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MORE ON EVALUATING DETERMINANTS

A. R. Moghaddamfar, S. M. H. Pooya, S. Navid Salehy and S. Nima Salehy

Abstract. This article provides a general technique for finding closed formulas for the determinants of families of matrices whose entries satisfy a three-term recurrence relation. The major purpose of this article is to generalize several published results about evaluating determinants. We also present new proofs for some known results due to Ch. Krattenthaler.

1. Introduction

Let $\alpha = (\alpha_i)_{i\geq 1}$, $\beta = (\beta_i)_{i\geq 1}$, $\lambda = (\lambda_i)_{i\geq 1}$, $\mu = (\mu_i)_{i\geq 1}$ and $\nu = (\nu_i)_{i\geq 1}$ be given sequences with $\alpha_1 = \beta_1$. Let $A = [a_{i,j}]_{1\leq i,j\leq n}$ be an $n \times n$ matrix whose entries satisfy the recurrence relations of the form

$$a_{i,j} = \lambda_{j-1}a_{i,j-1} + \mu_{j-1}a_{i-1,j-1} + \nu_{j-1}a_{i-1,j}, \quad 2 \leq i, j \leq n, \tag{1}$$

and the initial conditions $a_{1,j} = \alpha_j$ $(1 \leq j \leq n)$ and $a_{i,1} = \beta_i$ $(1 \leq i \leq n)$. The relative positions of the entries $a_{i,j-1}$, $a_{i-1,j-1}$, $a_{i-1,j}$ and $a_{i,j}$ in Eq. (1) form the "figure $\sqcap$" shape. We call a matrix a $\sqcap_{\lambda,\mu,\nu}$*-matrix* if its entries satisfy the recurrence Eq. (1). Moreover, a matrix is called a $\sqcap$*-matrix* if it is a $\sqcap_{\lambda,\mu,\nu}$-matrix for some sequences λ, μ and ν. In the case when $\mu_i = 0$ and $\lambda_i = \nu_i = 1$ for each i, the $\sqcap_{\lambda,\mu,\nu}$-matrix is called the *generalized Pascal triangle* associated with α, β and denoted by $P_{\beta,\alpha}(n)$ (see [1]). Also, if $\lambda_i = 0$, $\mu_i = a$ and $\nu_i = b$ for each i, where a and b are nozero constants, the $\sqcap_{\lambda,\mu,\nu}$-matrix is called the $7_{a,b}$*-matrix* (see [2]). In particular, $7_{1,1}$-matrices are called simply 7-matrices.

The determinant of a square matrix plays an important role in different areas of mathematics. Often, the solution of a particular problem depends on the explicit computation of a determinant. In recent years, a special attention has been paid to the problem of symbolic evaluation of determinants, see [3–13] for a full description.

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In [6], we investigated the 7-matrices and evaluated the determinants of certain 7-matrices with various choices for the first row and column. In this article, we study a more general case. We first present a factorization of a certain $\sqcap_{\lambda,\mu,\nu}$-matrix associated with arbitrary sequences λ, μ and ν, and then we find a generalization of results obtained before in [6] or newly presented. The main technique for proving our results is *matrix factorization of recursive arrays* (see Theorems 1 and 2). It is worth to mention that, using the results in this article, we obtain alternative proofs for some Theorems in [5].

We conclude the introduction with notation and terminology to be used throughout the article. Given a matrix A, we denote by $\mathrm{R}_i(A)$ and $\mathrm{C}_j(A)$ the row i and the column j of A, respectively. We use the notation A^T for the transpose of A. We also denote by

$$A(\{i_1, i_2, \ldots, i_k\}, \{j_1, j_2, \ldots, j_k\})$$

the submatrix of A obtained by deleting rows $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots,$ j_k. All our matrices start with row 1 and column 1. Given g_1 and g_2, a *Gibonacci sequence* (*generalized Fibonacci sequence*) $g_1, g_2, g_3, \ldots$ is defined recursively by $g_n = g_{n-1} + g_{n-2}$ for $n \geq 3$. As it is customary, an empty sum (e.g. a sum of the form $\sum_{i=m}^{n} f(i)$ where $n < m$) is taken to be 0.

2. Main results

We begin with the following theorem.

THEOREM 1. *Let $\alpha = (\alpha_i)_{i\geq 1}$, $\lambda = (\lambda_i)_{i\geq 1}$, $\mu = (\mu_i)_{i\geq 1}$ and $\nu = (\nu_i)_{i\geq 1}$ be given sequences and let $A = [a_{i,j}]_{1\leq i,j\leq n}$ be a $\sqcap_{\lambda,\mu,\nu}$-matrix with the initial conditions:*

$$a_{i,1} = \alpha_1 x^{i-1} + d\sum_{l=0}^{i-2} x^l \quad \text{and} \quad a_{1,j} = \alpha_j \qquad \text{for} \quad 1 \leq i, j \leq n.$$

Then we have the following factorization:

$$A = L \cdot B,$$

where $L = [L_{i,j}]_{1\leq i,j\leq n}$ is a 7-matrix with the initial conditions

$$L_{i,1} = x^{i-1} \quad \text{and} \quad L_{1,j} = 0 \quad \text{for} \quad 1 \leq i \leq n,\ 2 \leq j \leq n,$$

and $B = [B_{i,j}]_{1\leq i,j\leq n}$ is a matrix given by the recurrences

$$B_{i,j} = \begin{cases} \lambda_{j-1}B_{2,j-1} + (\mu_{j-1} + x\lambda_{j-1})B_{1,j-1} + (\nu_{j-1} - x)B_{1,j}, & \text{if } j \geq i = 2, \\ \lambda_{j-1}B_{i,j-1} + (\mu_{j-1} + \lambda_{j-1})B_{i-1,j-1} + (\nu_{j-1} - 1)B_{i-1,j}, & \text{if } i \geq 3,\ j \geq 2, \end{cases} \tag{2}$$

and the initial conditions $B_{1,j} = \alpha_j$, $1 \leq j \leq n$, $B_{2,1} = d$ and $B_{i,1} = 0$, $3 \leq i \leq n$. In particular, we have $\det A = \det B$.

Proof. It suffices to show each of the following:

$$\mathrm{R}_1(L \cdot B) = \mathrm{R}_1(A) = (\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n), \tag{3}$$

$$\mathrm{C}_1(L \cdot B) = \mathrm{C}_1(A) = (\alpha_1,\ \alpha_1 x + d,\ \ldots,\ \alpha_1 x^{n-1} + \sum_{l=0}^{n-2} dx^l)^T, \tag{4}$$

and

$$(L\cdot B)_{i,j} = \lambda_{j-1}(L\cdot B)_{i,j-1}+\mu_{j-1}(L\cdot B)_{i-1,j-1}+\nu_{j-1}(L\cdot B)_{i-1,j}, \quad 2 \le i,j \le n. \tag{5}$$

First, we suppose that $j = 1$. Clearly

$$(L \cdot B)_{1,j} = \sum_{k=1}^{n} L_{1,k}B_{k,j} = L_{1,1}B_{1,j} = \alpha_j,$$

and so Eq. (3) holds.

Next, we suppose that $j = 1$, and we obtain

$$(L \cdot B)_{i,1} = \sum_{k=1}^{n} L_{i,k}B_{k,1} = L_{i,1}B_{1,1} + L_{i,2}B_{2,1} = x^{i-1}\alpha_1 + \Big(\sum_{l=0}^{i-2} x^l\Big)d,$$

which implies that Eq. (4) holds.

To establish Eq. (5), recall that

$$L_{i,j} = L_{i-1,j-1} + L_{i-1,j}, \quad 2 \le i,j \le n. \tag{6}$$

Thus, for $2 \le i,j \le n$, we have that

$$\begin{aligned}
(L \cdot B)_{i,j} &= \sum_{k=1}^{n} L_{i,k}B_{k,j} \\
&= L_{i,1}B_{1,j} + \sum_{k=2}^{n} L_{i,k}B_{k,j} \\
&= L_{i,1}B_{1,j} + \sum_{k=2}^{n} (L_{i-1,k-1} + L_{i-1,k})B_{k,j} \quad \text{(by Eq. (6))} \\
&= L_{i,1}B_{1,j} + \sum_{k=2}^{n} L_{i-1,k-1}B_{k,j} + \sum_{k=2}^{n} L_{i-1,k}B_{k,j} \\
&= L_{i,1}B_{1,j} + \sum_{k=2}^{n} L_{i-1,k-1}B_{k,j} + \sum_{k=1}^{n} L_{i-1,k}B_{k,j} - L_{i-1,1}B_{1,j} \\
&= (L_{i,1} - L_{i-1,1})B_{1,j} + \sum_{k=2}^{n} L_{i-1,k-1}B_{k,j} + \sum_{k=1}^{n} L_{i-1,k}B_{k,j} \\
&= (L_{i,1} - L_{i-1,1})B_{1,j} + \sum_{k=3}^{n} L_{i-1,k-1}B_{k,j} + L_{i-1,1}B_{2,j} + \sum_{k=1}^{n} L_{i-1,k}B_{k,j}.
\end{aligned} \tag{7}$$

On the other hand, by computation we obtain

$$\begin{aligned}
\sum_{k=3}^{n} L_{i-1,k-1}B_{k,j} &= \sum_{k=3}^{n} L_{i-1,k-1}[\lambda_{j-1}B_{k,j-1} + (\lambda_{j-1} + \mu_{j-1})B_{k-1,j-1} \\
&\quad + (\nu_{j-1} - 1)B_{k-1,j}] \quad \text{(by Eq. (2))} \\
&= \lambda_{j-1} \sum_{k=3}^{n} L_{i-1,k-1}B_{k,j-1} + (\lambda_{j-1} + \mu_{j-1}) \sum_{k=3}^{n} L_{i-1,k-1}B_{k-1,j-1} \\
&\quad + (\nu_{j-1} - 1) \sum_{k=3}^{n} L_{i-1,k-1}B_{k-1,j} \\
&= \lambda_{j-1} \sum_{k=3}^{n} (L_{i,k} - L_{i-1,k})B_{k,j-1} + (\lambda_{j-1} + \mu_{j-1}) \sum_{k=2}^{n-1} L_{i-1,k}B_{k,j-1}
\end{aligned}$$

$$
\begin{aligned}
&+ (\nu_{j-1} - 1) \sum_{k=2}^{n-1} L_{i-1,k} B_{k,j} \quad \text{(by Eq. (6))} \\
&= \lambda_{j-1} \sum_{k=1}^{n} L_{i,k} B_{k,j-1} - \lambda_{j-1} \sum_{k=1}^{2} L_{i,k} B_{k,j-1} \\
&\quad - \lambda_{j-1} \sum_{k=1}^{n} L_{i-1,k} B_{k,j-1} + \lambda_{j-1} \sum_{k=1}^{2} L_{i-1,k} B_{k,-1} \\
&\quad + (\lambda_{j-1} + \mu_{j-1}) \sum_{k=1}^{n} L_{i-1,k} B_{k,j-1} - (\lambda_{j-1} + \mu_{j-1}) L_{i-1,1} B_{1,j-1} \\
&\quad + (\nu_{j-1} - 1) \sum_{k=1}^{n} L_{i-1,k} B_{k,j} - (\nu_{j-1} - 1) L_{i-1,1} B_{1,j} \ \text{(note } L_{i-1,n} = 0) \\
&= \lambda_{j-1} \sum_{k=1}^{n} L_{i,k} B_{k,j-1} - \lambda_{j-1} \sum_{k=1}^{2} L_{i,k} B_{k,j-1} + \lambda_{j-1} L_{i-1,2} B_{2,j-1} \\
&\quad + \mu_{j-1} \sum_{k=1}^{n} L_{i-1,k} B_{k,j-1} - \mu_{j-1} L_{i-1,1} B_{1,j-1} \\
&\quad + \nu_{j-1} \sum_{k=1}^{n} L_{i-1,k} B_{k,j} - \sum_{k=1}^{n} L_{i-1,k} B_{k,j} - (\nu_{j-1} - 1) L_{i-1,1} B_{1,j}.
\end{aligned}
$$

After having substituted this in Eq. (7), we get

$$
\begin{aligned}
(L \cdot B)_{i,j} &= \lambda_{j-1} \sum_{k=1}^{n} L_{i,k} B_{k,j-1} + \mu_{j-1} \sum_{k=1}^{n} L_{i-1,k} B_{k,j-1} \\
&\quad + \nu_{j-1} \sum_{k=1}^{n} L_{i-1,k} B_{k,j} + \Psi(i,j) \\
&= \lambda_{j-1} (L \cdot B)_{i,j-1} + \mu_{j-1} (L \cdot B)_{i-1,j-1} + \nu_{j-1} (L \cdot B)_{i-1,j} + \Psi(i,j),
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi(i,j) &= L_{i,1} B_{1,j} + L_{i-1,1} B_{2,j} - \lambda_{j-1} \sum_{k=1}^{2} L_{i,k} B_{k,j-1} \\
&\quad + \lambda_{j-1} L_{i-1,2} B_{2,j-1} - \mu_{j-1} L_{i-1,1} B_{1,j-1} - \nu_{j-1} L_{i-1,1} B_{1,j}.
\end{aligned}
$$

Now we must prove that $\Psi(i,j) = 0$. To do this, we observe that

$$
\begin{aligned}
\Psi(i,j) &= x L_{i-1,1} B_{1,j} + L_{i-1,1} B_{2,j} - \lambda_{j-1} x L_{i-1,1} B_{1,j-1} + \lambda_{j-1} (L_{i-1,2} - L_{i,2}) B_{2,j-1} \\
&\quad - \mu_{j-1} L_{i-1,1} B_{1,j-1} - \nu_{j-1} L_{i-1,1} B_{1,j} \quad \text{(notice that } L_{i,1} = x L_{i-1,1}) \\
&= L_{i-1,1} \big\{ B_{2,j} - \lambda_{j-1} B_{2,j-1} - (\mu_{j-1} + \lambda_{j-1} x) B_{1,j-1} - (\nu_{j-1} - x) B_{1,j} \big\} \\
&\qquad \text{(by Eq. (6))} \\
&= 0. \quad \text{(by Eq. (2))}
\end{aligned}
$$

The proof is thus complete. ■

Remark. This result generalizes Theorem 2.1 in [6]. Indeed, if we take $x = 1$, $\lambda_i = 0$ and $\mu_i = \nu_i = 1$ for each i, then we obtain Theorem 2.1 in [6].

Theorem 2. *Let $\alpha = (\alpha_i)_{i \geq 1}$ be an arbitrary sequence and let $\beta = (\beta_i)_{i \geq 1}$ be the sequence satisfying $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$ and linear recursion $\beta_i = \beta_{i-2} + k\beta_{i-1}$ for*

$i \geq 3$, where k is a constant. Let $\lambda = (\lambda_i)_{i\geq 1}$, $\mu = (\mu_i)_{i\geq 1}$ and $\nu = (\nu_i)_{i\geq 1}$ be three sequences with $\lambda_i = 0$, $\mu_1 = 1$ and $\nu_i = k$ for each i. Suppose that $A = [a_{i,j}]_{1\leq i,j\leq n}$ is the $\sqcap_{\lambda,\mu,\nu}$-matrix with the initial conditions:

$$a_{1,j} = \alpha_j \quad \text{and} \quad a_{i,1} = \beta_i \qquad \text{for} \quad 1 \leq i, j \leq n.$$

Then we have

$$\det A = \begin{cases} \alpha_1, & \text{if } n = 1, \\ rs^{n-2} \prod_{i=3}^{n-1} \mu_i^{n-i}, & \text{if } n \geq 2, \end{cases}$$

where $r = \alpha_1^2 - \alpha_2^2 + k\alpha_1\alpha_2$ and $s = \mu_2\alpha_1 + \mu_2 k\alpha_2 - \alpha_3$.

Proof. For $n \leq 3$ the result is straightforward. Hence, we assume that $n \geq 4$. We claim that

$$A = L \cdot \tilde{U},$$

where $L = [L_{i,j}]_{1\leq i,j\leq n}$ is a lower triangular matrix by the recurrence

$$L_{i,j} = L_{i-1,j-1} + kL_{i-1,j}, \quad 2 \leq i, j \leq n, \tag{8}$$

and the initial conditions $L_{1,1} = 1$, $L_{2,1} = 0$, and

$$L_{i,1} = L_{i-2,1} + kL_{i-1,1}, \quad 3 \leq i \leq n, \tag{9}$$

and $L_{1,j} = 0$, $2 \leq j \leq n$, and where $\tilde{U} = [\tilde{U}_{i,j}]_{1\leq i,j\leq n}$ with

$$\tilde{U}_{i,j} = \begin{cases} \alpha_j, & \text{if } i = 1, j \geq 1, \\ \alpha_2, & \text{if } i = 2, j = 1, \\ 0, & \text{if } i \geq 3, j = 1, \\ \mu_{j-1}\tilde{U}_{1,j-1} + k\tilde{U}_{1,j}, & \text{if } i = 2, 2 \leq j \leq n, \\ \mu_{j-1}\tilde{U}_{2,j-1} - \tilde{U}_{1,j}, & \text{if } i = 3, 2 \leq j \leq n, \\ \mu_{j-1}\tilde{U}_{i-1,j-1}, & \text{if } 4 \leq i \leq n,\ 2 \leq j \leq n. \end{cases} \tag{10}$$

Note that L is a $7_{1,k}$-matrix. For instance, when $n = 4$, the matrices L and $\tilde{U}$ are given by

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & k & 1 & 0 \\ k & 1+k^2 & 2k & 1 \end{pmatrix},$$

and

$$\tilde{U} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & \alpha_1 + k\alpha_2 & \mu_2\alpha_2 + k\alpha_3 & \mu_3\alpha_3 + k\alpha_4 \\ 0 & 0 & \mu_2\alpha_1 + \mu_2 k\alpha_2 - \alpha_3 & \mu_3(\mu_2\alpha_2 + k\alpha_3) - \alpha_4 \\ 0 & 0 & 0 & \mu_3(\mu_2\alpha_1 + \mu_2 k\alpha_2 - \alpha_3) \end{pmatrix}.$$

Moreover, by the structure of L, we have

$$L_{i,2} = L_{i+1,1}, \quad \text{for all} \quad 1 \leq i \leq n-1. \tag{11}$$

Hence, we can easily deduce that

$$L_{i,2} = L_{i-2,2} + kL_{i-1,2}, \quad 3 \leq i \leq n. \tag{12}$$

The matrix L is a lower triangular one with 1's on its diagonal, and so $\det(L) = 1$. Furthermore, by the structure of $\tilde{U}$, one can partition $\tilde{U}$ in the following manner

$$\tilde{U} = \left(\begin{array}{c|c} T & * \\ \hline 0 & U \end{array}\right),$$

where $T = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_1 + k\alpha_2 \end{pmatrix}$ and $U = \tilde{U}(\{1,2\},\{1,2\})$ is an upper triangular matrix with diagonal entries

$$\begin{aligned}
U_{1,1} &= \tilde{U}_{3,3} = s, \\
U_{2,2} &= \tilde{U}_{4,4} = \mu_3 \tilde{U}_{3,3} = \mu_3 s, \\
&\vdots \\
U_{i,i} &= \tilde{U}_{i+2,i+2} = \mu_{i+1}\tilde{U}_{i+1,i+1} = s\prod_{l=3}^{i+1}\mu_l, \\
&\vdots \\
U_{n-2,n-2} &= \tilde{U}_{n,n} = \mu_{n-1}\tilde{U}_{n-1,n-1} = s\prod_{l=3}^{n-1}\mu_l,
\end{aligned}$$

where $s = \mu_2\alpha_1 + \mu_2 k\alpha_2 - \alpha_3$. Let $r := \det T$. Therefore, we obtain

$$\det\tilde{U} = \det T \times \det U = r \times s^{n-2}\prod_{l=3}^{n-1}\mu_l^{n-l}.$$

Finally, it follows that

$$\det A = \det L \times \det\tilde{U} = \det\tilde{U} = rs^{n-2}\prod_{l=3}^{n-1}\mu_l^{n-l},$$

as desired.

The proof of the claim requires some calculations, which we handle case by case.

Case 1. $i = 1, \quad 1 \leq j \leq n$. In this case, we have

$$(L\cdot\tilde{U})_{1,j} = \sum_{l=1}^{n} L_{1,l}\tilde{U}_{l,j} = L_{1,1}\tilde{U}_{1,j} = \alpha_j.$$

Case 2. $(i,j) = (2,1)$. Here, we obtain

$$(L\cdot\tilde{U})_{2,1} = \sum_{l=1}^{n} L_{2,l}\tilde{U}_{l,1} = L_{2,1}\tilde{U}_{1,1} + L_{2,2}\tilde{U}_{2,1} = \alpha_2.$$

Case 3. $3 \leq i \leq n, \ j = 1$. Now, by easy calculations we get

$$(L\cdot\tilde{U})_{i,1} = \sum_{l=1}^{n} L_{i,l}\tilde{U}_{l,1} = L_{i,1}\tilde{U}_{1,1} + L_{i,2}\tilde{U}_{2,1}$$

$$
\begin{aligned}
&= (L_{i-2,1} + kL_{i-1,1})\tilde{U}_{1,1} + (L_{i-2,2} + kL_{i-1,2})\tilde{U}_{2,1} \quad \text{(by Eqs. (9) and (12))} \\
&= (L_{i-2,1}\tilde{U}_{1,1} + L_{i-2,2}\tilde{U}_{2,1}) + k(L_{i-1,1}\tilde{U}_{1,1} + L_{i-1,2}\tilde{U}_{2,1}) \\
&= \sum_{l=1}^{n} L_{i-2,l}\tilde{U}_{l,1} + k\sum_{l=1}^{n} L_{i-1,l}\tilde{U}_{l,1} \\
&= (L \cdot \tilde{U})_{i-2,1} + k(L \cdot \tilde{U})_{i-1,1}.
\end{aligned}
$$

Case 4. $2 \le i, j \le n$. Here, the sum in question can be calculated as follows

$$
\begin{aligned}
(L \cdot \tilde{U})_{i,j} &= \sum_{l=1}^{n} L_{i,l}\tilde{U}_{l,j} = \sum_{l=1}^{2} L_{i,l}\tilde{U}_{l,j} + \sum_{l=3}^{n} L_{i,l}\tilde{U}_{l,j} \\
&= \sum_{l=1}^{2} L_{i,l}\tilde{U}_{l,j} + \sum_{l=3}^{n}(L_{i-1,l-1} + kL_{i-1,l})\tilde{U}_{l,j} \quad \text{(by Eq. (8))} \\
&= \sum_{l=1}^{2} L_{i,l}\tilde{U}_{l,j} + \sum_{l=3}^{n} L_{i-1,l-1}\tilde{U}_{l,j} + k\sum_{l=3}^{n} L_{i-1,l}\tilde{U}_{l,j} \\
&= \sum_{l=1}^{2} L_{i,l}\tilde{U}_{l,j} + L_{i-1,2}\tilde{U}_{3,j} + \sum_{l=4}^{n} L_{i-1,l-1}\tilde{U}_{l,j} \\
&\quad + k\sum_{l=1}^{n} L_{i-1,l}\tilde{U}_{l,j} - k\sum_{l=1}^{2} L_{i-1,l}\tilde{U}_{l,j} \\
&= \sum_{l=1}^{2} L_{i,l}\tilde{U}_{l,j} + L_{i-1,2}(\mu_{j-1}\tilde{U}_{2,j-1} - \tilde{U}_{1,j}) + \mu_{j-1}\sum_{l=4}^{n} L_{i-1,l-1}\tilde{U}_{l-1,j-1} \\
&\quad + k(L \cdot \tilde{U})_{i-1,j} - k\sum_{l=1}^{2} L_{i-1,l}\tilde{U}_{l,j} \quad \text{(by Eq. (10))} \\
&= (L_{i,1} - L_{i-1,2})\tilde{U}_{1,j} + (L_{i,2}\tilde{U}_{2,j} + \mu_{j-1}L_{i-1,2}\tilde{U}_{2,j-1}) \\
&\quad + \mu_{j-1}\sum_{l=3}^{n} L_{i-1,l}\tilde{U}_{l,j-1} + k(L \cdot \tilde{U})_{i-1,j} - k\sum_{l=1}^{2} L_{i-1,l}\tilde{U}_{l,j} \\
&\quad \text{(note that } L_{i,1} - L_{i-1,2} = 0 \text{ by Eq. (11) and also } L_{i-1,n} = 0) \\
&= L_{i,2}\tilde{U}_{2,j} + \mu_{j-1}\sum_{l=2}^{n} L_{i-1,l}\tilde{U}_{l,j-1} + k(L \cdot \tilde{U})_{i-1,j} - k\sum_{l=1}^{2} L_{i-1,l}\tilde{U}_{l,j} \\
&= L_{i,2}\tilde{U}_{2,j} + \mu_{j-1}(L \cdot \tilde{U})_{i-1,j-1} - \mu_{j-1}L_{i-1,1}\tilde{U}_{1,j-1} + k(L \cdot \tilde{U})_{i-1,j} \\
&\quad - k\sum_{l=1}^{2} L_{i-1,l}\tilde{U}_{l,j} \\
&= \mu_{j-1}(L \cdot \tilde{U})_{i-1,j-1} + k(L \cdot \tilde{U})_{i-1,j} + (L_{i,2} - kL_{i-1,2})\tilde{U}_{2,j} \\
&\quad - L_{i-1,1}(\mu_{j-1}\tilde{U}_{1,j-1} + k\tilde{U}_{1,j}) \\
&= \mu_{j-1}(L \cdot \tilde{U})_{i-1,j-1} + k(L \cdot \tilde{U})_{i-1,j} + L_{i-1,1}\tilde{U}_{2,j} - L_{i-1,1}\tilde{U}_{2,j} \\
&\quad \text{(by Eqs. (8) and (10))} \\
&= \mu_{j-1}(L \cdot \tilde{U})_{i-1,j-1} + k(L \cdot \tilde{U})_{i-1,j}.
\end{aligned}
$$

This completes the proof of the theorem. ∎

3. Some applications

In this section we derive several consequences of Theorems 1 and 2. Several of these give simpler proofs of known results from [5].

Corollary 1. [5, Theorem 1] *In Theorem 1, if we take $\nu_i = \lambda_i = 1$, $\mu_i = \mu$, $\alpha_i = y^{i-1}$ for each i and $d = 0$, then we have*

$$\det A = (1+\mu)^{\binom{n-1}{2}}(\mu + x + y - xy)^{n-1}.$$

Proof. By Theorem 1, with $\nu_i = \lambda_i = 1$, $\mu_i = \mu$, $\alpha_i = y^{i-1}$ for each i and $d = 0$, we deduce that $\det A = \det B$, where $B = [B_{i,j}]_{1 \le i,j \le n}$ is a matrix given by the recurrence

$$B_{i,j} = \begin{cases} B_{2,j-1} + (\mu + x)B_{1,j-1} + (1-x)B_{1,j}, & \text{if } j \ge i = 2, \\ B_{i,j-1} + (\mu+1)B_{i-1,j-1}, & \text{if } i \ge 3,\ j \ge 2, \end{cases}$$

and the initial conditions $B_{1,j} = \alpha_j$, $1 \le j \le n$, and $B_{i,1} = 0$, $2 \le i \le n$. Evidently, B is an upper triangular matrix with diagonal entries:

$$B_{1,1} = 1, \quad B_{i,i} = (1+\mu)^{i-2}(\mu + x + y - xy), \quad \text{for } \ i = 2, 3, \dots, n.$$

Now, we can easily obtain

$$\det A = \det B = \prod_{i=1}^{n} B_{i,i} = \prod_{i=2}^{n}(1+\mu)^{i-2}(\mu+x+y-xy) = (1+\mu)^{\binom{n-1}{2}}(\mu+x+y-xy)^{n-1},$$

as desired. ■

Corollary 2. [5, Theorem 3] *In Theorem 1, if we take $\nu_i = \lambda_i = 1$, $\mu_i = \mu$, $\alpha_i = 1 - i$ for all i, and $d = x = 1$, then we have*

$$\det_{1 \le i,j \le m}[a_{i,j}] = \begin{cases} (1+\mu)^{2n(n-1)}, & \text{if } m = 2n \ge 2, \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Let A denote the matrix $[a_{i,j}]_{1 \le i,j \le m}$. By Theorem 1, with $\nu_i = \lambda_i = 1$, $\mu_i = \mu$, $\alpha_i = 1 - i$ for each i, and $d = x = 1$, we deduce that $\det A = \det B$, where $B = [B_{i,j}]_{1 \le i,j \le m}$ is a matrix given by the recurrence

$$B_{i,j} = B_{i,j-1} + (1+\mu)B_{i-1,j-1}, \quad 2 \le i,j \le m,$$

and the initial conditions $B_{1,j} = \alpha_j$, $1 \le j \le m$, and $B_{2,1} = 1$, $B_{i,1} = 0$ for $3 \le i \le m$. To obtain $\det B$, we factorize B as follows

$$B = C \cdot U,$$

where $C = [C_{i,j}]_{1 \le i,j \le m}$ is a matrix given by the recurrence

$$C_{i,j} = (1+\mu)C_{i-1,j-1}, \quad \text{for } \ 2 \le i,j \le m, \tag{13}$$

and the initial conditions $C_{1,1} = 0$, $C_{1,2} = -1$, $C_{2,1} = 1$ and $C_{i,1} = C_{1,i} = 0$ for $3 \le i \le m$, and where $U = [U_{i,j}]_{1 \le i,j \le m}$ is an upper triangular matrix by the recurrence

$$U_{i,j} = U_{i-1,j-1} + U_{i,j-1}, \quad \text{for } \ 2 \le i,j \le m, \tag{14}$$

and the initial conditions $U_{1,j} = 1$, $1 \le j \le m$ and $U_{i,1} = 0$, $2 \le i \le m$.

For instance, when $m=6$, the matrices C and U are given by

$$C=\begin{bmatrix}0 & -1 & 0 & 0 & 0 & 0\\ 1 & 0 & -\tilde{\mu} & 0 & 0 & 0\\ 0 & \tilde{\mu} & 0 & -\tilde{\mu}^2 & 0 & 0\\ 0 & 0 & \tilde{\mu}^2 & 0 & -\tilde{\mu}^3 & 0\\ 0 & 0 & 0 & \tilde{\mu}^3 & 0 & -\tilde{\mu}^4\\ 0 & 0 & 0 & 0 & \tilde{\mu}^4 & 0\end{bmatrix},$$

where $\tilde{\mu}=1+\mu$, and

$$U=\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3 & 4 & 5\\ 0 & 0 & 1 & 3 & 6 & 10\\ 0 & 0 & 0 & 1 & 4 & 10\\ 0 & 0 & 0 & 0 & 1 & 5\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}.$$

For the proof of the claimed factorization we compute the $(i,j)-$entry of $C\cdot U$, that is

$$(C\cdot U)_{i,j}=\sum_{k=1}^{m}C_{i,k}U_{k,j}.$$

First, we observe that

$$\begin{aligned}(C\cdot U)_{2,1}&=\sum_{k=1}^{m}C_{2,k}U_{k,1}=1,\\ (C\cdot U)_{i,1}&=\sum_{k=1}^{m}C_{i,k}U_{k,1}=0,\quad 3\le i\le m,\\ (C\cdot U)_{1,j}&=\sum_{k=1}^{m}C_{1,k}U_{k,j}=C_{1,1}U_{1,j}+C_{1,2}U_{2,j}=-(j-1),\quad 1\le j\le m.\end{aligned}$$

Next, we show that

$$(C\cdot U)_{i,j}=(1+\mu)(C\cdot U)_{i-1,j-1}+(C\cdot U)_{i,j-1},\quad 2\le i,j\le m.$$

To do this, by easy computations as before we observe that

$$\begin{aligned}(C\cdot U)_{i,j}&=\sum_{k=1}^{m}C_{i,k}U_{k,j}\\ &=C_{i,1}U_{1,j}+\sum_{k=2}^{m}C_{i,k}U_{k,j}\\ &=C_{i,1}U_{1,j}+\sum_{k=2}^{m}C_{i,k}(U_{k-1,j-1}+U_{k,j-1})\quad \text{(by Eq. (14))}\\ &=C_{i,1}U_{1,j}+\sum_{k=2}^{m}C_{i,k}U_{k-1,j-1}+\sum_{k=2}^{m}C_{i,k}U_{k,j-1}\\ &=C_{i,1}U_{1,j}+(1+\mu)\sum_{k=2}^{m}C_{i-1,k-1}U_{k-1,j-1}\\ &\quad+\sum_{k=1}^{m}C_{i,k}U_{k,j-1}-C_{i,1}U_{1,j-1}\quad \text{(by Eq. (13))}\end{aligned}$$

$$= C_{i,1}(U_{1,j} - U_{1,j-1}) + (1+\mu)\sum_{k=1}^{m-1} C_{i-1,k}U_{k,j-1} + (C\cdot U)_{i,j-1}$$
$$= (1+\mu)\sum_{k=1}^{m} C_{i-1,k}U_{k,j-1} + (C\cdot U)_{i,j-1},$$
$$\text{(note that } U_{m,j-1} = 0 \text{ and } U_{1,j} = U_{1,j-1} = 1)$$
$$= (1+\mu)(C\cdot U)_{i-1,j-1} + (C\cdot U)_{i,j-1}.$$

Now, by the claimed factorization, we conclude that

$$\det B = \det C\cdot U = \det C \cdot \det U = \det C.$$

In the sequel, we put $c(m) = \det[C_{i,j}]_{1\le i,j\le m}$. Now expanding the determinant along the last row and then the last column of the obtained minor, we get the following recurrence relation:

$$c(1) = 0,\ c(2) = 1, \quad \text{and} \quad c(m) = (1+\mu)^{2(m-2)}c(m-2), \quad (m \ge 3).$$

Now, if m is odd, then it is easy to see that $c(m) = 0$. In the case when m is even, say $m = 2n$, we have

$$\begin{aligned} c(m) &= (1+\mu)^{2(m-2)}\cdot(1+\mu)^{2(m-4)}\cdot(1+\mu)^{2(m-6)}\cdots(1+\mu)^{2(m-(m-2))}c(2) \\ &= (1+\mu)^{2((m-2)+(m-4)+\cdots+2)} \\ &= (1+\mu)^{\frac{m(m-2)}{2}} = (1+\mu)^{2n(n-1)}, \end{aligned}$$

which completes the proof of the corollary. ■

Corollary 3. [6, Theorem 2.4] *In Theorem 2, if we take $\mu_i = 1$ and $k = 1$, then we have*

$$\det_{1\le i,j\le n}[a_{i,j}] = \begin{cases} \alpha_1, & \text{if } n = 1, \\ (\alpha_1^2 - \alpha_2^2 + \alpha_1\alpha_2)(\alpha_1+\alpha_2-\alpha_3)^{n-2}, & \text{if } n \ge 2. \end{cases}$$

In the following corollary we examine the determinants of generalized Pascal triangles associated with a linear homogeneous recurrence relation of order 2 and an arithmetic sequence.

Corollary 4. *Let $\alpha = (\alpha_i)_{i\ge1}$ be the Gibonacci sequence with $\alpha_1 = a$, $\alpha_2 = b$, and let $\beta = (\beta_i)_{i\ge1}$ be the arithmetic sequence with the first term $\beta_1 = a$ and the common difference b. Then*

$$\det P_{\alpha,\beta}(n) = \begin{cases} a, & \text{if } \ n = 1, \\ (a+b)^{n-2}(a^2+ab-b^2), & \text{if } \ n \ge 2. \end{cases}$$

Proof. By Theorem 1, with $x = 1$, $\nu_i = \lambda_i = 1$ and $\mu_i = 0$, we deduce that

$$\det P_{\beta,\alpha}(n) = \det B,$$

where $B = (B_{i,j})_{1\le i,j\le n}$ is a matrix given by the recurrence

$$B_{i,j} = B_{i,j-1} + B_{i-1,j-1}, \quad 2 \le i,j \le n,$$

and the initial conditions $B_{1,j} = \alpha_j$, $1 \le j \le n$, $B_{2,1} = b$ and $B_{i,1} = 0$, $3 \le i \le n$. Now, we consider B^T the transpose of B. Using Corollary 3, we deduce that

$$\det(B^T) = \begin{cases} a, & \text{if} \quad n = 1, \\ (a+b)^{n-2}(a^2 + ab - b^2), & \text{if} \quad n \ge 2. \end{cases}$$

Now since $P_{\alpha,\beta}(n) = P_{\beta,\alpha}(n)^T$, one can easily see that the equality

$$\det P_{\alpha,\beta}(n) = \det P_{\beta,\alpha}(n) = \det B = \det B^T,$$

holds, as desired. ∎

Now consider the following sequence:

$$\omega^m(n) = (\omega_i^m)_{1 \le i \le n} = (\underbrace{1, 1, \ldots, 1}_{m-\text{times}}, \underbrace{0, 0, \ldots, 0}_{(n-2m)-\text{times}}, \underbrace{1, 1, \ldots, 1}_{m-\text{times}}),$$

where m, n are two natural numbers with $n \ge 2m$. Then, we have the following corollary.

Corollary 5. *In Theorem 2, if we take $\alpha_i = \omega_i^m$, then we have*

(*i*) *If $m = 1$, then*

$$\det_{1 \le i,j \le n} [a_{i,j}] = \begin{cases} k, & \text{if } n = 2, \\ (\mu_2 - 1)^{n-2}, & \text{if } n = 3, \\ \prod_{l=2}^{n-1} \mu_l^{n-l}, & \text{if } n \ge 4. \end{cases}$$

(*ii*) *If $m = 2$, then*

$$\det_{1 \le i,j \le n} [a_{i,j}] = \begin{cases} k\mu_3(\mu_2 + k\mu_2 - 1)^2, & \text{if } n = 4, \\ k(1+k)^{n-2} \prod_{l=2}^{n-1} \mu_l^{n-l}, & \text{if } n \ge 5. \end{cases}$$

(*iii*) *If $m \ge 3$, then*

$$\det_{1 \le i,j \le n} [a_{i,j}] = k(\mu_2 + k\mu_2 - 1)^{n-2} \prod_{l=3}^{n-1} \mu_l^{n-l}.$$

Remark. The above Corollary generalizes Corollary 2.5 in [6]. Indeed, if we take $\mu_i = 1$ and $k = 1$, then for $m \ge 2$, we have

$$\det_{1 \le i,j \le n} [a_{i,j}] = \begin{cases} 1, & \text{if } m = 2,\ n = 4, \\ 2^{n-2}, & \text{if } m = 2,\ n \ge 5, \\ 1, & \text{if } m \ge 3,\ n \ge 2m, \end{cases}$$

which is the Corollary 2.5 in [6].

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Department of Mathematics, K. N. Toosi University of Technology, P.O.Box 16315-1618, Tehran, Iran

and

Research Institute for Fundamental Sciences, Tabriz, Iran

E-mail: `moghadam@kntu.ac.ir, moghadam@mail.ipm.ir`