МАТЕМАТИЧКИ ВЕСНИК
64, 2 (2012), 97–107
June 2012

оригинални научни рад
research paper

ON COMPACT-COVERING AND SEQUENCE-COVERING IMAGES OF METRIC SPACES

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Abstract. In this paper we study the characterizations of compact-covering and 1-sequence-covering (resp. 2-sequence-covering) images of metric spaces and give a positive answer to the following question: How to characterize first countable spaces whose each compact subset is metrizable by certain images of metric spaces?

1. Introduction

To find internal characterizations of certain images of metric spaces is one of the central problems in General Topology. In 1973, E. Michael and K. Nagami [16] obtained a characterization of compact-covering and open images of metric spaces.

It is well known that the compact-covering and open images of metric spaces are the first countable spaces whose each compact subset is metrizable. However, its inverse does not hold [16]. For the first countable spaces whose each compact subset is metrizable, how to characterize them by certain images of metric spaces?

The sequence-covering maps play an important role on mapping theory about metric spaces [6, 9]. In this paper, we give the characterization of a compact-covering and 1-sequence-covering (resp. 2-sequence-covering) image of a metric space, and positively answer the question posed by S. Lin in [11, Question 2.6.5].

All spaces considered here are T_2 and all maps are continuous and onto. The letter N is the set of all positive natural numbers. Readers may refer to [10] for unstated definition and terminology.

2. Compact-covering and 1-sequence-covering images

In this section some characterizations of images of metric spaces by compact-covering and 1-sequence-covering maps are given.

2010 AMS Subject Classification: 54C10; 54D70; 54E40; 54E20; 54E99.

Keywords and phrases: Metrizable spaces; compact-covering maps; 1-sequence-covering maps; *so*-network; snf-countable spaces.

Supported by the NSFC (No. 10971185) and NSF (2009J01013) of Fujian Province of China.

DEFINITION 2.1. [4] Let X be a space, and $P \subset X$.

(1) A convergent sequence $\{x_n\}$ in X is called eventually in P, if for each sequence $\{x_n\}$ converging to x, there is an $m \in N$ such that $\{x\} \cup \{x_n : n \geq m\} \subset P$;

(2) P is called a *sequential neighborhood* of x in X, if for each sequence $\{x_n\}$ converging to x, $\{x_n\}$ is eventually in P;

(3) P is called a *sequentially open set* in X, if P is a sequential neighborhood of each of its points;

(4) X is called a *sequential space*, if each sequentially open set is open in X.

DEFINITION 2.2. Let $\mathcal{P} = \bigcup_{x\in X} \mathcal{P}_x$ be a cover of a space X such that for each $x \in X$, (a) if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$; (b) the family $\mathcal{P}_x$ is a network of x in X, i.e., $x \in \bigcap \mathcal{P}_x$, and if $x \in U$ and U is open in X, then $P \subset U$ for some $P \in \mathcal{P}_x$.

The family $\mathcal{P}$ is called a *weak base* for X [3], if for every $G \subset X$, the set G must be open in X whenever for each $x \in G$ there exists $P \in \mathcal{P}_x$ such that $P \subset G$, where $\mathcal{P}_x$ is called a weak base at $x \in X$. The family $\mathcal{P}$ is an *sn-network of* X [8], if each element of $\mathcal{P}_x$ is a sequential neighborhood of x for each $x \in X$, where $\mathcal{P}_x$ is called an *sn*-network at $x \in X$. A space X is *snf (resp. gf)-countable*, if X has an *sn*-network (*resp.* a weak base) $\mathcal{P}$ such that each $\mathcal{P}_x$ is countable.

We know that each gf-countable space is sequential by [18].

DEFINITION 2.3. Let X be a space. Then X is called a *k-space* [5], if for every subset A of X such that $K \cap A$ is closed in K for each compact subset K in X, A is closed in X. A space X is *Fréchet* [4], if whenever $x \in \mathrm{cl}(A) \subset X$, there is a sequence in A converging to the point x.

It is easy to check that the following relations:

first-countable spaces $\Rightarrow$ Fréchet spaces $\Rightarrow$ sequential spaces $\Rightarrow$ k-spaces.

DEFINITION 2.5. Let $f : X \to Y$ be a map.

(1) f is a *compact-covering map* [15], if each compact subset of Y is the image of some compact subset of X under f;

(2) f is a *sequence-covering map* [17], if whenever $\{y_n\}$ is a convergent sequence in Y, there exists a convergent sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$;

(3) f is a 1-*sequence-covering map* [8], if for each $y \in Y$, there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to $y \in Y$, there is a sequence $\{x_n\}$ converging to $x \in X$ with each $x_n \in f^{-1}(y_n)$;

(4) f is a *pseudo-open map* [2], if whenever $f^{-1}(y) \subset U$ and U is open in X, then $y \in \mathrm{Int}(f(U))$;

(5) f is an *almost-open map* [1], if for each $y \in Y$, there is $x \in f^{-1}(y)$ such that whenever U is a neighborhood of x in X, then $f(U)$ is a neighborhood of $f(y)$ in Y.

Obviously, we have the following relations:

open maps ⇒ almost-open maps ⇒ pseudo-open maps ⇒ quotient maps.

LEMMA 2.5. [17] *Let* $f : X \to Y$ *be a map. If* Y *is a sequential space and* f *is a sequence-covering map, then* f *is a quotient map.*

LEMMA 2.6. [12] *A space* X *is a 1-sequence-covering image of a metric space if and only if* X *is an* snf*-countable space.*

COROLLARY 2.7. [12] *A space* X *is a 1-sequence-covering and quotient image of a metric space if and only if* X *is a* gf*-countable space.*

LEMMA 2.8. [16] *A space* X *is a compact-covering image of a metric space if and only if each compact subset of* X *is metrizable.*

LEMMA 2.9. [11] *Let* $f{:}X \to Y$ *be a map.*

(1) *If* Y *is a* k*-space and* f *is a compact-covering map, then* f *is a quotient map;*

(2) *If* Y *is a Fréchet space and* f *is a quotient map, then* f *is a pseudo-open map.*

LEMMA 2.10. [9] *Let* $f : X \to Y$ *be a map. If* X *is a first-countable space, then* f *is an almost-open map if and only if* f *is a 1-sequence-covering and pseudo-open map.*

THEOREM 2.11. *The following are equivalent for a space* X*.*

(1) X *is a compact-covering and 1-sequence-covering image of a metric space;*

(2) X *is a compact-covering image of a metric space, and a 1-sequence-covering image of a metric space;*

(3) X *is an* snf*-countable space of which all compact subsets are metrizable.*

Proof. (1) ⇒ (2) is obvious. (2) ⇒ (3) by Lemmas 2.6 and 2.8. Next, we show that (3) ⇒ (1).

Let X be an snf-countable space all compact subsets of which are metrizable. By Lemma 2.6, there are a metric space M_1 and a 1-sequence-covering map $f : M_1 \to X$. By Lemma 2.8, there are a metric space M_2 and a compact-covering map $g : M_2 \to X$. Put $M = M_1 \oplus M_2$, and define $h : M \to X$ by $h|M_1 = f$ and $h|M_2 = g$. Then M is a metric space and h is a compact-covering and 1-sequence-covering map. In fact, for any compact subset K of X, $g : M_2 \to X$ is a compact-covering map. So there exists a compact subset L of M_2 such that $g(L) = K$. Since $L \subset M_2 \subset M$, we have $h(L) = g(L) = K$. For each $x \in X$, there exists an $\alpha \in f^{-1}(x) \cap M_1 \subset M$ such that whenever $\{x_n\}$ is a sequence in X converging to x, there exists a sequence $\{\alpha_n\}$ converging to α in M_1 with each $\alpha_n \in f^{-1}(x_n) \subset h^{-1}(x_n)$. Therefore, h is a compact-covering and 1-sequence-covering map. ■

S. Lin [11] posed a question as follow: For the first countable spaces whose each compact subset is metrizable, how to characterize them by certain images of metric spaces? We can answer this question by the following corollary.

Corollary 2.12. *The following are equivalent for a space X.*

(1) *X is a compact-covering and almost-open image of a metric space;*

(2) *X is a compact-covering image of a metric space, and an almost-open image of a metric space;*

(3) *X is a first-countable space whose every compact subset is metrizable.*

Proof. (1)$\Rightarrow$(2) is obvious.

(2)$\Rightarrow$(3). It is easy to check that the first countability is preserved by almost-open maps.

(3)$\Rightarrow$(1). In view of Theorem 2.11, the space X is a compact-covering and 1-sequence-covering image of a metric space. Thus, by Lemmas 2.5, 2.9 and 2.10, the space X is a compact-covering and almost-open image of a metric space. ■

By Lemma 2.5, Corollary 2.7 and Theorem 2.11, we have the following corollary.

Corollary 2.13. *The following are equivalent for a space X.*

(1) *X is a compact-covering, 1-sequence-covering and quotient image of a metric space;*

(2) *X is a compact-covering image of a metric space, and a 1-sequence-covering and quotient image of a metric space;*

(3) *X is a gf-countable space whose each compact subset is metrizable.*

Finally, we give some examples to show some relations between compact-covering images of metric spaces and 1-sequence-covering images of metric spaces.

Example 2.14. There exists a non-snf-countable space whose each compact subset is metrizable.

Proof. Let X be the sequential fan S_ω [10, Example 3.1.8]. Then X is a Fréchet space, which is not first-countable. Thus X is not snf-countable. Since each compact subset of X is countable and X is T_2, it is easy to check that each compact subset of X is metrizable. ■

Example 2.15. There is an snf-countable space whose each compact subset is metrizable, but it is not a k-space.

Proof. Put $X = N \cup \{p\}$, $p \in \beta N - N$. Endow X with the subspace topology of Stone-Čech βN. Since each compact subset of X is finite and X is T_2, each compact subset of X is metrizable. Since any convergent sequence in βN is trivial, the space βN is snf-countable. Then X is snf-countable. Since N is not a closed subspace of X, it is obvious that X is not a k-space. ■

We give a new definition for the following Example.

Let A be a non-empty subset in X. A countable family $\{V_n\}_{n\in N}$ of subsets of X is called a countable *sn-network* of A in X if it satisfies that:

(1) for each open set V in X with $A \subset V$, there is an $n \in N$ such that $A \subset V_n \subset V$;

(2) for each $n \in N, V_n$ is a sequential neighborhood of each point in A.

EXAMPLE 2.16. There is a first-countable space X whose each compact subset is metrizable, but some compact subset of X does not have a countable sn-network in X.

Proof. Let X be the butterfly space [14]. Then X is first-countable such that all compact subsets of X are metrizable, and the compact subset $I \times \{0\}$ in X doesn't have a countable neighborhood base. Since X is first-countable, each sequential neighborhood of a point $x \in X$ is a neighborhood of x in X, which implies that $I \times \{0\}$ doesn't have a countable sn-network in X. ∎

3. Compact-covering and 2-sequence-covering images

In this section, we mainly discuss some characterizations of images of metric spaces by compact-covering and 2-sequence-covering maps.

DEFINITION 3.1. [8] Let $f : X \to Y$ be a map. f is a 2-*sequence-covering map*, if for each $y \in Y$ and each $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to $y \in Y$, there is a sequence $\{x_n\}$ converging to $x \in X$ with each $x_n \in f^{-1}(y_n)$.

Obviously, we have the following relations:

2-sequence-covering maps $\Rightarrow$ 1-sequence-covering maps $\Rightarrow$ sequence-covering maps.

DEFINITION 3.2. [10] Let A be a non-empty subset of a space X. A countable family $\{V_n\}_{n \in N}$ of open subsets of X is called a countable neighborhood base of A in X if for each open set V in X with $A \subset V$, there is an $n \in N$ such that $K \subset V_n \subset V$.

DEFINITION 3.3. [16] Suppose that A is a non-empty subset of a space X, and $\mathcal{B}$ is a family of open subsets of X. The family $\mathcal{B}$ is called an *outer base* of A in X if for each $x \in A$ and an open set V in X with $x \in V$, there exists $B \in \mathcal{B}$ such that $x \in B \subset V$.

Similarly, we have the following two definitions.

DEFINITION 3.4. Let A be a non-empty subset of a space X. A countable family $\{V_n\}_{n \in N}$ of subsets of X is called a countable *so-network* of A in X if it satisfies that:

(1) for each open set V in X with $A \subset V$, there is an $n \in N$ such that $K \subset V_n \subset V$;

(2) for each $n \in N, V_n$ is a sequentially open set in X.

DEFINITION 3.5. Suppose A is a non-empty subset of a space X and $\mathcal{B}$ is a family of subsets of X, $\mathcal{B}$ is called an *outer so-network* of A in X if it satisfies that:

(1) each element of $\mathcal{B}$ is a sequentially open set in X;

(2) for each $x \in A \cap V$ and V is open in X, there exists $B \in \mathcal{B}$ such that $x \in B \subset V$.

Lemma 3.6. [8] *Let $f : X \to Y$ be a map and $\{y_n\}$ be a sequence converging to $y \in Y$. If $\{B_m\}_{m \in N}$ is a decreasing network at some point $x \in f^{-1}(y)$ in X, and $\{y_n\}$ is eventually in $f(B_m)$ for every $m \in N$, then there is a sequence $\{x_n\}$ converging to x such that each $x_n \in f^{-1}(y_n)$.*

Lemma 3.7. [9] *A space X is Fréchet if and only if for each $x \in X$, each sequential neighborhood of x in X is a neighborhood of x in X.*

Lemma 3.8. [9] *Let $f : X \to Y$ be a map. If X is first-countable, then f is an open map if and only if f is a 2-sequence-covering and quotient map.*

Now, we recall the concepts of Ponomarev's system and the CC-property.

Assume that $\mathcal{P}$ is a network of a space X. Put $\mathcal{P} = \{P_\alpha\}_{\alpha \in \Lambda}$, and endow Λ with the discrete topology. Put

$$M = \{\alpha = (\alpha_i) \in \Lambda^\omega : \{P_{\alpha_i}\}_{i \in N} \text{ is a network at some } x_\alpha \in X\}.$$

Then M, which is a subspace of the Tychonoff product space Λ^ω, is a metric space. Define a function $f : M \to X$ by $f(\alpha) = x_\alpha$. Then $f(\alpha) \in \bigcap_{i \in N} P_{\alpha_i}$, and f is well defined. $(f, M, X, \mathcal{P})$ is called a *Ponomarev's system* [13].

Let K be a subset of X. $\mathcal{F}$ is called a *cfp-covering* [19] of K, if $\mathcal{F}$ is a cover of K in X such that it can be precisely refined by some finite cover of K consisting of closed subsets of K.

Let $\mathcal{P}$ be a collection of subsets of X, and K be a subset of X. We say that $\mathcal{P}$ has the *CC-property* [13] on K, if whenever C is a non-empty compact subset of K, and V a neighborhood of C in X, then there exists a subset $\mathcal{F}$ of $\mathcal{P}$ such that $\mathcal{F}$ is a cfp-cover of C and $\bigcup \mathcal{F} \subset V$.

Lemma 3.9. [10] *Suppose $(f, M, X, \mathcal{P})$ is a Ponomarev's system. If K is a compact subset of X, and there exists a countable subfamily $\mathcal{P}_K$ of $\mathcal{P}$ such that $\mathcal{P}_K$ has the CC-property on K, then there is a compact subset L of M such that $f(L) = K$.*

Lemma 3.10. *Suppose each compact subset of a space X is metrizable, and K is a subset of X. If $\mathcal{B}$ is an outer so-network of K in X, then $\mathcal{B}$ has the CC-property on K.*

Proof. Let $H \subset K \cap V$, where H is a compact subset of K and V is a neighborhood of H in X. If $x \in H$, then there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset V$. By Lemma 3.7 and the metrizability of H, we have

$$x \in \operatorname{int}_H(B_x \cap H) \subset B_x \cap H \subset B_x.$$

Since H is regular, there is an open set V_x of H such that

$$x \in V_x \subset \operatorname{cl}_H(V_x) = \overline{V}_x \subset \operatorname{int}_H(B_x \cap H) \subset B_x.$$

Since $\{V_x\}_{x \in H}$ is an open cover of H, it has a finite subcover $\{V_{x_i}\}_{i \leq n}$ such that $H = \bigcup_{i \leq n} \overline{V}_{x_i} \subset \bigcup_{i \leq n} B_{x_i} \subset V$ and $\{B_{x_i}\}_{i \leq n}$ is precisely refined by $\{\overline{V}_{x_i}\}_{i \leq n}$. Therefore, the family $\mathcal{B}$ has the CC-property on K. ∎

Lemma 3.11. *Assume that every compact subset of a space X has a countable outer so-network, then X is a compact-covering and 2-sequence-covering image of a metric space.*

Proof. Let $\mathcal{B}_K$ be a countable outer *so*-network of a compact subset K in X. Let $\mathcal{B} = \bigcup\{\mathcal{B}_K : K \text{ is a compact subset of } X\}$, and denote $\mathcal{B} = \{B_\alpha : \alpha \in A\}$. Let $(f, M, X, \mathcal{B})$ be the Ponomarev's system. Since $\mathcal{B}_K|_K$ is a countable network of K, the compact subset K is metrizable. In view of Lemmas 3.9 and 3.10, it is easy to see that $f : X \to Y$ is a compact-covering map. In the following we shall prove that f is a 2-sequence-covering map.

For each $x \in X$ and $\beta = (\alpha_i) \in f^{-1}(x)$, $\{B_{\alpha_i}\}_{i\in N} \subset \mathcal{B}$ is an *sn*-network of x in X. For each $n \in N$, put $C_n = \{(\gamma_i) \in M : \gamma_i = \alpha_i \text{ whenever } i \le n\}$. Then $\{C_n : n \in N\}$ is a decreasing neighborhood base of β in M. Then $f(C_n) = \bigcap_{i\le n} B_{\alpha_i}$ for $n \in N$.

In fact, if $\gamma = (\gamma_i) \in C_n$, then $f(\gamma) \in \bigcap_{i\in N} B_{\gamma_i} \subset \bigcap_{i\le n} B_{\alpha_i}$ and $f(C_n) \subset \bigcap_{i\le n} B_{\alpha_i}$. Suppose that $z \in \bigcap_{i\le n} B_{\alpha_i}$. Choose a countable subfamily $\{B_{\delta_i}\}_{i\in N}$ of $\mathcal{B}$ such that

(1) $\delta_i = \alpha_i$ whenever $i \le n$;

(2) $\{B_{\delta_i}\}_{i\in N}$ is a network at z in X.

Put $\delta = (\delta_i) \in A^\omega$. Then $\delta \in C_n$ and $z = f(\delta) \in f(C_n)$. Thus $\bigcap_{i\le n} B_{\alpha_i} \subset f(C_n)$. So $f(C_n) = \bigcap_{i\le n} B_{\alpha_i}$.

Assume that $\{x_j\}$ is a sequence in X converging to the point x. Since $f(C_n)$ is a sequential neighborhood of x in X, by Lemma 3.6, there exists a sequence $\{\beta_j\}$ with each $\beta_j \in f^{-1}(x_j)$ such that $\{\beta_j\}$ converges to β. Therefore, the map f is compact-covering and 2-sequence-covering. ■

Lemma 3.12. *If a space X is a compact-covering and 2-sequence-covering image of a metric space, then every compact subset of X is metrizable and has a countable so-network in X.*

Proof. Assume that $f : M \to X$ is a compact-covering and 2-sequence-covering map. For each compact subset K of X, there exists a compact subset L in M such that $f(L) = K$. Suppose that $\{V_n\}_{n\in N}$ is a decreasing countable open neighborhood base of L in M. In the following we shall prove that $\{f(V_n)\}_{n\in N}$ is a countable *so*-network of K in X.

(1) For each $n \in N$, $K = f(L) \subset f(V_n)$ by $L \subset V_n$. Assume that U is an open set of X with $K \subset U$, then $L \subset f^{-1}(K) \subset f^{-1}(U) \subset M$. There exists a $k \in N$ such that $L \subset V_k \subset f^{-1}(U)$, so $K = f(L) \subset f(V_k) \subset U$.

(2) For each $y \in f(V_n)$, there is $x_y \in V_n$ such that $f(x_y) = y$. Since f is a 2-sequence-covering map, for each sequence $\{y_i\}$ converging to y, there is $x_i \in f^{-1}(y_i)$ such that the sequence $\{x_i\}$ converges to $x_y \in V_n$. Therefore, $\{x_i\}$ is eventually in V_n and $\{y_i\}$ is eventually in $f(V_n)$, that is, $f(V_n)$ is a sequentially open set in X. By Lemma 2.8, every compact subset of X is metrizable. ■

Lemma 3.13. *If each compact subset of a space X is metrizable and has a countable so-network in X, then it has a countable outer so-network in X.*

Proof. Assume that K is a metrizable and compact subset of X. Then K has a countable base $\{U_n\}_{n\in N}$ in K. Suppose that $\{V_n\}_{n\in N}$ is a countable *so*-network of K in X. Let

$$A = \{(n,m) \in N^2 : \overline{U}_m \subset U_n\}.$$

For each $(n,m,k) \in A \times N$, $\overline{U}_m \cap (K \setminus U_n) = \varnothing$ by $\overline{U}_m \subset U_n$.

There exists an open set $U_{n,m}$ in X such that

$$\overline{U}_m \subset U_{n,m} \subset \overline{U}_{n,m} \subset X \setminus (K \setminus U_n).$$

Put $W(n,m,k) = U_{n,m} \cap V_k$, $\Lambda = \{F : F \text{ is a finite subset of } A \times N\}$ and $\mathcal{H} = \{\cap\{W(n,m,k) : (n,m,k) \in F\}\}_{F\in\Lambda}$. Then $\mathcal{H}$ is a countable family of subsets of X. For each $x \in K$, define $B_x = \{\alpha \in A\times N : x \in W(\alpha)\}$ and $H(F) = \cap\{W(\alpha) : \alpha \in F\}$ with $F \subset B_x$.

Let $\mathcal{H}_x = \{H(F) : F \subset B_x \text{ and } F \text{ is finite}\}$. Then $\mathcal{H} = \bigcup_{x\in K} \mathcal{H}_x$. In the following we shall prove that $\mathcal{H}$ is an outer *so*-network of K in X.

(1) For any $H(F_1), H(F_2) \in \mathcal{H}_x$, we can obtain that $F_1 \subset B_x, F_2 \subset B_x$ and F_1, F_2 are both finite by the definition of $\mathcal{H}_x$. Let $F = F_1 \cup F_2$. Then $H(F) \in \mathcal{H}_x$ and $H(F) \subset H(F_1) \cap H(F_2)$.

(2) Suppose that U is an open neighborhood of x in X and there doesn't exist any finite subset F of B_x such that $x \in H(F) \subset U$. Choose a point $p(F) \in H(F)\setminus U$. Put

$$Q(F) = \{p(F') : F' \text{ is a finite subset of } B_x \text{ and } F \subset F'\}.$$

Then $U \cap Q(F) = \varnothing$ and $K \cap \overline{Q(F)} \neq \varnothing$. Otherwise, there exists a $k \in N$ such that $V_k \cap \overline{Q(F)} = \varnothing$. Because K is a regular space and $\{U_n\}_{n\in N}$ is a base of K, there exists $(n,m) \in N^2$ such that $x \in \overline{U}_m \subset U_n$. Denote $\alpha = (n,m,k)$, $F' = F \cup \{\alpha\}$, then $\alpha \in B_x$ and $p(F') \in W(\alpha) \cap Q(F) \subset V_k \cap Q(F) = \varnothing$, which is a contradiction.

If $F_1 \subset F_2$, then $Q(F_2) \subset Q(F_1)$ and $\{K \cap \overline{Q(F)} : F \text{ is a finite subset of } B_x\}$ is closed under finite intersections. Since K is a compact set, $K \cap (\cap\{\overline{Q(F)} : F$ is a finite subset of $B_x\}) \neq \varnothing$.

On the other hand, for any $y \in K\setminus\{x\}$, since K is a regular space, there exists $(n,m) \in N^2$ such that $x \in U_m \subset \overline{U}_m \subset U_n \subset K \setminus \{y\}$, thus $\overline{U}_{n,m} \subset X \setminus (K \setminus U_n) \subset X \setminus \{y\}$. For any $k \in N$, let $\alpha = (n,m,k)$, then $\alpha \in B_x$ and $y \notin \overline{U}_{n,m}$.

Since $Q(\{\alpha\}) = \{p(F') : F' \text{ is a finite subset of } B_x \text{ and } \alpha \in F'\} \subset H(\{\alpha\}) = W(\alpha) \subset U_{n,m}$, $y \notin \overline{Q(\{\alpha\})}$, $(K\setminus\{x\}) \cap (\cap\{\overline{Q(F)} : F$ is a finite subset of $B_x\}) = \varnothing$ and $\cap\{K \cap \overline{Q(F)} : F \text{ is a finite subset of } B_x\} = \{x\} \subset U$. Because K is compact, there exists a finite subset F of B_x such that $x \in K \cap \overline{Q(F)} \subset U$, and thus $U \cap Q(F) \neq \varnothing$, which is a contradiction. Therefore, there exists a finite subset F of B_x such that $H(F) \subset U$ and $x \in H(F) \subset U$.

In a word, we have proved that $\mathcal{H}_x$ is a network at x in X.

(3) For any $H(F) \in \mathcal{H}_x$, $H(F) = \cap\{W(\alpha) : \alpha \in F\}$, so we only need to prove that $W(\alpha)$ is a sequentially open set in X. By the definition of $W(\alpha)$, we have $x \in W(\alpha) = U_{n,m} \cap V_k$ in which $U_{n,m}$ is an open set of X and V_k is a sequentially open set of K in X. Therefore, $W(\alpha)$ is a sequentially open set in X, that is, $H(F)$ is a sequentially open set in X.

By (1)$\sim$(3), we can obtain that if each compact subset of X is metrizable and has a countable *so*-network in X, then it has a countable outer *so*-network in X. ■

The following Theorem holds by Lemmas 3.11, 3.12 and 3.13.

THEOREM 3.14. *The following are equivalent for a space X.*

(1) *X is a compact-covering and 2-sequence-covering image of a metric space;*

(2) *Each compact subset of X is metrizable and has a countable so-network in X;*

(3) *Every compact subset of X has a countable outer so-network in X.*

Theorem 3.14 extends a result of Michael and Nagami [16], see Corollary 3.15.

COROLLARY 3.15. [16] *The following are equivalent for a space X.*

(1) *X is a compact-covering and open image of a metric space;*

(2) *Each compact subset of X is metrizable and has a countable neighborhood base in X;*

(3) *Every compact subset of X has a countable outer base in X.*

Finally, we give some examples to show the relations between compact-covering images of metric spaces and 2-sequence-covering images of metric spaces.

EXAMPLE 3.16. There exists a space X whose every compact subset is metrizable and has a countable *sn*-network in X, but X is not a compact-covering and 2-sequence-covering image of any metric space.

Proof. Let $S_1 = \{0\} \cup \{\frac{1}{n} : n \in N\}$ and $I = [0,1]$. Define $X = I \times S_1$ and $Y = I \times (S_1 - \{0\})$.

Endow X with the following topology [7]: Y has the usual Euclidean topology as a subspace of X. Define a typical neighborhood of $(t,0) \in X$ to be of the form

$$\{(t,0)\} \cup (\cup\{V(t,k) : k \geq n\}), n \in N;$$

where $V(t,k)$ is a open neighborhood of $(t,\frac{1}{k})$ in $I \times \{\frac{1}{k}\}$.

Put $M = (\oplus\{I \times \{\frac{1}{n} : n \in N\}) \oplus (\oplus\{\{t\} \times S_1 : t \in I\})$ and define f from M onto X such that f is an obvious mapping. Then f is a compact-covering, 1-sequence-covering, quotient, and two-to-one mapping from the locally compact metric M onto X [9, Example 1.5.4].

In the following we shall prove that each compact subset of X has a countable *sn*-network in X. Suppose that K is a compact subset of X. Since $I \times \{0\}$ is a closed discrete subspace of X and any compact subset of a discrete space is finite, we have $K \cap (I \times \{0\})$ is finite, denote it by $\{(t_i,0)\}_{i \leq m}$. Put $K_0 = \bigcup_{i \leq m}(\{t_i\} \times S_1)$.

If $K \setminus (K_0 \cup (\bigcup_{j\le n}(I \times \{\frac{1}{j}\}))) \neq \varnothing$ for each $n \in N$, then there exist a sequence $\{j_n\} \subset N$ and $\{x_n\} \subset X$ such that $j_m \neq j_n$ if $m \neq n$, $j_n \to +\infty$ and $x_n \in K \cap (I \times \{\frac{1}{j_n}\}) \setminus K_0$ for each $n \in N$. Suppose that x is an accumulation point of $\{x_n\}$ in K. Then $x \in K \cap (I \times \{0\})$. Thus, there exists $i \le m$ such that $x = (t_i, 0)$. For each $n \in N$, we choose an open neighborhood $V(t_i, j_n)$ of $(t_i, \frac{1}{j_n})$ in $I \times \{\frac{1}{j_n}\}$ such that $x_n \notin V(t_i, j_n)$. For each $k \in N \setminus \{j_n : n \in N\}$, put $V(t_i, k) = I \times \{\frac{1}{k}\}$. Define

$$W = \{x\} \cup (\bigcup_{k\in N} V(t_i, k)),$$

then $x \in W$ and W is open in X and $x_n \notin W$, which is a contradiction.

Therefore, there exists an $n \in N$ such that $K \subset K_0 \cup (\bigcup_{j\le n}(I \times \{\frac{1}{j}\}))$, i.e.,

$$K = (\bigcup_{j\le n}((I \times \{\frac{1}{j}\}) \cap K)) \cup (\bigcup_{i\le m}((\{t_i\} \times S_1) \cap K)).$$

Since $I \times \{\frac{1}{j}\}$ is a closed subspace of X and K is a compact subset of X, $(I \times \{\frac{1}{j}\}) \cap K$ is a compact subset of $I \times \{\frac{1}{j}\}$. Thus $(I \times \{\frac{1}{j}\}) \cap K$ has a countable neighborhood base $\{V_n\}_{n\in N}$ in $I \times \{\frac{1}{j}\}$. For each $(t_i, \frac{1}{j}) \in (I \times \{\frac{1}{j}\}) \cap K$, if there is a sequence $\{x_n\}$ converging to $(t_i, \frac{1}{j})$ in X, then $\{x_n\}$ is eventually in an open neighborhood $V(t_i, j)$ of $(t_i, \frac{1}{j})$ in $I \times \{\frac{1}{j}\}$. Therefore, it is easy to check that $\{V_n\}_{n\in N}$ is a countable sn-network of $(I \times \{\frac{1}{j}\}) \cap K$ in X. Similarly, it is easy to prove that $(\{t_i\} \times S_1) \cap K$ has a countable sn-network in X. Hence, K has a countable sn-network in X.

Since $(0, 0) \in \overline{(I - \{0\}) \times S_1}$ and for any sequence $\{x_n\}$ converging to (0,0) in X, there are at most finitely many elements which are not in $\{0\} \times S_1$, every sequence in $(I - \{0\}) \times S_1$ can't converge to (0,0). Thus, X is not a Fréchet space. Since sequential spaces are preserved by quotient maps, X is a sequential space. If X is a compact-covering and 2-sequence-covering image of a metric space, by Lemma 3.8, f is an open map. Hence, X is a first-countable space, but X is not a Fréchet space, which is a contradiction. ∎

Example 3.17. A compact-covering and 2-sequence-covering image of a metric space need not to be a quotient image of a metric space.

Proof. Let $X = N \cup \{p\}$, $p \in \beta N \backslash N$. Endow X with discrete topology. Then X is a metric space. Put $Y = N \cup \{p\}$, endow Y with the subspace topology of βN.

Define $f : X \to Y$ by $f(x) = x$ for each $x \in X$. Since each compact subset of Y is finite, f is a compact-covering map. It is easy to check that f is a 2-sequence-covering map. Since Y is not a k-space, Y is not a quotient image of any metric space. ∎

Acknowledgements. The author would like to thank Professor Lin Shou for the very constructive and valuable comments. The author is also very grateful to the referee for his helpful comments.

REFERENCES

[**1**] A.V. Arhangel'skiĭ, *On open and almost-open mappings of topological spaces* (in Russian), Dokl. Akad. Nauk SSSR. **147** (1962), 999–1002.

[**2**] A.V. Arhangel'skiĭ, *Some type of quotient mappings and the relations between classes of topological spaces* (in Russian), Dokl. Akad. Nauk SSSR. **153** (1963), 743–746.

[**3**] A.V. Arhangel'skiĭ, *Mappings and spaces*, Russian. Math. Survey **21** (1966), 115–162.

[**4**] S.P. Franklin, *Spaces in which sequences suffice*, Fund. Math. **57** (1965), 107–115.

[**5**] D. Gale, *Compact sets of functions and function rings*, Proc. Amer. Math. Soc. **1** (1950), 303–308.

[**6**] Y. Ge, *Weak forms of open mappings and strong forms of sequence-covering mappings*, Mat. Vesnik **59** (2007), 1–8.

[**7**] G. Gruenhage, E.A. Michael, Y. Tanaka, *Spaces determined by point-countable covers*, Pacific J. Math. **113** (1984), 303–332.

[**8**] S. Lin, *On sequence-covering s-mapping*, Adv. Math. (China) **25** (1996), 548–551.

[**9**] S. Lin, *Point-Countable Covers and Sequence-Covering Mappings*, Beijing: Chinese Science Press, 2002 (in Chinese).

[**10**] S. Lin, *The Topology of Metric Spaces and Function Spaces*, Beijing: Chinese Science Press, 2004 (in Chinese).

[**11**] S. Lin, *Generalized Metric Spaces and Mappings*, Beijing: Chinese Scientific Publishers, 2007 (in Chinese).

[**12**] S. Lin, P. Yan, *On sequence-covering compact mapping*, Acta. Math. Sinica. **44** (2001), 175–182.

[**13**] S. Lin, P. Yan, *Note on cfp-covers*, Comment. Math. Univ. Carolinae **44** (2003), 295–306.

[**14**] L.F. McAuley, *A relation between perfect separability and normality in semimetric spaces*, Pacific J. Math. **6** (1956), 315–326.

[**15**] E. Michael, *$\aleph_0$-spaces,* J. Math. Mech. **5** (1966), 983–1002.

[**16**] E. Michael, K. Nagami, *Compact-covering images of metric space*, Proc. Amer. Math. Soc. **37** (1973), 260–266.

[**17**] F. Siwiec, *Sequence-covering and countably bi-quotient maps*, General. Topology. Appl. **1** (1971), 143–154.

[**18**] F. Siwiec, *On defining a space by a weak base*, Pacific J. Math. **52** (1974), 233–245.

[**19**] P. Yan, *The compact images of metric spaces*, J. Math. Study **30** (1997), 185–187.

(received 08.10.2010; in revised form 13.12.2010; available online 20.01.2011)

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