cdot)Af\||_{\infty,[0,a]}}{6a}\cdot\left[\left(\frac{6t_0^4+a^4}{4}\right)-at_0^3\right].$$

Next we call

$$T_n^*(t)f:=C(t)(f)-\sum_{k=1}^{n-1}\frac{t^{2k}}{(2k)!}A^kf,$$

$t\in\mathbb{R}$. Notice that $T_n^*(0)f=f$. Furthermore we have

$$T_n^*(t)f-T_n^*(0)f=T_n^*(t)f-f=T_n(t)f,$$

for all $t\in\mathbb{R}$. Now we prove

Theorem 4. *Let* $p,q>1$ *be such that* $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$(i)\quad \left\|\frac{1}{a}\int_0^a T_n^*(t)f\,dt-f\right\|\leq\frac{\||C(\cdot)A^nf\||_{q,[0,a]}\,a^{2n-\frac{1}{q}}}{(2n-1)!\,(p(2n-1)+1)^{\frac{1}{p}}\left(2n+\frac{1}{p}\right)},$$

$$(ii)\quad \left\|\int_0^a M_n(t)f\,dt\right\|\leq\frac{\||C(\cdot)A^nf\||_{q,[0,a]}\,a^{2n+\frac{1}{p}+1}}{(2n)!\,(2pn+1)^{\frac{1}{p}}\left(2n+\frac{1}{p}+1\right)}.$$

Proof. (i) Set $m=2n-1$, then

$$T_n(t)f=\int_0^t\frac{(t-s)^m}{m!}C(s)A^nf\,ds.$$

By Hölder's inequality we obtain

$$\|T_n(t)f\| = \left\|\int_0^t \frac{(t-s)^m}{m!} C(s) A^n f\, ds\right\|$$
$$\leq \frac{1}{m!}\int_0^t (t-s)^m \|C(s) A^n f\|\, ds$$
$$\leq \frac{1}{m!}\left(\int_0^t (t-s)^{pm}\, ds\right)^{\frac{1}{p}}\left(\int_0^t \|C(s) A^n f\|^q\, ds\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{m!}\left(\frac{t^{m+\frac{1}{p}}}{(pm+1)^{\frac{1}{p}}}\right)\left(\int_0^a \|C(s) A^n f\|^q\, ds\right)^{\frac{1}{q}}$$
$$= \frac{1}{m!}\left(\frac{t^{m+\frac{1}{p}}}{(pm+1)^{\frac{1}{p}}}\right)\|\|C(s) A^n f\|\|_{q,[0,a]},$$

for all $t \in [0,a]$. That is we have

$$\|T_n(t)f\| \leq \frac{1}{m!}\left(\frac{t^{m+\frac{1}{p}}}{(pm+1)^{\frac{1}{p}}}\right)\|\|C(s) A^n f\|\|_{q,[0,a]},$$

for all $t \in [0,a]$. Therefore

$$\left\|\frac{1}{a}\int_0^a T_n^*(t) f\, dt - T_n^*(0) f\right\| = \left\|\frac{1}{a}\int_0^a (T_n^*(t) f - T_n^*(0) f)\, dt\right\|$$
$$= \left\|\frac{1}{a}\int_0^a T_n(t) f\, dt\right\| \leq \frac{1}{a}\int_0^a \|T_n(t) f\|\, dt$$
$$\leq \left(\frac{\|\|C(s) A^n f\|\|_{q,[0,a]}}{am!\,(pm+1)^{\frac{1}{p}}}\right)\int_0^a t^{m+\frac{1}{p}} dt = \frac{\|\|C(s) A^n f\|\|_{q,[0,a]}\, a^{m+\frac{1}{p}}}{m!\,(pm+1)^{\frac{1}{p}}\left(m+\frac{1}{p}+1\right)}.$$

We have proved that

$$\left\|\frac{1}{a}\int_0^a T_n^*(t) f\, dt - T_n^*(0) f\right\| \leq \frac{\|\|C(s) A^n f\|\|_{q,[0,a]}\, a^{m+\frac{1}{p}}}{m!\,(pm+1)^{\frac{1}{p}}\left(m+\frac{1}{p}+1\right)}$$
$$= \frac{\|\|C(s) A^n f\|\|_{q,[0,a]}\, a^{2n-\frac{1}{q}}}{(2n-1)!\,(p(2n-1)+1)^{\frac{1}{p}}\left(2n+\frac{1}{p}\right)},$$

establishing the claim.

(ii) Set $m = 2n$, then similarly we get

$$\left\|\frac{1}{a}\int_0^a M_n(t) f\, dt\right\| \leq \frac{\|\|C(s) A^n f\|\|_{q,[0,a]}\, a^{m+\frac{1}{p}}}{m!\,(pm+1)^{\frac{1}{p}}\left(m+\frac{1}{p}+1\right)}$$
$$= \frac{\|\|C(s) A^n f\|\|_{q,[0,a]}\, a^{2n+\frac{1}{p}}}{(2n)!\,(2pn+1)^{\frac{1}{p}}\left(2n+\frac{1}{p}+1\right)},$$

proving the claim. ■

Theorem 5.

$$(i)\quad \left\| \frac{1}{a}\int_0^a T_n^*(t)\, f\, dt - f \right\| \le \frac{a^{2n-1}}{(2n-1)!} \left\| \left\| C(\cdot)\, A^n f \right\| \right\|_{1,[0,a]}.$$

$$(ii)\quad \left\| \frac{1}{a}\int_0^a M_n(t)\, f\, dt \right\| \le \frac{a^{2n+1}}{(2n)!} \left\| \left\| C(\cdot)\, A^n f \right\| \right\|_{1,[0,a]}.$$

Proof. Set $m = 2n-1$. We notice that

$$\|T_n(t)\, f\| = \left\| \int_0^t \frac{(t-s)^m}{m!} C(s)\, A^n f\, ds \right\|$$
$$\le \frac{1}{m!}\int_0^t (t-s)^m \|C(s)\, A^n f\|\, ds \le \frac{a^m}{m!}\int_0^t \|C(s)\, A^n f\|\, ds$$
$$= \frac{a^m}{m!}\int_0^a \|C(s)\, A^n f\|\, ds \le \frac{a^m}{m!} \left\| \left\| C(s)\, A^n f \right\| \right\|_{1,[0,a]},$$

for all $t \in [0,a]$, i.e.

$$\|T_n(t)\, f\| \le \frac{a^m}{m!} \left\| \left\| C(s)\, A^n f \right\| \right\|_{1,[0,a]},$$

for all $t \in [0,a]$. Therefore

$$\left\| \frac{1}{a}\int_0^a T_n(t)\, f\, dt \right\| \le \frac{1}{a}\int_0^a \|T_n(t)\, f\|\, dt \le \frac{a^m}{m!} \left\| \left\| C(s)\, A^n f \right\| \right\|_{1,[0,a]}$$
$$= \frac{a^{2n-1}}{(2n-1)!} \left\| \left\| C(s)\, A^n f \right\| \right\|_{1,[0,a]},$$

proving the claim.

(ii) Set $m = 2n$. Then similarly we get

$$\left\| \frac{1}{a}\int_0^a M_n(t)\, f\, dt \right\| \le \frac{a^{2n}}{(2n)!} \left\| \left\| C(s)\, A^n f \right\| \right\|_{1,[0,a]},$$

proving the claim. ■

In the case that $n = p = q = 2$, we get

Corollary 6 (to Theorem 4).

$$(i)\quad \left\| \frac{1}{a}\int_0^a \left(C(t)\, f - \frac{t^2}{2} A f \right) dt - f \right\| \le \frac{\left\| \left\| C(\cdot)\, A^2 f \right\| \right\|_{2,[0,a]} a^{3.5}}{27\sqrt{7}}.$$

$$(ii)\quad \left\| \int_0^a \left(S(t)\, f - ft - \frac{t^3}{6} A f \right) dt \right\| \le \frac{\left\| \left\| C(\cdot)\, A^2 f \right\| \right\|_{2,[0,a]} a^{5.5}}{396}.$$

Next, we prove Ostrowski type inequality on the general interval $[a,b]$, $0 \le a < b$.

THEOREM 7. *Let* $t_0 \in [a, b]$.

$$(i) \quad \left\| \frac{1}{b-a} \int_a^b T_n(t) f \, dt - T_n(t_0) f \right\| \leq \frac{\||C(\cdot) A^n f\||_{\infty,[0,b]}}{(2n)!\,(b-a)} \times \left[\frac{4n t_0^{2n+1}}{2n+1} - t_0^{2n}(a+b) + \left(\frac{a^{2n+1} + b^{2n+1}}{2n+1} \right) \right].$$

$$(ii) \quad \left\| \frac{1}{b-a} \int_a^b M_n(t) f \, dt - M_n(t_0) f \right\| \leq \frac{\||C(\cdot) A^n f\||_{\infty,[0,b]}}{(2n+1)!\,(b-a)} \times \left[\left(\frac{2(2n+1) t_0^{2(n+1)} + a^{2(n+1)} + b^{2(n+1)}}{2(n+1)} \right) - (a+b) t_0^{2n+1} \right].$$

Proof. i) Set $m := 2n-1$, and $E_n(t) f := T_n(t) f - T_n(t_0) f$, where $t, t_0 \in [a, b]$, with $a \geq 0$. As in the proof of Theorem 2 we obtain

$$\|E_n(t) f\| \leq \frac{\||C(\cdot) A^n f\||_{\infty,[0,b]}}{(m+1)!} \left| t^{m+1} - t_0^{m+1} \right|, \quad \text{for all } t, t_0 \in [a, b].$$

Next we observe

$$\begin{aligned}
\left\| \frac{1}{b-a} \int_a^b T_n(t) f \, dt - T_n(t_0) f \right\| &= \frac{1}{b-a} \left\| \int_a^b (T_n(t) f - T_n(t_0) f) \, dt \right\| \\
&\leq \frac{1}{b-a} \int_a^b \|T_n(t) f - T_n(t_0) f\| \, dt = \frac{1}{b-a} \int_a^b \|E_n(t) f\| \, dt \\
&\leq \frac{\||C(\cdot) A^n f\||_{\infty,[0,b]}}{(m+1)!\,(b-a)} \int_a^b \left| t^{m+1} - t_0^{m+1} \right| dt \\
&= \frac{\||C(\cdot) A^n f\||_{\infty,[0,b]}}{(m+1)!\,(b-a)} \left[\int_a^{t_0} \left(t_0^{m+1} - t^{m+1} \right) dt + \int_{t_0}^b \left(t^{m+1} - t_0^{m+1} \right) dt \right] \\
&= \frac{\||C(\cdot) A^n f\||_{\infty,[0,b]}}{(m+1)!\,(b-a)} \left[t_0^{m+1} [2t_0 - a - b] + \left(\frac{a^{m+2} + b^{m+2} - 2t_0^{m+2}}{m+2} \right) \right] \\
&= \frac{\||C(\cdot) A^n f\||_{\infty,[0,b]}}{(2n)!\,(b-a)} \left[t_0^{2n} [2t_0 - a - b] + \left(\frac{a^{2n+1} + b^{2n+1} - 2t_0^{2n+1}}{2n+1} \right) \right] \\
&= \frac{\||C(\cdot) A^n f\||_{\infty,[0,b]}}{(2n)!\,(b-a)} \left[\frac{4n t_0^{2n+1}}{2n+1} - t_0^{2n}(a+b) + \left(\frac{a^{2n+1} + b^{2n+1}}{2n+1} \right) \right],
\end{aligned}$$

proving the claim.

(ii) Set $m = 2n$. Then similarly we find

$$\left\| \frac{1}{b-a} \int_a^b M_n(t) f\, dt - M_n(t_0) f \right\|$$

$$\leq \frac{\| \|C(\cdot) A^n f\| \|_{\infty,[0,b]}}{(m+1)!\,(b-a)} \left[t_0^{m+1} [2t_0 - a - b] + \left(\frac{a^{m+2} + b^{m+2} - 2t_0^{m+2}}{m+2} \right) \right]$$

$$= \frac{\| \|C(\cdot) A^n f\| \|_{\infty,[0,b]}}{(2n+1)!\,(b-a)} \left[\left(\frac{2(2n+1)\, t_0^{2(n+1)} + a^{2(n+1)} + b^{2(n+1)}}{2(n+1)} \right) - (a+b)\, t_0^{2n+1} \right],$$

proving the claim. ∎

When $n = 1$ we get

Proposition 8. *Let $f \in D(A)$, $0 \leq a < b$, $t_0 \in [0, a]$. Then*

(*i*)
$$\left\| \frac{1}{b-a} \int_a^b C(t) f\, dt - C(t_0) f \right\| \leq \frac{\| \|C(\cdot) A f\| \|_{\infty,[0,b]}}{2(b-a)} \cdot \left[\frac{4t_0^3}{3} - t_0^2 (a+b) + \left(\frac{a^3 + b^3}{3} \right) \right],$$

(*ii*)
$$\left\| \frac{1}{b-a} \int_a^b S(t) f\, dt - S(t_0) f + f \left(t_0 - \frac{a+b}{2} \right) \right\| \leq \frac{\| \|C(\cdot) A f\| \|_{\infty,[0,b]}}{6(b-a)} \cdot \left[\left(\frac{6t_0^4 + a^4 + b^4}{4} \right) - (a+b)\, t_0^3 \right].$$

4. Applications

Let X be the Banach space of odd, 2π-periodic real functions in the space of bounded uniformly continuous functions from $\mathbb{R}$ into itself: $BUC(\mathbb{R})$. Let $A := \frac{d^2}{dx^2}$ with $D(A^n) = \{f \in X : f^{(2k)} \in X,\ k = 1, \ldots, n\}$, $n \in \mathbb{N}$. A generates a Cosine function C^* given by (see [6], p. 121)

$$[C^*(t) f](x) = \frac{1}{2} [f(x+t) + f(x-t)], \quad \text{for all } \ x, t \in \mathbb{R}.$$

The corresponding Sine function S^* is given by

$$[S^*(t) f](x) = \frac{1}{2} \left[\int_0^t f(x+s)\, ds + \int_0^t f(x-s)\, ds \right], \quad \text{for all } \ x, t \in \mathbb{R}.$$

Here we consider $f \in D(A^n)$, $n \in \mathbb{N}$, as above. By (2) we obtain

$$\overline{T_n}(t) f := \frac{1}{2} [f(\cdot + t) + f(\cdot - t)] - \sum_{k=0}^{n-1} \frac{t^{2k}}{(2k)!} f^{(2k)}$$

$$= \int_0^t \frac{(t-s)^{2n-1}}{2(2n-1)!} \left[f^{(2n)}(\cdot + s) + f^{(2n)}(\cdot - s) \right] ds, \quad \text{for all } t \in \mathbb{R}.$$

By (3) we get

$$\overline{M_n}(t) f := \frac{1}{2}\left[\int_0^t f(\cdot + s)\, ds + \int_0^t f(\cdot - s)\, ds\right] - \sum_{k=1}^{n} \frac{t^{2k-1}}{(2k-1)!} f^{(2(k-1))}$$
$$= \int_0^t \frac{(t-s)^{2n}}{2\,(2n)!}\left[f^{(2n)}(\cdot + s) + f^{(2n)}(\cdot - s)\right] ds, \quad \text{for all } t \in \mathbb{R}.$$

Let $g \in BUC(\mathbb{R})$, we define $\|g\| = \|g\|_\infty := \sup_{x\in\mathbb{R}} |g(x)| < \infty$.

Notice also that

$$\|\|C^*(s)A^n f\|_\infty\|_\infty = \|\|C^*(s) f^{(2n)}\|_\infty\|_\infty$$
$$= \frac{1}{2}\|\|(f^{(2n)}(\cdot + s) + f^{(2n)}(\cdot - s))\|_\infty\|_\infty$$
$$\leq \frac{1}{2}[\|\|f^{(2n)}(\cdot + s)\|_\infty\|_\infty + \|\|f^{(2n)}(\cdot - s)\|_\infty\|_\infty] \leq \|f^{(2n)}\|_\infty < \infty.$$

We have the following applications

Corollary 9 (to Theorem 7). *Let $t_0 \in [a,b]$. Then*

$$(i) \quad \left\|\frac{1}{b-a}\int_a^b \overline{T_n}(t) f\, dt - \overline{T_n}(t_0) f\right\|_\infty$$
$$\leq \frac{\|f^{(2n)}\|_\infty}{(2n)!\,(b-a)} \cdot \left[\frac{4nt_0^{2n+1}}{2n+1} - t_0^{2n}(a+b) + \left(\frac{a^{2n+1} + b^{2n+1}}{2n+1}\right)\right],$$

$$(ii) \quad \left\|\frac{1}{b-a}\int_a^b \overline{M_n}(t) f\, dt - \overline{M_n}(t_0) f\right\|_\infty$$
$$\leq \frac{\|f^{(2n)}\|_\infty}{(2n+1)!\,(b-a)} \cdot \left[\left(\frac{2\,(2n+1)\, t_0^{2(n+1)} + a^{2(n+1)} + b^{2(n+1)}}{2\,(n+1)}\right) - (a+b)\, t_0^{2n+1}\right].$$

Corollary 10 (to Proposition 8). *Let $f \in X : f^{(2)} \in X$, $0 \leq a < b$, $t_0 \in [a,b]$. Then*

$$(i) \quad \left\|\frac{1}{b-a}\int_a^b [f(\cdot + t) + f(\cdot - t)]\, dt - [f(\cdot + t_0) + f(\cdot - t_0)]\right\|_\infty$$
$$\leq \frac{\|f^{(2)}\|_\infty}{(b-a)}\left[\frac{4t_0^3}{3} - t_0^2(a+b) + \left(\frac{a^3+b^3}{3}\right)\right],$$

$$(ii) \quad \left\|\frac{1}{b-a}\int_a^b S^*(t) f\, dt - S^*(t_0) f + f(\cdot) \cdot \left(t_0 - \frac{a+b}{2}\right)\right\|_\infty$$
$$\leq \frac{\|f^{(2)}\|_\infty}{6\,(b-a)}\left[\left(\frac{6t_0^4 + a^4 + b^4}{4}\right) - (a+b)\, t_0^3\right].$$

We finish with

COROLLARY 11 (to Corollary 6).

(i)
$$\left\|\frac{1}{2a}\int_0^a\left[f(\cdot+t)+f(\cdot-t)-t^2f^{(2)}(\cdot)\right]dt-f(\cdot)\right\|_\infty \le \frac{\left\|\left\|f^{(4)}(\cdot+t)+f^{(4)}(\cdot-t)\right\|_\infty\right\|_{2,[0,a]}a^{3.5}}{54\sqrt{7}},$$

(ii)
$$\left\|\int_0^a\left(\left[S^*(t)f\right](x)-f(x)t-\frac{t^3}{6}f^{(2)}(x)\right)dt\right\|_{\infty,x} \le \frac{\left\|\left\|\left[C^*(t)f^{(4)}\right](x)\right\|_{\infty,x}\right\|_{2,[0,a],t}a^{5.5}}{396}.$$

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(received 10.06.2011; in revised form 28.09.2011; available online 01.11.2011)

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A.

E-mail: ganastss@memphis.edu