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A NOTE ON SEQUENCE-COVERING π-IMAGES OF METRIC SPACES

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Abstract. In this paper, we prove that a space is a sequence-covering π-image of a metric space if and only if it has a σ-strong network consisting of cs-covers (or sn-covers) if and only if it is a Cauchy sn-symmetric space.

1. Introduction and definitions

To find internal characterizations of certain images of metric spaces is one of the central problems in general topology. Some characterizations of quotient π-images (open π-images, pseudo-open π-images, sequentially-quotient π-images, weak-open π-images) of metric spaces are obtained in [2–5, 7, 13, 15].

The purpose of this paper is to investigate sequence-covering π-images of metric spaces. We prove that a space is a sequence-covering π-image of a metric space if and only if it is has a σ-strong network consisting of cs-covers (or sn-covers) if and only if it is a Cauchy sn-symmetric space.

Throughout this paper all spaces are Hausdorff, and all mappings are continuous and surjective. N denotes the set of all natural numbers. $\tau(X)$ denotes a topology on X. For a collection $\mathcal{P}$ of subsets of a space X and a mapping $f : X \to Y$, we denote $\{f(P) : P \in \mathcal{P}\}$ by $f(\mathcal{P})$, $\mathcal{P}_x = \{P \in \mathcal{P} : x \in P\}$ and $st(x, \mathcal{P}) = \bigcup \mathcal{P}_x$. For the usual product space $\prod_{i\in N} X_i$, π_i denotes the projective $\prod_{i\in N} X_i$ onto X_i. For a sequence $\{x_n\}$ in a space X, we denote $\langle x_n\rangle = \{x_n : n \in N\}$.

Definition 1.1. [9] Let $f : X \to Y$ be a mapping. f is is called a sequence-covering mapping, if whenever $\{y_n\}$ is a convergent sequence in Y, then there exists a convergent sequence $\{x_n\}$ in X such that each $x_n \in f^{-1}(y_n)$.

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Definition 1.2. [11] Let X be a space, and $P \subset X$. Then,

(1) A sequence $\{x_n\}$ in X is called eventually in P, if $\{x_n\}$ converges to x, and there exists $m \in N$ such that $\{x\} \cup \{x_n : n \geq m\} \subset P$.

(2) P is called a sequential neighborhood of x in X, if $x \in P$, and whenever a sequence $\{x_n\}$ in X converges to x , then $\{x_n\}$ is eventually in P.

(3) P is called sequential open in X, if P is a sequential neighborhood of each of its points.

(4) X is called a sequential space, if any sequential open subset of X is open in X.

Definition 1.3. [12] Let $\mathcal{P}$ be a collection of subsets of a space X and $x \in X$.

(1) $\mathcal{P}$ is called a network of x in X, if $x \in \bigcap \mathcal{P}$ and for each neighborhood U of x, there exists $P \in \mathcal{P}$ such that $P \subset U$.

(2) $\mathcal{P}$ is called a *sn*-network of x in X, if $\mathcal{P}$ is a network of x in X and each element of $\mathcal{P}$ is also a sequential neighborhood of x.

(3) $\mathcal{P}$ is called a *cs*-cover for X, if $\mathcal{P}$ is a cover for X, and every convergent sequence in X is eventually in some element of $\mathcal{P}$.

(4) $\mathcal{P}$ is called an *sn*-cover for X, if $\mathcal{P}$ is a cover for X, every element of $\mathcal{P}$ is a sequential neighborhood of some point in X, and for each $x \in X$ there exists a sequential neighborhood P of x in X such that $P \in \mathcal{P}$.

Definition 1.4. [2] Let $\{\mathcal{P}_n\}$ be a sequence of covers of a space X.

(1) $\bigcup\{\mathcal{P}_n : n \in N\}$ is called a σ-strong network for X, if $\langle st(x, \mathcal{P}_n)\rangle$ is a network of x in X for each $x \in X$.

(2) $\bigcup\{\mathcal{P}_n : n \in N\}$ is called a σ-strong network consisting of p-covers, if $\bigcup\{\mathcal{P}_n : n \in N\}$ is a σ-strong network for X and each $\mathcal{P}_n$ satisfies property p.

Definition 1.5. Let X be a set. A non-negative real valued function d defined on $X \times X$ is called a d-function on X, if $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for any $x \in X$.

Let d be a d-function on a space X. In this paper we write $B(x, 1/n) = \{y \in X : d(x, y) < 1/n\}$ and $d(A) = \sup\{d(x, y) : x, y \in A\}$, where $x \in X$, $n \in N$ and $A \subset X$.

Definition 1.6. [8] Let d be a d-function on a space X. (X, d) is called an *sn*-symmetric space, if d satisfies the condition: $\{B(x, 1/n) : n \in N\}$ is an *sn*-network of x in X for any $x \in X$, where d is called an *sn*-symmetric on X.

Definition 1.7. [6] Let (X, d) be a metric space and let $f : X \to Y$ be a mapping. f is called a π-mapping with respect to d, if for each $y \in Y$ and each open neighborhood V in Y, $d(f^{-1}(y), X \setminus f^{-1}(V)) > 0$.

Definition 1.8. [1] Let (X, d) be an *sn*-symmetric space, Then,

(1) a sequence $\{x_n\}$ in X is called d-Cauchy, if for each $\varepsilon > 0$, there exists $k \in N$ such that $d(x_m, x_n) < \varepsilon$ for all $n, m > k$.

(2) X is called a Cauchy sn-symmetric space, if each convergent sequence in X is d-Cauchy.

2. Main results

Lemma 2.1. [13] *Let (X, d) be an sn-symmetric space, $n \in N$ and $x \in X$. Put $\mathcal{P}_n = \{A \subset X : d(A) < 1/n\}$, then $st(x, \mathcal{P}_n) = B(x, 1/n)$.*

Theorem 2.2. *The following are equivalent for a space X:*

(1) X is a sequence-covering π-image of a metric space;

(2) X has a σ-strong network consisting of cs-covers;

(3) X has a σ-strong network consisting of sn-covers;

(4) X is a Cauchy sn-symmetric space.

Proof: (1)⇔(2)⇔(3) hold by Theorem 3.1.7 in [12]. We only need to prove (2)⇔(4).

(2)⇒(4). Suppose $\bigcup\{\mathcal{P}_n : n \in N\}$ is a σ-strong network consisting of cs-covers for X. We can assume that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \in N$.

For each $x, y \in X$, denote

$$t(x, y) = \min\{n : x \notin st(y, \mathcal{P}_n)\} \quad (x \neq y).$$

We define $d(x,y) = \begin{cases} 0, x = y \\ 2^{-t(x,y)}, x \neq y; \end{cases}$ then d is a d-function on X.

Claim. For each $x, y \in X, x \in st(y, \mathcal{P}_n)$ if and only if $t(x, y) > n$.

In fact, the 'if' part is obvious. The only if part: Suppose $x \in st(y, \mathcal{P}_n)$ but $t(x, y) \leq n$, since $\mathcal{P}_n$ refine $\mathcal{P}_{t(x,y)}$, then $st(y, \mathcal{P}_n) \subset st(y, \mathcal{P}_{t(x,y)})$. Note that $x \notin st(y, \mathcal{P}_{t(x,y)})$, so $x \notin st(y, \mathcal{P}_n)$, a contradiction.

For each $x \in X$ and $n \in N, st(x, \mathcal{P}_n) = B(x, 1/2^n)$ by the Claim.

Because $\bigcup\{\mathcal{P}_n : n \in N\}$ is a σ-strong network for X, then (X, d) is a sn-symmetric space.

For each sequence $\{x_n\}$ in X converging to $x \in X$ and $\varepsilon > 0$, there exists $k \in N$ such that $1/2^k < \varepsilon$. Since $\mathcal{P}_k$ is a cs-cover for X, then there exist $P \in \mathcal{P}_k$ and $l \in N$ such that $\{x\} \cup \{x_n : n \geq l\} \subset P$. If $n, m \geq l$, then $x_n, x_m \in P$, so $x_n \in st(x_m, \mathcal{P}_k)$. Thus $t(x_n, x_m) > k$ by the Claim.

Hence

$$d(x_n, x_m) = 1/2^{t(x_n,x_m)} < 1/2^k < \varepsilon \;\; if \; n, m \geq l.$$

Therefore $\{x_n\}$ is d-Cauchy. This implies that X is a Cauchy sn-symmetric space.

(4)⇒(2). Suppose X is a Cauchy sn-symmetric space. For each $n \in N$, put

$$\mathcal{P}_n = \{A \subset X : d(A) < 1/n\}$$

By Lemma 2.1, $st(x, \mathcal{P}_n) = B(x, 1/n)$ for each $x \in X$, so $\langle st(x, \mathcal{P}_n)\rangle$ is a network of x in X for each $x \in X$. Thus $\bigcup\{\mathcal{P}_n : n \in N\}$ is a σ-strong network for X.

For each $n \in N$ and each sequence $\{x_i\}$ converging to $x \in X$, since $\{x_i\}$ is d-Cauchy, then there exists $m_1 \in N$ such that $d(x_i, x_j) < 1/(n+1)$ for all $i, j \geq m_1$. Since X is a sn-symmetric space, then $\{B(x, 1/i) : i \in N\}$ is an sn-network of x in X. So $B(x, 1/(n+1))$ is a sequential neighborhood of x in X. Thus there exists $m_2 \in N$ such that $d(x, x_i) < 1/(n+1)$ for all $i \geq m_2$. Put

$$P = \{x\} \cup \{x_i : i \geq m\} \quad \text{where } m = m_1 + m_2,$$

then $P \in \mathcal{P}_n$.

Obviously, $\{x_i\}$ is eventually in P. Hence each $\mathcal{P}_n$ is a cs-cover for X. Therefore, X has a σ-strong network consisting of cs-covers. ∎

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