МАТЕМАТИЧКИ ВЕСНИК
64, 1 (2012), 33–38
March 2012

оригинални научни рад
research paper

INEQUALITIES INVOLVING A CLASS OF ANALYTIC FUNCTIONS

Rabha W. Ibrahim

Abstract. In the present paper, we define a new class of normalized analytic functions based on the Φ-like type. Some inequalities are introduced based on the concept of the subordination in the unit disk. The convexity techniques are used to obtain the main result. Some applications are posed on classes of functions such as starlike functions, convex functions, starlike and convex functions of complex order.

1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the unit disk $U = \{z : |z| < 1\}$ and for $a \in \mathbb{C}$ (set of complex numbers) and $n \in \mathbb{N}$ (set of natural numbers), let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}$ be the class of functions f, analytic in U and normalized by the conditions $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}$ is called starlike of order μ if it satisfies the following inequality

$$\Re\Big\{\frac{zf'(z)}{f(z)}\Big\} > \mu, \quad (z \in U)$$

for some $0 \le \mu < 1$. We denoted this class $\mathcal{S}(\mu)$. A function $f \in \mathcal{A}$ is called convex of order μ if it satisfies the following inequality

$$\Re\Big\{\frac{zf''(z)}{f'(z)} + 1\Big\} > \mu, \quad (z \in U)$$

for some $0 \le \mu < 1$. We denoted this class $\mathcal{C}(\mu)$. We note that $f \in \mathcal{C}(\mu)$ if and only if $zf' \in \mathcal{S}(\mu)$.

Let f be analytic in U, g analytic and univalent in U and $f(0) = g(0)$. Then, by the symbol $f(z) \prec g(z)$ (f subordinate to g) in U, we shall mean $f(U) \subset g(U)$.

Let $\phi\colon \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z)), zp'(z)) \prec h(z)$ then p is called a solution of

2010 AMS Subject Classification: 30C99, 30C45.

Keywords and phrases: Univalent functions; Starlike functions; Convex functions; Differential subordination; Superordination; Unit disk; Starlike and convex functions of complex order; Φ-like functions; Convexity techniques.

the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, $p \prec q$. If p and $\phi(p(z)), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \phi(p(z)), zp'(z))$ then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$. We use the notation $s(g) = \{f \in \mathcal{H}(U) : f \prec g\}$, for details (see [1]).

The function $f \in \mathcal{A}$ is called Φ-like if

$$\Re\Big\{\frac{zf'(z)}{\Phi(f(z))}\Big\} > 0, \ \ z \in U.$$

This concept was introduced by Brickman [2] and established that a function $f \in \mathcal{A}$ is univalent if and only if f is Φ-like for some Φ.

Definition 1. Let Φ be analytic function in a domain containing $f(U)$, $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(\omega) \neq 0$ for $\omega \in f(U) - 0$. Let $q(z)$ be a fixed analytic function in U, $q(0) = 1$. The function $f \in \mathcal{A}$ is called Φ-like with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z), \ z \in U.$$

Ruscheweyh [3] investigated this general class of Φ-like functions.

Let X be a locally convex linear topological space. For a subset $U \subset X$ the closed convex hull of U is defined as the intersection of all closed convex sets containing U and will be denoted by $co(U)$. If $U \subset V \subset X$ then U is called an extremal subset of V provided that whenever $u = tx + (1-t)y$ where $u \in U$, $x, y \in V$ and $t \in (0,1)$ then $x, y \in U$. An extremal subset of U consisting of one point is called an extreme point of U. The set of the extreme points of U will be denoted by $\mathcal{E}(U)$.

Remark 1. If $L\colon \mathcal{H}(U) \to \mathcal{H}(U)$ is an invertible linear map and $F \subset \mathcal{H}(U)$ is a compact subset, then $L(co(F)) = co(L(F))$ and the set $\mathcal{E}(co(F))$ is in one-to-one correspondence with $\mathcal{E}L(co(F))$.

In the present paper, we estimate the upper and lower bound for functions $f \in \mathcal{A}$ in the class $\mathcal{A}^p_m(\alpha_1, \dots, \alpha_p; \Phi_1, \dots, \Phi_p)$, $p \geq 1$, $m \in \mathbb{N}$ which satisfy the condition

$$\Re\Big\{\sqrt{[}m]1 + \alpha_1\frac{zf'(z)}{\Phi_1(f(z))} + \alpha_2\frac{zf'(z)}{\Phi_2(f(z))} + \cdots + \alpha_p\frac{zf'(z)}{\Phi_p(f(z))} - (\alpha_1 + \cdots + \alpha_p)\Big\} > 0, \tag{1}$$

where α_j, $j = 1, \dots, p$, are complex numbers and Φ_j satisfies Definition 1. In order to obtain our results, we need the following technical lemmas.

Lemma 1. [4] *Suppose that F_α is defined by the equality*

$$F_\alpha(z) = \Big(\frac{1+cz}{1-z}\Big)^\alpha, \quad (|c| \leq 1,\, c \neq -1).$$

If $\alpha \geq 1$ then $co(s(F^{\alpha}))$ consists of all functions in $\mathcal{H}(U)$ represented by

$$f(z) = \int_0^{2\pi} \Big(\frac{1+cze^{-it}}{1-ze^{-it}}\Big)^{\alpha}\, d\mu(t),$$

where μ is a positive measure on $[0, 2\pi]$ having the property $\mu([0, 2\pi]) = 1$ and

$$\mathcal{E}(co(s(F^{\alpha}))) = \Big\{\frac{1+cze^{-it}}{1-ze^{-it}} \;\Big|\; t \in [0, 2\pi]\Big\}.$$

Lemma 2. [4] *If $J\colon \mathcal{H}(U) \to \mathbb{R}$ is a real-valued, continuous convex functional and $\mathcal{F}$ is a compact subset of $\mathcal{H}(U)$, then*

$$\max\{J(f) : f \in co(\mathcal{F})\} = \max\{J(f) : f \in \mathcal{F}\} = \max\{J(f) : f \in \mathcal{E}(co(\mathcal{F}))\}.$$

In the particular case if J is a linear map then we also have:

$$\min\{J(f) : f \in co(\mathcal{F})\} = \min\{J(f) : f \in \mathcal{F}\} = \min\{J(f) : f \in \mathcal{E}(co(\mathcal{F}))\}.$$

Lemma 3. [5] *For $z \in U$ we have*

$$\Re\Big\{\sum_{n=1}^{j} \frac{z^n}{n+2}\Big\} > -\frac{1}{3}, \quad (z \in U).$$

2. The main result

Our main result is the following:

Theorem 1. *Let $f \in \mathcal{A}_m^p(\alpha_1, \dots, \alpha_p; \Phi_1, \dots, \Phi_p)$, $p \geq 1$, $m \in \mathbb{N}$ and α_j, $j = 1, \dots, p$, are complex numbers. Then*

$$1 + \inf_{z \in U} \Re\Big(\sum_{n=1}^{\infty} \frac{\sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1}}{Q(n)} z^n\Big) < \Re\Big(1 + \alpha_p\Big[\frac{zf'(z)}{\Phi_p(f(z))} - 1\Big]\Big)$$
$$< 1 + \sup_{z \in U} \Re\Big(\sum_{n=1}^{\infty} \frac{\sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1}}{Q(n)} z^n\Big), \quad (2)$$

where

$$Q(u) := 1 + \alpha_1 u + \alpha_2 u(u-1) + \dots + \alpha_p u(u-1) \cdot \ldots \cdot (u-p+1).$$

Proof. Since $f \in \mathcal{A}_m^p(\alpha_1, \dots, \alpha_p; \Phi_1, \dots, \Phi_p)$, we have

$$\sqrt{[m]}1 + \alpha_1 \frac{zf'(z)}{\Phi_1(f(z))} + \alpha_2 \frac{zf'(z)}{\Phi_2(f(z))} + \dots + \alpha_p \frac{zf'(z)}{\Phi_p(f(z))} - (\alpha_1 + \dots + \alpha_p) \prec \frac{1+z}{1-z}$$

which is equivalent to

$$1+\alpha_1\frac{zf'(z)}{\Phi_1(f(z))}+\alpha_2\frac{zf'(z)}{\Phi_2(f(z))}+\cdots+\alpha_p\frac{zf'(z)}{\Phi_p(f(z))}-(\alpha_1+\cdots+\alpha_p)\prec\Big(\frac{1+z}{1-z}\Big)^m.$$

In view of Lemma 1,

$$\begin{aligned}1+\alpha_1\frac{zf'(z)}{\Phi_1(f(z))}+\alpha_2\frac{zf'(z)}{\Phi_2(f(z))}+\cdots+\alpha_p\frac{zf'(z)}{\Phi_p(f(z))}-(\alpha_1+\cdots+\alpha_p)\\ =\int_0^{2\pi}\Big(\frac{1+ze^{-it}}{1-ze^{-it}}\Big)^{\alpha}d\mu(t):=h(z),\end{aligned}\tag{3}$$

where μ is a positive measure on $[0,2\pi]$ having the property $\mu([0,2\pi])=1$. Assume that

$$1+\alpha_p\Big[\frac{zf'(z)}{\Phi_p(f(z))}-1\Big]=1+\sum_{n=1}^{\infty}b_nz^n,\quad (z\in U).\tag{4}$$

Then

$$\begin{aligned}1+\alpha_1\frac{zf'(z)}{\Phi_1(f(z))}+\alpha_2\frac{zf'(z)}{\Phi_2(f(z))}+\cdots+\alpha_p\frac{zf'(z)}{\Phi_p(f(z))}-(\alpha_1+\cdots+\alpha_p)\\ =1+\sum_{n=1}^{\infty}b_nQ(n)z^n.\end{aligned}\tag{5}$$

On the other hand, we observe that

$$\int_0^{2\pi}\Big(\frac{1+ze^{-it}}{1-ze^{-it}}\Big)^{\alpha}d\mu(t)=1+\sum_{n=1}^{\infty}\Big[\sum_{k=0}^{n}C_m^kC_{m+n-k-1}^{m-1}\Big]z^n\int_0^{2\pi}e^{-int}\,d\mu(t)\tag{6}$$

such that $C_j^i=0$, $i>j$ and

$$\sum_{k=0}^{n}C_m^kC_{m+n-k-1}^{m-1}=\sum_{k=0}^{m}C_m^kC_{m+n-k-1}^{m-1}.\tag{7}$$

Thus (6) and (7) imply

$$1+\sum_{n=1}^{\infty}b_nQ(n)z^n=1+\sum_{n=1}^{\infty}\Big[\sum_{k=0}^{m}C_m^kC_{m+n-k-1}^{m-1}\Big]z^n\int_0^{2\pi}e^{-int}\,d\mu(t)$$

and

$$b_n=\frac{1}{Q(n)}\Big(\sum_{k=0}^{m}C_m^kC_{m+n-k-1}^{m-1}\Big)\int_0^{2\pi}e^{-int}\,d\mu(t).$$

Consequently,

$$1+\alpha_p[\frac{zf'(z)}{\Phi_p(f(z))}-1]=1+\sum_{n=1}^{\infty}\frac{1}{Q(n)}\Big(\sum_{k=0}^{m}C_m^kC_{m+n-k-1}^{m-1}\Big)z^n\int_0^{2\pi}e^{-int}\,d\mu(t).$$

If

$$\mathfrak{H}=\Big\{h\in\mathcal{H}(U):h(z)=\int_0^{2\pi}\Big(\frac{1+ze^{-it}}{1-ze^{-it}}\Big)^{\alpha}d\mu(t),\ \ \mu([0,2\pi])=1\Big\}$$

and

$$\mathfrak{Q} = \Big\{ q \in \mathcal{H}(U) : q(z) = 1 + \alpha_p [\frac{zf'(z)}{\Phi_p(f(z))} - 1], \;\; f \in \mathcal{A}_m^p(\alpha_1, \dots, \alpha_p; \Phi_1, \dots, \Phi_p) \Big\}$$

then the correspondence $L \colon \mathfrak{H} \to \mathfrak{Q}$, $L(h) = q(z)$ defines an invertible linear map and according to Remark 1, the extreme points of the class $\mathfrak{Q}$ are

$$q_t(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{Q(n)} \Big(\sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1} \Big) z^n e^{-int}, \quad (z \in U, t \in [0, 2\pi)).$$

Hence in view of Lemma 2 this implies the assertion of Theorem 1. ∎

3. Applications

THEOREM 2. *Let f be starlike in U. Then*

$$\frac{2}{3} < \Re \Big\{ \frac{zf'(z)}{f(z)} \Big\} < 2.$$

Proof. Since f is starlike, then $\Re\{\frac{zf'(z)}{f(z)}\} > 0$ implies that $f \in \mathcal{A}_1^1(1, 0, \dots, 0; f)$. Thus according to Theorem 1, we obtain the following inequalities:

$$1 + \inf_{z \in U} \Re \Big(\sum_{n=1}^{\infty} \frac{1}{n+1} z^n \Big) \le \Re \Big\{ \frac{zf'(z)}{f(z)} \Big\} \le 1 + \sup_{z \in U} \Re \Big(\sum_{n=1}^{\infty} \frac{1}{n+1} z^n \Big).$$

In virtue of Lemma 3 it yields

$$\begin{aligned} 1 + \inf_{z \in U} \Re \Big(\sum_{n=1}^{\infty} \frac{1}{n+1} z^n \Big) &\ge 1 + \inf_{z \in U} \Re \Big(\sum_{n=1}^{j} \frac{1}{n+1} z^n \Big) \\ &\ge 1 + \inf_{z \in U} \Re \Big(\sum_{n=1}^{j} \frac{1}{n+2} z^n \Big) > \frac{2}{3}. \end{aligned}$$

On the other hand, and by residue theory, we pose

$$1 + \sup_{z \in U} \Re \Big(\sum_{n=1}^{\infty} \frac{1}{n+1} z^n \Big) < 1 + \sup\nolimits_{z \in U} \Re \Big(\sum_{n=1}^{\infty} z^n \Big) = 1 + \sup\nolimits_{z \in U} \Re \Big(\frac{z}{1-z} \Big) = 2.$$

Hence the proof. ∎

THEOREM 3. *Assume that $f \in \mathcal{A}$ and satisfies*

$$\Re \Big(\frac{zf'(z)}{f(z)} \Big) > \frac{1}{2}. \tag{8}$$

Then

$$\frac{e^{2\pi} + 2\pi e^{\pi} - 1}{e^{2\pi} - 1} \le 1 + \Re \Big(\frac{zf'(z)}{f(z)} \Big) \le \frac{e^{2\pi}(1+\pi) - (1-\pi)}{e^{2\pi} - 1}.$$

The result is sharp.

Proof. Condition (8) implies that $f \in \mathcal{A}_1^2(1,1,0,\dots,0;f,f)$. Thus according to Theorem 1, we have the following inequalities:

$$1+\inf_{z\in U}\Re\Big(\sum_{n=1}^{\infty}\frac{2}{n^2+1}z^n\Big)\le \Re\Big\{\frac{zf'(z)}{f(z)}\Big\}\le 1+\sup_{z\in U}\Re\Big(\sum_{n=1}^{\infty}\frac{2}{n^2+1}z^n\Big).$$

The minimum and maximum principle for harmonic functions imply that

$$1+\inf_{z\in U}\Re\Big(\sum_{n=1}^{\infty}\frac{2}{n^2+1}z^n\Big)=1+\inf_{t\in[0,2\pi]}\Re\Big(\sum_{n=1}^{\infty}\frac{2}{n^2+1}e^{int}\Big)$$

and

$$1+\sup_{z\in U}\Re\Big(\sum_{n=1}^{\infty}\frac{2}{n^2+1}z^n\Big)=1+\sup_{t\in[0,2\pi]}\Re\Big(\sum_{n=1}^{\infty}\frac{2}{n^2+1}e^{int}\Big).$$

Again by using the residue theory we deduce that

$$1+\Re\Big(\sum_{n=1}^{\infty}\frac{2}{n^2+1}e^{int}\Big)=\frac{\pi(e^t+e^{2\pi-t})}{e^{2\pi}-1},\quad (t\in[0,2\pi])$$

consequently, we obtain the desired assertion. ■

In the same manner we have the following result:

THEOREM 4. *Assume that* $f \in \mathcal{A}_1^1(\frac{1}{b},0\dots,0;f)$, $\quad b\in\mathbb{C}\backslash\{0\}$. *Then*

$$1+\frac{1}{b}\Re\Big\{\frac{zf'(z)}{f(z)}-1\Big\}\ge 1+\sum_{k=1}^{\infty}\frac{(-1)^k\Re(b)}{\Re(b)+k}.$$

The result is sharp.

REFERENCES

[**1**] S.S. Miller, P.T. Mocanu, *Differential Subordinantions: Theory and Applications*, Pure and Applied Mathematics No.225 Dekker, New York, 2000.

[**2**] L. Brickman, *Φ-like analytic functions*, I, Bull. Amer. Math. Soc. **79** (1973), 555–558.

[**3**] St. Ruscheweyh, *A subordination theorem for Φ-like functions*, J. London Math. Soc. **13** (1976), 275–280.

[**4**] D.J. Hallenbeck, T.H. Mac Gregor, *Linear Problems and Convexity Techniques in Geometric Function Theory*, Pitman Advanced Publishing Program, 1984.

[**5**] J.M. Jahangiri, K. Farahmand, *Partial sums of functions of bounded turning*, Inter. J. Math. Math. Sci. **1** (2004), 45–47.

(received 13.09.2010; available online 20.01.2011)

School of Mathematical Sciences, Faculty of Sciences and Technology, UKM, Malaysia

E-mail: `rabhaibrahim@yahoo.com`