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# THE THEOREMS OF URQUHART AND STEINER-LEHMUS IN THE POINCARÉ BALL MODEL OF HYPERBOLIC GEOMETRY

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**Abstract.** In [Comput. Math. Appl. 41 (2001), 135–147], A.A. Ungar employs the Möbius gyrovector spaces for the introduction of the hyperbolic trigonometry. This A.A. Ungar's work, plays a major role in translating some theorems in Euclidean geometry to corresponding theorems in hyperbolic geometry. In this paper we present (i) the hyperbolic Breusch's lemma, (ii) the hyperbolic Urquhart's theorem, and (iii) the hyperbolic Steiner-Lehmus theorem in the Poincaré ball model of hyperbolic geometry by employing results from A.A. Ungar's work.

## 1. Introduction

This paper is inspired by the beautiful papers [9, 15] by A.A. Ungar on hyperbolic trigonometry. A.A. Ungar showed that the hyperbolic sine and the hyperbolic cosine rules are valid in the Poincaré ball model of hyperbolic geometry in a form analogous to their Euclidean counterparts. In this paper we shall apply hyperbolic trigonometry to the study of the hyperbolic Breusch's Lemma, the hyperbolic Urquhart's theorem and the hyperbolic Steiner-Lehmus theorem in the Poincaré ball model of hyperbolic geometry. In the Poincaré ball model, a gyroline (or, a hyperbolic line) is an Euclidean semicircular arc that intersects the boundary of the ball orthogonally.

## 2. Möbius transformations of the disc

In complex analysis Möbius transformations are well known and fundamental. The most general Möbius transformation of the complex open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex $z$-plane

$$z \mapsto e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta} (z_0 \oplus z)$$

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defines the Möbiüs addition $\oplus$ in the disc, allowing the Möbius transformation of the disc to be viewed as a Möbius *left gyrotranslation*

$$z \mapsto z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta$ is a real number, $z_0 \in \mathbb{D}$, and $\overline{z_0}$ is the complex conjugate of $z_0$. Möbius substraction "$\ominus$" is given by $a \ominus z = a \oplus (-z)$, clearly $z \ominus z = 0$ and $\ominus z = -z$. Möbius addition $\oplus$ is a binary operation in the disc $\mathbb{D}$, but clearly it is neither commutative nor associative. Möbius addition $\oplus$ gives rise to the groupoid $(\mathbb{D}, \oplus)$ studied by A.A. Ungar in several books including [8, 10, 13, 15, 16]. Möbius addition is analogous to the common vector addition $+$ in Euclidean plane geometry. Since Möbius addition $\oplus$ is not associative, the groupoid $(\mathbb{D}, \oplus)$ is not a group. However, it has a group-like structure that we present below.

The breakdown of commutativity in Möbius addition is "repaired" by the introduction of gyrator,

$$gyr : \mathbb{D} \times \mathbb{D} \to Aut(\mathbb{D}, \oplus)$$

which gives rise to gyrations,

$$gyr\,[a,b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b} \tag{1}$$

where $Aut(\mathbb{D}, \oplus)$ is the automorphism group of the groupoid $(\mathbb{D}, \oplus)$. Therefore, the *gyrocommutative law* of Möbius addition $\oplus$ follows from the definition of gyration in (1),

$$a \oplus b = gyr\,[a,b]\,(b \oplus a)\,. \tag{2}$$

Coincidentally, the gyration $gyr[a,b]$ that repairs the breakdown of the commutative law of $\oplus$ in (2), repairs the breakdown of the associative law of $\oplus$ as well, giving rise to the respective *left* and *right gyroassociative laws*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr\,[a,b]\,c$$
$$(a \oplus b) \oplus c = a \oplus (b \oplus gyr\,[b,a]\,c)$$

for all $a, b, c \in \mathbb{D}$.

DEFINITION 1. A groupoid $(\mathbb{G}, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms

| | | |
|---|---|---|
| $(G1)$ | $0 \oplus a = 0$ | left identity property |
| $(G2)$ | $\ominus a \oplus a = 0$ | left inverse property |
| $(G3)$ | $a \oplus (b \oplus c) = (a \oplus b) \oplus gyr\,[a,b]\,c$ | left gyroassociative law |
| $(G4)$ | $gyr\,[a,b] \in Aut(\mathbb{G}, \oplus)$ | gyroautomorphism |
| $(G5)$ | $gyr\,[a,b] = gyr\,[a \oplus b, b]$ | left loop property |

for all $a, b, c \in \mathbb{G}$.

Additionally, if the binary operation "$\oplus$" obeys the gyrocommutative law

$$(G6) \quad a \oplus b = gyr\,[a,b]\,(b \oplus a) \quad \text{gyrocommutative Law}$$

for all $a, b, c \in \mathbb{G}$, then $(\mathbb{G}, \oplus)$ is called a gyrocommutative gyrogroup. It is easy to see that $-a = \ominus a$ for all elements $a$ of $\mathbb{G}$.

Clearly, with these properties, one can now readily check that the Möbius complex disc groupoid $(\mathbb{D}, \oplus)$ is a gyrocommutative gyrogroup.

The axioms in Definition 1 imply the right identity property, the right inverse property, the right gyroassociative law and the right loop property. We refer readers to [8, 10, 13, 15, 16] for more details about gyrogroups.

## 3. Möbius gyrogroups: From disc to the ball

Let us identify complex numbers of the complex plane $\mathbb{C}$ with vectors of the Euclidean plane $\mathbb{R}^2$ in the usual way:

$$\mathbb{C} \ni u = u_1 + iu_2 = (u_1, u_2) = \mathbf{u} \in \mathbb{R}^2.$$

Then the equations

$$\mathbf{u} \cdot \mathbf{v} = \mathrm{Re}(\overline{u}v), \qquad \|\mathbf{u}\| = |u|\,. \tag{3}$$

give the inner product and the norm in $\mathbb{R}^2$, so that Möbius addition in the disc $\mathbb{D}$ of $\mathbb{C}$ becomes Möbius addition in the disc $\mathbb{R}^2_1 = \left\{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\| < 1\right\}$ of $\mathbb{R}^2$. Indeed, we get from (3) that

$$\begin{aligned} u \oplus v &= \frac{u+v}{1+\bar{u}v} = \frac{(1+u\bar{v})(u+v)}{(1+\bar{u}v)(1+u\bar{v})} \\ &= \frac{(1+\bar{u}v+u\bar{v}+|v|^2)u + (1-|u|^2)v}{1+\bar{u}v+u\bar{v}+|u|^2|v|^2} \\ &= \frac{(1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2)\mathbf{u}+(1-\|\mathbf{u}\|^2)\mathbf{v}}{1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2\|\mathbf{v}\|^2} = \mathbf{u} \oplus \mathbf{v} \end{aligned} \tag{4}$$

for all $u, v \in \mathbb{D}$ and all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2_1$.

## 4. Möbius addition in the ball

Let $\mathbb{V}$ be any real inner-product space and

$$\mathbb{V}_s = \{v \in \mathbb{V} : \|v\| < s\}$$

be the open ball of $\mathbb{V}$ with radius $s > 0$. Möbius addition in $\mathbb{V}_s$ is motivated by (4). It is given by the equation

$$\mathbf{u} \oplus \mathbf{v} = \frac{(1 + (2/s^2)\mathbf{u}\cdot\mathbf{v} + (1/s^2)\|\mathbf{v}\|^2)\mathbf{u} + (1 - (1/s^2)\|\mathbf{u}\|^2)\mathbf{v}}{1 + (2/s^2)\mathbf{u}\cdot\mathbf{v} + (1/s^4)\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{5}$$

where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{V}_s$ inherits from its space $\mathbb{V}$ and where, ambiguously, $+$ denotes both addition of real numbers on the real line and addition of vectors in $\mathbb{V}$.

Without loss of generality, we may assume that $s = 1$ in (5). However we prefer to keep $s$ as a free positive parameter in order to exhibit the results that in the limit as $s \to \infty$, when the ball $\mathbb{V}_s$ expands to the whole of its real inner product space $\mathbb{V}$, and Möbius addition $\oplus$ reduces to vector addition $+$ in $\mathbb{V}$, i.e.,

$$\lim_{s \longrightarrow \infty} \mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v} \quad \text{and} \quad \lim_{s \to \infty} \mathbb{V}_s = \mathbb{V}.$$

Möbius scalar multiplication is given by the equation

$$\begin{aligned} r \otimes \mathbf{v} &= s \frac{(1 + \|\mathbf{v}\| / s)^r - (1 - \|\mathbf{v}\| / s)^r}{(1 + \|\mathbf{v}\| / s)^r + (1 - \|\mathbf{v}\| / s)^r} \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= s \tanh\left(r \tanh^{-1} \|\mathbf{v}\| / s\right) \frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned}$$

where $r \in \mathbb{R}$, $\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$, $\mathbf{v} \neq 0$ and $r \otimes 0 = 0$.

Möbius scalar multiplication possesses the following properties:

$$\begin{aligned} n \otimes \mathbf{v} &= v \oplus v \oplus \cdots \oplus v && n\text{-terms} \\ (r_1 + r_2) \otimes \mathbf{v} &= r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v} && \text{scalar distribute law} \\ (r_1 r_2) \otimes \mathbf{v} &= r_1 \otimes (r_2 \otimes \mathbf{v}) && \text{scalar associative law} \\ r \otimes (r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}) &= r \otimes (r_1 \otimes \mathbf{v}) \oplus r \otimes (r_2 \otimes \mathbf{v}) && \text{monodistributive law} \\ \|r \otimes \mathbf{v}\| &= |r| \otimes \|\mathbf{v}\| && \text{homogeneity property} \\ \frac{|r| \otimes \mathbf{v}}{\|r \otimes \mathbf{v}\|} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} && \text{scaling property} \\ gyr\,[\mathbf{a}, \mathbf{b}]\,(r \otimes \mathbf{v}) &= r \otimes gyr\,[\mathbf{a}, \mathbf{b}]\,\mathbf{v} && \text{gyroautomorphism property} \\ 1 \otimes \mathbf{v} &= \mathbf{v} && \text{multiplicative unit property} \end{aligned}$$

Definition 2 (Möbius gyrovector spaces). Let $(\mathbb{V}_s, \oplus)$ be a Möbius gyrogroup equipped with scalar multiplication $\otimes$. The triple $(\mathbb{V}_s, \oplus, \otimes)$ is called a Möbius gyrovector space.

## 5. Möbius geodesics and angles

As it is well known from Euclidean geometry, the straight line passing though two given points $A$ and $B$ of a vector space $\mathbb{R}^n$ can be represented by the expression

$$A + (-A + B)\, t$$

$t \in \mathbb{R}$. Obviously it passes through $A$ when $t = 0$, and through $B$ when $t = 1$.

In full analogy with Euclidean geometry, the unique Möbius geodesic passing though two given points $\mathbf{A}$ and $\mathbf{B}$ of a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ is represented by the parametric gyrovector equation

$$L_{\mathbf{AB}} = \mathbf{A} \oplus (\ominus \mathbf{A} \oplus \mathbf{B}) \otimes t$$

with parameter $t \in \mathbb{R}$. It passes through $\mathbf{A}$ when $t = 0$, and through $\mathbf{B}$ when $t = 1$. The gyroline $L_{\mathbf{AB}}$ turns out to be a circular arc that intersects the boundary of the ball $\mathbb{V}_s$ orthogonally. The gyromidpoint $M_{\mathbf{AB}}$ of the points $\mathbf{A}$ and $\mathbf{B}$ corresponds to the parameter $t = 1/2$ of the gyroline $L_{\mathbf{A}B}$, see [11],

$$M_{\mathbf{AB}} = \mathbf{A} \oplus (\ominus\mathbf{A} \oplus \mathbf{B}) \otimes \frac{1}{2}.$$

The measure of a Möbius angle between two intersecting geodesic rays equals the measure of the Euclidean angle between corresponding intersecting tangent lines, as shown in Fig. 1.

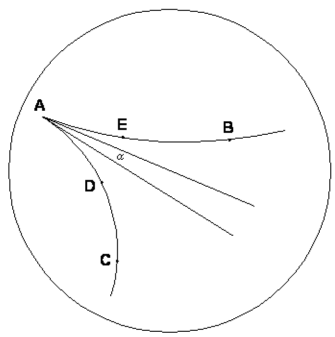


Fig 1. The unique 2-dimensional geodesics that passes through two given points and the hyperbolic angle between two intersecting geodesics rays in a Möbius gyrovector plane $\left(\mathbb{R}^2_s, \oplus, \otimes\right)$. For the non-zero gyrovectors $\ominus\mathbf{A} \oplus \mathbf{B}$ and $\ominus\mathbf{A} \oplus \mathbf{C}$ or equivalently $\ominus\mathbf{A} \oplus \mathbf{E}$ and $\ominus\mathbf{A} \oplus \mathbf{D}$ the measure of the gyroangle $\alpha$ given by the equation $\cos\alpha = \frac{\ominus\mathbf{A}\oplus\mathbf{B}}{\|\ominus\mathbf{A}\oplus\mathbf{B}\|} \cdot \frac{\ominus\mathbf{A}\oplus\mathbf{C}}{\|\ominus\mathbf{A}\oplus\mathbf{C}\|}$ or equivalently by the equation $\cos\alpha = \frac{\ominus\mathbf{A}\oplus\mathbf{E}}{\|\ominus\mathbf{A}\oplus\mathbf{E}\|} \cdot \frac{\ominus\mathbf{A}\oplus\mathbf{D}}{\|\ominus\mathbf{A}\oplus\mathbf{D}\|}$.

The hyperbolic angle is invariant under left gyrotranslations and rotations, see [7].

Definition 3. The hyperbolic distance function in $\mathbb{R}^n_s$, given by the equation

$$d(\mathbf{A}, \mathbf{B}) = \|\mathbf{A} \ominus \mathbf{B}\|.$$

for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^n_s$.

## 6. Gyrotriangles and gyrotrigonometry in Möbius gyrovector space

Definition 4. A gyrotriangle $\Delta\mathbf{ABC}$ in a gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ is a gyrovector space object formed by the three points $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{V}_s$, called the vertices of the gyrotriangle, and the gyrovectors $\ominus\mathbf{A} \oplus \mathbf{B}$, $\ominus\mathbf{B} \oplus \mathbf{C}$ and $\ominus\mathbf{C} \oplus \mathbf{A}$, called the sides of the gyrotriangle. These are respectively the sides opposite to the vertices $\mathbf{C}, \mathbf{A}$ and $\mathbf{B}$. The gyrotriangle sides generate the three gyrotriangle gyroangles $\alpha, \beta$ and $\gamma$ at the respective vertices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$, as shown in Fig. 2 below.

Theorem 5. [10] *Let $\Delta\mathbf{ABC}$ be a gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ with vertices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$, corresponding gyroangles $\alpha, \beta, \gamma$, $0 < \alpha +$*

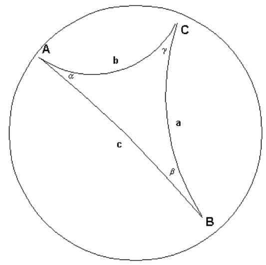


Fig. 2. A gyrotriangle in a Möbius gyrovector plane $\left(\mathbb{R}^2_s, \oplus, \otimes\right)$.

$\beta + \gamma < \pi$, *and side gyrolenghts* $\|\ominus\mathbf{B}\oplus\mathbf{C}\|, \|\ominus\mathbf{C}\oplus\mathbf{A}\|, \|\ominus\mathbf{A}\oplus\mathbf{B}\|$. *The side gyrolengths of the gyrotriangle* $\Delta\mathbf{ABC}$ *are determined by its gyroangles according to the AAA to SSS conversion equations*

$$\left(\frac{\|\ominus\mathbf{B}\oplus\mathbf{C}\|}{s}\right)^2 = \frac{\cos\alpha + \cos(\beta+\gamma)}{\cos\alpha + \cos(\beta-\gamma)}$$
$$\left(\frac{\|\ominus\mathbf{C}\oplus\mathbf{A}\|}{s}\right)^2 = \frac{\cos\beta + \cos(\alpha+\gamma)}{\cos\beta + \cos(\alpha-\gamma)}$$
$$\left(\frac{\|\ominus\mathbf{A}\oplus\mathbf{B}\|}{s}\right)^2 = \frac{\cos\gamma + \cos(\alpha+\beta)}{\cos\gamma + \cos(\alpha-\beta)}.$$

The hyperbolic law of cosine and the hyperbolic law of sine can be recast in a form fully analogous to the form of their Euclidean counterparts. Let us use the notation $\|\mathbf{a}\|_M = \gamma_{\mathbf{a}}^2\|\mathbf{a}\|$ where $\gamma_{\mathbf{a}}$ is the gamma factor

$$\gamma_{\mathbf{a}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{a}\|^2}{s^2}}},$$

so that, conversely

$$\frac{\|\mathbf{a}\|}{s} = \frac{2\left(\|\mathbf{a}\|_M/s\right)}{1+\sqrt{1+4\left(\|\mathbf{a}\|_M/s\right)^2}}.$$

THEOREM 6. [9] *Let* $\Delta\mathbf{ABC}$ *be a gyrotriangle in a Möbius gyrovector space* $(\mathbb{V}_s, \oplus, \otimes)$ *with vertices* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{V}_s$ *and sides* $\mathbf{a} = \ominus\mathbf{B}\oplus\mathbf{C}$ , $\mathbf{b} = \ominus\mathbf{C}\oplus\mathbf{A}$ *and* $\mathbf{c} = \ominus\mathbf{A}\oplus\mathbf{B}$ *with hyperbolic angles* $\alpha, \beta$ *and* $\gamma$ *at the vertices* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$. *Then we have the hyperbolic law of sine,*

$$\frac{\|\mathbf{a}\|_M}{\sin\alpha} = \frac{\|\mathbf{b}\|_M}{\sin\beta} = \frac{\|\mathbf{c}\|_M}{\sin\gamma}.$$

THEOREM 7. [9] *Let* $\Delta\mathbf{ABC}$ *be a gyrotriangle in a Möbius gyrovector space* $(\mathbb{V}_s, \oplus, \otimes)$ *with vertices* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{V}_s$ *and sides* $\mathbf{a} = \ominus\mathbf{B}\oplus\mathbf{C}$ , $\mathbf{b} = \ominus\mathbf{C}\oplus\mathbf{A}$ *and*

$\mathbf{c} = \ominus\mathbf{A} \oplus \mathbf{B}$ *with hyperbolic angles* $\alpha, \beta$ *and* $\gamma$ *at the vertices* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$. *Then we have the hyperbolic law of cosine,*

$$\frac{1}{s}c^2 = \frac{1}{s}a^2 \oplus \frac{1}{s}b^2 \ominus \frac{1}{s}\frac{2ab\cos\gamma}{\left(1+\frac{a^2}{s^2}\right)\left(1+\frac{b^2}{s^2}\right) - \frac{2}{s^2}ab\cos\gamma}.$$

*where* $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$.

Theorem 8. [9] *Let* $\Delta\mathbf{ABC}$ *be a gyrotriangle in a Möbius gyrovector space* $(\mathbb{V}_s, \oplus, \otimes)$ *with vertices* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{V}_s$ *and sides* $\mathbf{a} = \ominus\mathbf{B} \oplus \mathbf{C}$, $\mathbf{b} = \ominus\mathbf{C} \oplus \mathbf{A}$ *and* $\mathbf{c} = \ominus\mathbf{A} \oplus \mathbf{B}$ *with hyperbolic angles* $\alpha, \beta$ *and* $\gamma$ *at the vertices* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$. *If* $\gamma = \pi/2$ *then we have the hyperbolic Pythagorean identity,*

$$\frac{1}{s}c^2 = \frac{1}{s}a^2 \oplus \frac{1}{s}b^2$$

*where* $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$.

## 7. The theorems of Urquhart and Steiner-Lehmus in the Poincaré ball model of hyperbolic geometry

In literature, Urquhart's theorem is also known as the most elementary theorem of Euclidean geometry since it involves only the concept of straight line and distance. Urquhart discovered this result when considering some of the fundamental concepts of the theory of special relativity. The origin and some history of this theorem, we refer [5].

Many authors give the proof of this theorem in different ways. In [18], an elementary synthetic proof of Urquhart's theorem has been posted at Professor's Wu online forum at Berkeley. In [17], K.S. Williams and in [3], M. Hajja gave the proofs which only involved the sine formula for triangles and a few simple trigonometric identities.

In [15], A.A. Ungar proved the hyperbolic Breusch's lemma, the hyperbolic Urquhart's theorem and hyperbolic Steiner-Lehmus theorem in the Einstein gyrovector plane $\left(\mathbb{R}^2_s, \oplus, \otimes\right)$, but there is no attempt made for obtaining these results in the Poincaré ball model of hyperbolic geometry. In this paper, we give affirmative answer of the problem explained above.

Theorem 9. (Breusch's Lemma in Euclidean Geometry) *Let* $\Delta ABC_k$ , $k = 1, 2$, *be two triangles in Euclidean plane* $\mathbb{R}^2$ *with common side* $AB$, *with sides* $a_k, b_k, c_k$, *and with angles* $\alpha_k, \beta_k, \gamma_k$, *as shown in Fig. 3. Then*

$$a_1 + b_1 = a_2 + b_2 \Leftrightarrow \tan\frac{\alpha_1}{2}\tan\frac{\beta_1}{2} = \tan\frac{\alpha_2}{2}\tan\frac{\beta_2}{2}.$$

For the proof, we refer to [15] or [6].

Theorem 10. (Urquhart's Theorem in Euclidean geometry) *Let* $\Delta AD_1BD_2$ *be a concave quadrilateral in a Euclidean plane* $\mathbb{R}^2$, *and let* $AD_1$ *meet* $D_2B$ *at* $C_1$, *and* $AD_2$ *meet* $D_1B$ *at* $C_2$, *as shown in Fig. 4. Then*

$$|AC_1| + |C_1B| = |AC_2| + |C_2B| \iff |AD_1| + |D_1B| = |AD_2| + |D_2B|.$$

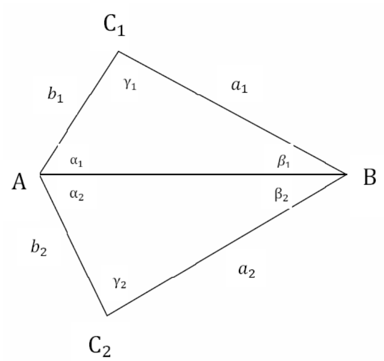


Fig. 3. The triangles $\Delta ABC_k$ , $k = 1, 2$, in Euclidean plane $R^2$ with common side $AB$, with sides $a_k, b_k, c_k$, and with angles $\alpha_k, \beta_k, \gamma_k$.

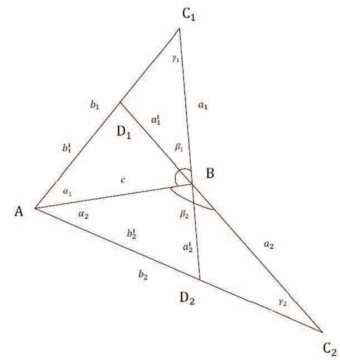


Fig. 4. A concave quadrilateral $AD_1BD_2$ in Euclidean plane $R^2$ satisfying $AD_1$ meet $D_2B$ at $C_1$, and $AD_2$ meet $D_1B$ at $C_2$.

For the proof, we refer to [15] or [3].

Let us give a trigonometric example which plays a major role in the proofs of the hyperbolic Breusch's lemma and hyperbolic Urquhart's theorem in the Poincaré ball model of hyperbolic geometry.

Example 11. Let $\Delta\mathbf{ABC}$ be a gyrotriangle in a Möbius gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^n_s$ and sides $\mathbf{a} = \ominus\mathbf{B} \oplus \mathbf{C}$ , $\mathbf{b} = \ominus\mathbf{C} \oplus \mathbf{A}$ and $\mathbf{c} = \ominus\mathbf{A} \oplus \mathbf{B}$ with hyperbolic angles $\alpha, \beta$ and $\gamma$ at the vertices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$. Then

$$\frac{\sin\alpha + \sin\beta}{\sin(\alpha+\beta)} = \frac{1}{c_s}\frac{\gamma_a^2 a_s + \gamma_b^2 b_s}{\gamma_a^2\gamma_b^2\left(1 - a_s^2 b_s^2\right)}$$

where $a_s = \|\mathbf{a}\|/s$, $b_s = \|\mathbf{b}\|/s$, $c_s = \|\mathbf{c}\|/s$.

Indeed, from the well known gyrotrigonometric functional identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$, and applying the following identities to gyrotriangle $\Delta\mathbf{ABC}$, see [10], we easily get to desired result:

$$\begin{aligned}
\frac{b_s c_s}{\gamma_a^2}\sin\alpha &= \frac{a_s c_s}{\gamma_b^2}\sin\beta = \frac{a_s b_s}{\gamma_c^2}\sin\gamma,\\
\cos\alpha &= \frac{-a_s^2 + b_s^2 + c_s^2 - a_s^2 b_s^2 c_s^2}{2 b_s c_s}\gamma_a^2,\\
\cos\beta &= \frac{a_s^2 - b_s^2 + c_s^2 - a_s^2 b_s^2 c_s^2}{2 a_s c_s}\gamma_b^2,\\
\cos\gamma &= \frac{a_s^2 + b_s^2 - c_s^2 - a_s^2 b_s^2 c_s^2}{2 a_s b_s}\gamma_c^2.
\end{aligned}$$

Let us give the hyperbolic Breusch's lemma, and the hyperbolic Urquhart's theorem in the Poincaré ball model of hyperbolic geometry.

THEOREM 12. (Breusch's Lemma in Hyperbolic geometry) *Let $\Delta\mathbf{ABC}_k$, $k = 1, 2$, be two gyrotriangles in a Möbius gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ with common side $\ominus\mathbf{A} \oplus \mathbf{B}$, with side gyrolenghts $a_k, b_k, c_k$, and with angles $\alpha_k, \beta_k, \gamma_k$, as similar to Fig. 3. Then*

$$\frac{\gamma^2_{a_1} a_{1_s} + \gamma^2_{b_1} b_{1_s}}{\gamma^2_{a_1}\gamma^2_{b_1}\left(1 - a^2_{1_s} b^2_{1_s}\right)} = \frac{\gamma^2_{a_2} a_{2_s} + \gamma^2_{b_2} b_{2_s}}{\gamma^2_{a_2}\gamma^2_{b_2}\left(1 - a^2_{2_s} b^2_{2_s}\right)} \iff \tan\frac{\alpha_1}{2}\tan\frac{\beta_1}{2} = \tan\frac{\alpha_2}{2}\tan\frac{\beta_2}{2}.$$

*Proof.* First of all, since any trigonometric identity is identical with a corresponding gyrotrigonometric identity, the following identity is valid in trigonometry when $\sin(\alpha + \gamma) \neq 0$, and hence in gyrotrigonometry as well:

$$\frac{\sin\alpha + \sin\gamma}{\sin(\alpha + \gamma)} = -1 + \frac{2}{1 - \tan\dfrac{\alpha}{2}\tan\dfrac{\gamma}{2}}.$$

Therefore, we get

$$\frac{\gamma^2_{a_1} a_{1_s} + \gamma^2_{b_1} b_{1_s}}{\gamma^2_{a_1}\gamma^2_{b_1}\left(1 - a^2_{1_s} b^2_{1_s}\right)} = \frac{\gamma^2_{a_2} a_{2_s} + \gamma^2_{b_2} b_{2_s}}{\gamma^2_{a_2}\gamma^2_{b_2}\left(1 - a^2_{2_s} b^2_{2_s}\right)} \Leftrightarrow \frac{\sin\alpha_1 + \sin\beta_1}{\sin(\alpha_1 + \beta_1)} = \frac{\sin\alpha_2 + \sin\beta_2}{\sin(\alpha_2 + \beta_2)}$$

$$\Leftrightarrow \tan\frac{\alpha_1}{2}\tan\frac{\beta_1}{2} = \tan\frac{\alpha_2}{2}\tan\frac{\beta_2}{2}.$$

THEOREM 13. (Urquhart's Theorem in Hyperbolic geometry) *Let $\Delta\mathbf{AD}_1\mathbf{BD}_2$ be a concave gyroquadrilateral in a Möbius gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ where $\alpha_k = \angle\mathbf{BAC}_k$, $\beta_k = \angle\mathbf{ABC}_k$, and $\gamma_k = \angle\mathbf{AC}_k\mathbf{B}$, and let $\mathbf{AD}_1$ meet $\mathbf{D}_2\mathbf{B}$ at $\mathbf{C}_1$, and $\mathbf{AD}_2$ meet $\mathbf{D}_1\mathbf{B}$ at $\mathbf{C}_2$, as shown in Fig. 5. Then*

$$\frac{\gamma^2_{a_1} a_{1_s} + \gamma^2_{b_1} b_{1_s}}{\gamma^2_{a_1}\gamma^2_{b_1}\left(1 - a^2_{1_s} b^2_{1_s}\right)} = \frac{\gamma^2_{a_2} a_{2_s} + \gamma^2_{b_2} b_{2_s}}{\gamma^2_{a_2}\gamma^2_{b_2}\left(1 - a^2_{2_s} b^2_{2_s}\right)} \Leftrightarrow \frac{\gamma^2_{a'_1} a'_{1_s} + \gamma^2_{b'_1} b'_{1_s}}{\gamma^2_{a'_1}\gamma^2_{b'_1}\left(1 - a'^2_{1_s} b'^2_{1_s}\right)} = \frac{\gamma^2_{a'_2} a'_{2_s} + \gamma^2_{b'_2} b'_{2_s}}{\gamma^2_{a'_2}\gamma^2_{b'_2}\left(1 - a'^2_{2_s} b'^2_{2_s}\right)}$$

*where $a_k = \|\ominus\mathbf{C}_k \oplus \mathbf{B}\|$, $b_k = \|\ominus\mathbf{C}_k \oplus \mathbf{A}\|$, $a'_k = \|\ominus\mathbf{D}_k \oplus \mathbf{B}\|$, $b'_k = \|\ominus\mathbf{D}_k \oplus \mathbf{A}\|$.*

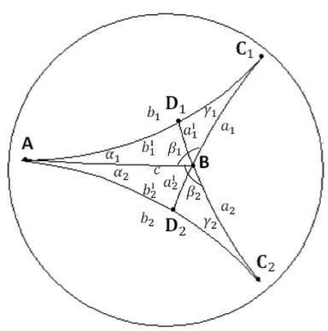


Fig. 5. A concave gyroquadrilateral $\mathbf{AD}_1\mathbf{BD}_2$ in a Möbius gyrovector space $\left(\mathbb{R}^n_s, \oplus, \otimes\right)$ with $\alpha_k = \angle\mathbf{BAC}_k$, $\beta_k = \angle\mathbf{ABC}_k$, and $\gamma_k = \angle\mathbf{AC}_k\mathbf{B}$, satisfying $\mathbf{AD}_1$ meet $\mathbf{D}_2\mathbf{B}$ at $\mathbf{C}_1$, and $\mathbf{AD}_2$ meet $\mathbf{D}_1\mathbf{B}$ at $\mathbf{C}_2$.

*Proof.* Applying the hyperbolic Breusch's lemma to each of the two gyrotriangles $\Delta\mathbf{ABC}_k$ and $\Delta\mathbf{ABD}_k$, $k=1,2$, we have

$$\frac{\gamma_{a_1}^2 a_{1_s} + \gamma_{b_1}^2 b_{1_s}}{\gamma_{a_1}^2 \gamma_{b_1}^2 \left(1 - a_{1_s}^2 b_{1_s}^2\right)} = \frac{\gamma_{a_2}^2 a_{2_s} + \gamma_{b_2}^2 b_{2_s}}{\gamma_{a_2}^2 \gamma_{b_2}^2 \left(1 - a_{2_s}^2 b_{2_s}^2\right)} \Leftrightarrow \tan\frac{\alpha_1}{2}\tan\frac{\beta_1}{2} = \tan\frac{\alpha_2}{2}\tan\frac{\beta_2}{2}$$

and

$$\frac{\gamma_{a_1^{|}}^2 a_{1_s}^{|} + \gamma_{b_1^{|}}^2 b_{1_s}^{|}}{\gamma_{a_1^{|}}^2 \gamma_{b_1^{|}}^2 \left(1 - a_{1_s}^{|2} b_{1_s}^{|2}\right)} = \frac{\gamma_{a_2^{|}}^2 a_{2_s}^{|} + \gamma_{b_2^{|}}^2 b_{2_s}^{|}}{\gamma_{a_2^{|}}^2 \gamma_{b_2^{|}}^2 \left(1 - a_{2_s}^{|2} b_{2_s}^{|2}\right)}$$
$$\Leftrightarrow \tan\frac{\alpha_1}{2}\tan\left(\frac{\pi}{2} - \frac{\beta_2}{2}\right) = \tan\frac{\alpha_2}{2}\tan\left(\frac{\pi}{2} - \frac{\beta_1}{2}\right) \tag{6}$$

respectively. Clearly, the right-hand side of (6) implies the equality

$$\tan\frac{\alpha_1}{2}\tan\frac{\beta_1}{2} = \tan\frac{\alpha_2}{2}\tan\frac{\beta_2}{2}$$

and so the proof is completed. ■

Theorem 14. (Steiner-Lehmus Theorem in Hyperbolic Geometry) *Let* $\Delta\mathbf{ABC}$ *be a gyrotriangle in a Möbius gyrovector space* $(\mathbb{R}_s^n, \oplus, \otimes)$ *having two equal internal gyroangle bisectors (each measured from a vertex to the opposite side). Then the gyrotriangle* $\Delta\mathbf{ABC}$ *is isosceles.*

*Proof.* Given a gyrotriangle $\Delta\mathbf{ABC}$, $\|\ominus\mathbf{B}\oplus\mathbf{E}\|$ and $\|\ominus\mathbf{C}\oplus\mathbf{D}\|$ are equal bisectors of the gyroangles $\angle\mathbf{ABC} := 4\alpha$ and $\angle\mathbf{ACB} := 4\beta$, respectively. Clearly, if we prove that the equality $\|\ominus\mathbf{B}\oplus\mathbf{D}\| = \|\ominus\mathbf{E}\oplus\mathbf{C}\|$ holds true, this implies the equality of the gyrolength $\|\ominus\mathbf{A}\oplus\mathbf{B}\|$ to the gyrolength $\|\ominus\mathbf{A}\oplus\mathbf{C}\|$. Without loss of generality, we may assume $\|\ominus\mathbf{B}\oplus\mathbf{D}\| \geq \|\ominus\mathbf{E}\oplus\mathbf{C}\|$. Now, to prove $\|\ominus\mathbf{B}\oplus\mathbf{D}\| = \|\ominus\mathbf{E}\oplus\mathbf{C}\|$, suppose the contrary that $\|\ominus\mathbf{B}\oplus\mathbf{D}\| > \|\ominus\mathbf{E}\oplus\mathbf{C}\|$. This implies $\|\ominus\mathbf{B}\oplus\mathbf{D}\|^2 > \|\ominus\mathbf{E}\oplus\mathbf{C}\|^2$ and applying the hyperbolic cosine law to the gyrotriangles $\triangle\mathbf{DBC}$ and $\triangle\mathbf{EBC}$ we get $\cos 2\alpha > \cos 2\beta$. Since $2\alpha, 2\beta \in I = (0, \pi/2)$ and cosine function is decreasing on $I$, we get $\beta > \alpha$. Since the tangent function is increasing on $I$, we get $\tan\beta > \tan\alpha$ and this implies

$$\frac{2}{1 - \tan\alpha\tan 2\beta} > \frac{2}{1 - \tan 2\alpha\tan\beta}.$$

Applying the hyperbolic Breusch's lemma to the gyrotriangles $\triangle\mathbf{EBC}$ and $\triangle\mathbf{DBC}$, we have

$$\frac{\gamma_{\ominus\mathbf{E}\oplus\mathbf{C}}^2 \|\ominus\mathbf{E}\oplus\mathbf{C}\|_s + \gamma_{\ominus\mathbf{B}\oplus\mathbf{E}}^2 \|\ominus\mathbf{B}\oplus\mathbf{E}\|_s}{\gamma_{\ominus\mathbf{E}\oplus\mathbf{C}}^2 \gamma_{\ominus\mathbf{B}\oplus\mathbf{E}}^2 \left(1 - \|\ominus\mathbf{E}\oplus\mathbf{C}\|_s^2 \|\ominus\mathbf{B}\oplus\mathbf{E}\|_s^2\right)}$$
$$> \frac{\gamma_{\ominus B\oplus\mathbf{D}}^2 \|\ominus\mathbf{B}\oplus\mathbf{D}\|_s + \gamma_{\ominus\mathbf{D}\oplus\mathbf{C}}^2 \|\ominus\mathbf{D}\oplus\mathbf{C}\|_s}{\gamma_{\ominus\mathbf{B}\oplus\mathbf{D}}^2 \gamma_{\ominus\mathbf{D}\oplus\mathbf{C}}^2 \left(1 - \|\ominus\mathbf{B}\oplus\mathbf{D}\|_s^2 \|\ominus\mathbf{D}\oplus\mathbf{C}\|_s^2\right)}.$$

Now define a function $f$ from $[0,s)$ to $\mathbb{R}$ by the rule

$$f(x)=\frac{\dfrac{xs}{s^2-x^2}+\dfrac{ks}{s^2-k^2}}{\dfrac{s^2}{s^2-x^2}\dfrac{s^2}{s^2-k^2}\left(1-\dfrac{x^2}{s^2}\dfrac{k^2}{s^2}\right)} \tag{7}$$

where $k$ and $s$ are fixed elements of $\mathbb{R}$ such that $0\le k<s$. A simple calculation shows that $f$ is an increasing function and therefore from (7), we get $\|\ominus\mathbf{E}\oplus\mathbf{C}\|>\|\ominus\mathbf{B}\oplus\mathbf{D}\|$ which is the desired contradiction.

Finally, applying the hyperbolic cosine law to the gyrotriangles $\triangle\mathbf{DBC}$ and $\triangle\mathbf{EBC}$, we get $2\alpha=2\beta$, i.e. $\|\ominus\mathbf{A}\oplus\mathbf{B}\|=\|\ominus\mathbf{A}\oplus\mathbf{C}\|$. ■

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