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OPERATOR REPRESENTATIONS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS AND CERTAIN POLYNOMIALS

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Abstract. A new technique is evolved to give operator representations of hypergeometric functions and certain polynomials.

1. Introduction

In 1731, Euler defined the derivative formula

$$D_x^n x^\lambda = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-n)} x^{\lambda-n}, \qquad D_x \equiv \frac{d}{dx}$$

where n is a positive integer. Its general form is

$$D_x^\mu x^\lambda = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-\mu)} x^{\lambda-\mu}, \tag{1.1}$$

where λ and μ are arbitrary complex numbers. Here (1.1) is given to facilitate the use of D^{-n}, i.e. replacing μ by $-n$ in (1.1), where n is a positive integer. Here series representations of different operators have been used to establish operational representation of hypergeometric functions and various known polynomials. The technique used and the results obtained are believed to be new.

2. Definitions and notation

In deriving the operator representations of hypergeometric functions and certain polynomials use has been made of the fact of the following notations:

If D_x denotes a derivative operator, then D_x^{-1} is nothing but the inverse operator of D_x. Now we can write the following:

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$$D_x^{-1} = \frac{1}{D_x}, \qquad D_x^{-n} = D_x^{-1} D_x^{-1} \dots (n \text{ times}) = (D_x^{-1})^n = \left(\frac{1}{D_x}\right)^n,$$

$$(\alpha)_n \; x^{-n} = x^{\alpha} (-D_x)^n \; x^{-\alpha}, \tag{2.1}$$

$$\frac{1}{(\alpha)_n} \; x^n = x^{-\alpha+1} (\frac{1}{D_x})^n \; x^{\alpha-1}. \tag{2.2}$$

Some results used in the proofs are

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{2.3}$$

$$(1-x)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n \; x^n}{n!}. \tag{2.4}$$

We also need the definitions of the following generalized hypergeometric functions and polynomials in terms of hypergeometric function and also their notations (see [4], [11], [12]).

Generalized hypergeometric function. The generalized hypergeometric function is defined as

$${}_pF_q \left[\begin{matrix} a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix} x \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_n \; x^n}{\prod_{j=1}^{q} (b_j)_n \; n!}. \tag{2.5}$$

Laguerre polynomial. It is denoted by the symbol $L_n^{(\alpha)}(x)$ and is defined as

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \; {}_1F_1 \left[\begin{matrix} -n; \\ 1+\alpha; \end{matrix} x \right].$$

Legendre polynomial. It is denoted by the symbol $P_n(x)$ and is defined as

$$P_n(x) = {}_2F_1 \left[\begin{matrix} -n, \; n+1 \; ; \\ 1; \end{matrix} \frac{1-x}{2} \right].$$

Jacobi polynomial. It is denoted by the symbol $P_n^{(\alpha,\beta)}(x)$ and is defined as

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} \; {}_2F_1 \left[\begin{matrix} -n, 1+\alpha+\beta+n \; ; \\ 1+\alpha; \end{matrix} \frac{1-x}{2} \right].$$

Ultraspherical polynomial. The special case of $\beta = \alpha$ of the Jacobi polynomial is called ultraspherical polynomial and is denoted by $P_n^{(\alpha,\alpha)}(x)$. It is thus defined as

$$P_n^{(\alpha,\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \; {}_2F_1 \left[\begin{matrix} -n, 1+2\alpha+n \; ; \\ 1+\alpha; \end{matrix} \frac{1-x}{2} \right].$$

Gegenbauer polynomial. The Gegenbauer polynomial $C_n^{\nu}(x)$ is the generalization of Legendre polynomial and is defined as

$$C_n^{\nu}(x) = \frac{(2\nu)_n}{n!} \; {}_2F_1 \left[\begin{matrix} -n, \; 2\nu+n \; ; \\ \nu+\frac{1}{2}; \end{matrix} \frac{1-x}{2} \right].$$

Bessel polynomial. Simple Bessel polynomial $y_n(x)$ is defined as

$$y_n(x) ={}_2 F_0 \left[\begin{matrix} -n, \; n+1 \; ; \\ -; \end{matrix} \frac{-x}{2}\right],$$

and the generalized Bessel polynomials $y_n(a,b,x)$ is defined as

$$y_n(a,b,x) ={}_2 F_0 \left[\begin{matrix} -n, \; a-1+n; \\ -; \end{matrix} \frac{-x}{b}\right].$$

Lagrange polynomial. It is denoted by the symbol $g_n^{(\alpha,\beta)}(x,y)$ and is defined by

$$g_n^{(\alpha,\beta)}(x,y) = \frac{(\alpha)_n}{n!}\; {}_2F_1 \left[\begin{matrix} -n, \beta; \\ 1-\alpha-n; \end{matrix} \frac{y}{x}\right].$$

Sylvester polynomial. It is denoted by the symbol $\varphi_n(x)$ and is defined as

$$\varphi_n(x) = \frac{x^n}{n!}\; {}_2F_0 \left[\begin{matrix} -n, \; x; \\ -; \end{matrix} x^{-1}\right].$$

Shively's pseudo Laguerre polynomial. It is denoted by the symbol $R_n(a,x)$ and is defined as

$$R_n(a,x) = \frac{(a)_{2n}}{n!(a)_n} {}_1F_1 \left[\begin{matrix} -n \; ; \\ a+n \; ; \end{matrix} \; x\right].$$

Hermite polynomial. It is denoted by the symbol $H_n(x)$ and is defined as

$$H_n(x) = (2x)^n \; {}_2F_0 \left[\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2} \; ; \\ -; \end{matrix} \; -\frac{1}{x^2}\right].$$

Bateman's $Z_n(x)$. Bateman's polynomial $Z_n(x)$ is defined as

$$Z_n(x) ={}_2 F_2 \left[\begin{matrix} -n, \; n+1 \; ; \\ 1, \; 1; \end{matrix} x\right].$$

Bateman's generalization of $Z_n(x)$. Bateman moved from $Z_n(x)$ to the more general polynomial

$${}_2F_2 \left[\begin{matrix} -n, \; 2v+n \; ; \\ v+\frac{1}{2}, 1+b; \end{matrix} t\right].$$

It may be remarked here that the above polynomial is the Gegenbauer type generalization of $Z_n(x)$. We will therefore adopt the symbol $Z_n^v(b,t)$. Thus we have

$$Z_n^v(b,t) ={}_2 F_2 \left[\begin{matrix} -n, 2v+n \; ; \\ v+\frac{1}{2}, \; 1+b; \end{matrix} t\right].$$

A Jacobi type generalization of $Z_n(x)$ may be denoted by the symbol $Z_n^{(\alpha,\beta)}(b,x)$ and is defined as

$$Z_n^{(\alpha,\beta)}(b,x) ={}_2 F_2 \left[\begin{matrix} -n, \; 1+\alpha+\beta+n \; ; \\ 1+\alpha, \; 1+b; \end{matrix} x\right].$$

Tchebycheff polynomials. The Tchebycheff polynomials $T_n(x)$ and $U_n(x)$ of the first and second kinds respectively are special ultraspherical polynomials. In details

$$T_n(x) = \frac{n!}{(\frac{1}{2})_n} P_n^{(-\frac{1}{2},-\frac{1}{2})}(x), \qquad U_n(x) = \frac{(n+1)!}{(\frac{3}{2})_n} P_n^{(\frac{1}{2},\frac{1}{2})}(x),$$

and in terms of hypergeometric function their definition will be as follows

$$T_n(x) ={}_2 F_1 \left[\begin{matrix} -n,\ n\ ; \\ \frac{1}{2}; \end{matrix} \frac{1-x}{2}\right], \quad U_n(x) = (n+1)\ {}_2F_1 \left[\begin{matrix} -n,\ n+2; \\ \frac{3}{2}; \end{matrix} \frac{1-x}{2}\right].$$

Cesáro polynomial. It is denoted by the symbol $g_n^{(s)}(x)$ and is defined as

$$g_n^{(s)}(x) = \binom{s+n}{n}\ {}_2F_1 \left[\begin{matrix} -n,\ 1\ ; \\ -s-n\ ; \end{matrix} x\right].$$

Bedient's polynomials. Bedient [2], in his study of some polynomials associated with Appell's F_2 and F_3, introduced

$$R_n(\beta,\gamma,x) = \frac{(\beta)_n(2x)^n}{n!}\ {}_3F_2 \left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}\ , \gamma-\beta; \\ \gamma, 1-\beta-n\ ; \end{matrix} \frac{1}{x^2}\right],$$

$$G_n(\alpha,\beta;x) = \frac{(\alpha)_n(\beta)_n(2x)^n}{n!(\alpha+\beta)_n}\ {}_3F_2 \left[\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2},\ 1-\alpha-\beta-n; \\ 1-\alpha-n, 1-\beta-n\ ; \end{matrix} \frac{1}{x^2}\right].$$

Rice polynomial. Rice polynomial $H_n(\xi,p,v)$ is defined as

$$H_n(\xi,p,v) =\ {}_3F_2 \left[\begin{matrix} -n,\ n+1\ , \xi; \\ 1,\ p; \end{matrix} v\right].$$

A Jacobi type generalization of Rice polynomial $H_n(\xi,p,v)$ is due to Khandekar [10] who denoted his generalized polynomial by the symbol $H_n^{(\alpha,\beta)}(\xi,p,v)$ and is defined as

$$H_n^{(\alpha,\beta)}(\xi,p,v) = \frac{(1+\alpha)_n}{n!}\ {}_3F_2 \left[\begin{matrix} -n,\ 1+\alpha+\beta+n\ , \xi; \\ 1+\alpha, p; \end{matrix} v\right].$$

Sister Celine's polynomial. Sister M. Celine denoted her polynomial by the symbol $f_n \left[\begin{matrix} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{matrix} x\right]$ and is defined as

$$f_n \left[\begin{matrix} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{matrix} x\right] ={}_{p+2} F_{q+2} \left[\begin{matrix} -n,\ n+1, a_1,\cdots,a_p; \\ 1,\ \frac{1}{2}\ ,\ b_1,\cdots,b_q; \end{matrix} x\right].$$

3. Operational representation

If $D_{x_i} \equiv \frac{\partial}{\partial x_i}$ and $D_{y_i} \equiv \frac{\partial}{\partial y_i}$, where $i = 1,2,3,\ldots$, by using these partial differential operators, we define the following function:

$$\biguplus_p^q [b_1, b_2 \ldots, b_p; c_1, c_2, \ldots, c_q; x_1, x_2, \ldots, x_q : y_1, y_2, \ldots, y_p]$$

$$= \prod_{j=1}^{P} y_j^{b_j} \prod_{i=1}^{q} x_i^{-c_i+1} \exp\left(\frac{(-1)^p \prod_{j=1}^{P} D_{y_j}}{\prod_{i=1}^{q} D_{x_i}}\right) \prod_{j=1}^{P} y_j^{-b_j} \prod_{i=1}^{q} x_i^{c_i-1}, \qquad (3.1)$$

and we can show that the result (3.1) is equivalent to the following:

$$\biguplus_{p}^{q}[b_1, b_2 \ldots, b_p; c_1, c_2, \ldots, c_q; x_1, x_2, \ldots, x_q : y_1, y_2, \ldots, y_p]$$
$$= {}_pF_q \left[b_1, b_2, \ldots, b_p; c_1, c_2, \ldots, c_q; \frac{x_1 x_2 \ldots x_q}{y_1 y_2 .. y_p}\right]. \quad (3.2)$$

Again we define another function, as follows:

$$\biguplus_{p+1}^{q}[a, b_1, b_2 \ldots, b_p; c_1, c_2, \ldots, c_q; x_1, x_2, \ldots, x_q : y_1, y_2, \ldots, y_p]$$
$$= \prod_{j=1}^{p} y_j^{b_j} \prod_{i=1}^{q} x_i^{-c_i+1} \left(1 - \frac{(-1)^p \prod_{j=1}^{p} D_{y_j}}{\prod_{i=1}^{q} D_{x_i}}\right)^{-a} \prod_{j=1}^{P} y_j^{-b_j} \prod_{i=1}^{q} x_i^{c_i-1}. \quad (3.3)$$

Further, we can show that the result (3.3) is equivalent to the following

$$\biguplus_{p+1}^{q}[b_1, b_2 \ldots, b_p; c_1, c_2, \ldots, c_q; x_1, x_2, \ldots, x_q : y_1, y_2, \ldots, y_p]$$
$$= {}_{p+1}F_q \left[a, b_1, b_2, \ldots, b_p; c_1, c_2, \ldots, c_q; \frac{x_1 x_2 \ldots x_q}{y_1 y_2 .. y_p}\right]. \quad (3.4)$$

Proof of (3.2). Taking the R.H.S of (3.1) and applying the result (2.3), we get

$$\prod_{j=1}^{p} y_j^{b_j} \prod_{i=1}^{q} x_i^{-c_i+1} \sum_{r=0}^{\infty} \frac{(-1)^{pr} \prod_{j=1}^{p} (D_{y_j})^r}{r! \prod_{i=1}^{q} (D_{x_i})^r} \prod_{j=1}^{p} y_j^{-b_j} \prod_{i=1}^{q} x_i^{c_i-1}$$
$$= \sum_{r=0}^{\infty} \frac{1}{r!} \prod_{j=1}^{p} \left\{ y_j^{b_j} (-D_{y_j})^r y_j^{-b_j} \right\} \prod_{i=1}^{q} \left\{ x_i^{-c_i+1} \left(\frac{1}{D_{x_i}}\right)^r x_i^{c_i-1} \right\}.$$

Now applying the results (2.1) and (2.2), we get

$$= \sum_{r=0}^{\infty} \frac{1}{r!} \prod_{j=1}^{p} \{(b_j)_r \; y_j^{-r}\} \prod_{i=1}^{q} \left\{ \frac{1}{(c_i)_r} \; x_i^r \right\}.$$

From (2.5), we get the result (3.2).

Proof of (3.4). Taking the R.H.S of (3.3) and applying the results (2.4), we get

$$\prod_{j=1}^{p} y_j^{b_j} \prod_{i=1}^{q} x_i^{-c_i+1} \sum_{r=0}^{\infty} \frac{(a)_r \; (-1)^{pr} \prod_{j=1}^{p} (D_{y_j})^r}{r! \prod_{i=1}^{q} (D_{x_i})^r} \prod_{j=1}^{p} y_j^{-b_j} \prod_{i=1}^{q} x_i^{c_i-1}$$
$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r!} \prod_{j=1}^{p} \left\{ y_j^{b_j} (-D_{y_j})^r y_j^{-b_j} \right\} \prod_{i=1}^{q} \left\{ x_i^{-c_i+1} \left(\frac{1}{D_{x_i}}\right)^r x_i^{c_i-1} \right\},$$

and now applying the results (2.1) and (2.2), we get the following:

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r!} \prod_{j=1}^{p} \left\{(b_j)_r \ y_j^{-r}\right\} \ \prod_{i=1}^{q} \ \left\{\frac{1}{(c_i)_r} \ x_i^r\right\}.$$

Again from (2.5), we get the result (3.4). ■

4. Operational representations of hypergeometric functions

By taking different values of p and q in (3.1)–(3.4), we get operational representations of hypergeometric functions. If we take $p = 1$ and $q = 0$ in (3.2), we get

$$\biguplus_{1}^{0}[a; -; - : x] = \ {}_1F_0\left[a; -; \frac{1}{x}\right],$$

and from (3.1)

$${}_1F_0\left[a; -; \frac{1}{x}\right] = x^a \ e^{-D_x} \ x^{-a}.$$

Similarly, we can define the following

$${}_1F_1\left[a; b; \frac{x}{y}\right] = y^a \ x^{-b+1} \ e^{\frac{-D_y}{D_x}} \ y^{-a}x^{b-1}$$

$${}_1F_1\left[a; b; x\right] = x^{-b+1}\left(1 - \frac{1}{D_x}\right)^{-a} x^{b-1}$$

$${}_2F_0\left[a, b; \frac{1}{x}\right] = x^a\left(1 + D_x\right)^{-b} x^{-a}$$

$${}_2F_0\left[a, b; \frac{1}{x}\right] = x^b\left(1 + D_x\right)^{-a} x^{-b}$$

$${}_2F_0\left[a, b; \frac{1}{xy}\right] = x^a y^b \ e^{D_x D_y} x^{-a} y^{-b}$$

$${}_2F_1\left[a, b; c; \frac{x}{y}\right] = x^{-c+1} y^b\left(1 + \frac{D_y}{D_x}\right)^{-a} x^{c-1} y^{-b}$$

$${}_2F_1\left[a, b; c; \frac{x}{y}\right] = x^{-c+1} y^a\left(1 + \frac{D_y}{D_x}\right)^{-b} x^{c-1} y^{-a}$$

$${}_2F_1\left[a, b; c; \frac{x}{yz}\right] = x^{-c+1} y^a z^b \ \exp\left(\frac{D_y D_z}{D_x}\right) x^{c-1} y^{-a} z^{-b}$$

$${}_2F_2\left[a, b; c, d; \frac{xy}{z}\right] = x^{-c+1} y^{-d+1} z^b\left(1 + \frac{D_z}{D_x D_y}\right)^{-a} x^{c-1} y^{d-1} z^{-b}$$

$${}_2F_2\left[a, b; c, d; \frac{xy}{z}\right] = x^{-c+1} y^{-d+1} z^a\left(1 + \frac{D_z}{D_x D_y}\right)^{-b} x^{c-1} y^{d-1} z^{-a}$$

$${}_2F_2\left[a, b; c, d; \frac{xy}{z}\right] = x^{-d+1} y^{-c+1} z^b\left(1 + \frac{D_z}{D_x D_y}\right)^{-a} x^{d-1} y^{c-1} z^{-b}$$

$$
{}_2F_2\left[a,b;c,d;\frac{xy}{zw}\right]=x^{-c+1}y^{-d+1}z^{a}\ w^{b}\ \exp\left(\frac{D_zD_w}{D_xD_y}\right)x^{c-1}y^{d-1}z^{-a}\ w^{-b}
$$

$$
{}_2F_2\left[a,b;c,d;\frac{xy}{zw}\right]=x^{-d+1}y^{-c+1}z^{b}\ w^{a}\ \exp\left(\frac{D_zD_w}{D_xD_y}\right)x^{d-1}y^{c-1}z^{-b}\ w^{-a}
$$

$$
{}_3F_0\left[a,b,c;-;\frac{1}{xy}\right]=x^{b}y^{c}\left(1-D_xD_y\right)^{-a}x^{-b}y^{-c}
$$

$$
{}_3F_0\left[a,b,c;-;\frac{1}{xy}\right]=x^{a}y^{c}\left(1-D_xD_y\right)^{-b}x^{-a}y^{-c}
$$

$$
{}_3F_0\left[a,b,c;-;\frac{1}{xy}\right]=x^{a}y^{b}\left(1-D_xD_y\right)^{-c}x^{-a}y^{-b}
$$

$$
{}_3F_0\left[a,b,c;-;\frac{1}{xyz}\right]=x^{a}y^{b}z^{c}\ \exp\left(D_xD_yD_z\right)x^{-a}y^{-b}z^{-c}
$$

$$
{}_3F_1\left[a,b,c;d;\frac{x}{yz}\right]=x^{-d+1}y^{a}z^{b}\left(1-\frac{D_yD_z}{D_x}\right)^{-c}x^{d-1}y^{-a}z^{-b}
$$

$$
{}_3F_1\left[a,b,c;d;\frac{x}{yz}\right]=x^{-d+1}y^{a}z^{c}\left(1-\frac{D_yD_z}{D_x}\right)^{-b}x^{d-1}y^{-a}z^{-c}
$$

$$
{}_3F_1\left[a,b,c;d;\frac{x}{yz}\right]=x^{-d+1}y^{b}z^{c}\left(1-\frac{D_yD_z}{D_x}\right)^{-a}x^{d-1}y^{-b}z^{-c}
$$

$$
{}_3F_1\left[a,b,c;d;\frac{x}{yzw}\right]=x^{-d+1}y^{b}z^{c}w^{a}\ \exp\ \left(\frac{-D_zD_yD_w}{D_x}\right)w^{-a}x^{d-1}y^{-b}z^{-c}.
$$

5. Operational representations of certain polynomials

The operational representations of several polynomials are given as follows

$$
L_n^{(\alpha)}(x)=\frac{x^{-\alpha}}{n!}(D_x-1)^{n}x^{n+\alpha}
$$

$$
L_n^{(\alpha)}(x)=\frac{(1+\alpha)_n}{n!}x^{-\alpha}\left(1-\frac{1}{D_x}\right)^{n}x^{\alpha}
$$

$$
L_n^{(\alpha)}(\frac{x}{y})=\frac{(1+\alpha)_n}{n!}x^{-\alpha}y^{-n}e^{\frac{-D_y}{D_x}}x^{\alpha}y^{n}
$$

$$
P_n\left(1-\frac{2x}{y}\right)=\frac{y^{-n}}{n!}D_x^{n}\left(1+\frac{D_y}{D_x}\right)^{-n-1}x^{n}y^{n}
$$

$$
P_n\left(1-\frac{2x}{y}\right)=y^{-n}\left(1+\frac{D_y}{D_x}\right)^{-n-1}x^{0}y^{n}
$$

$$
P_n\left(1-\frac{2x}{y}\right)=y^{n+1}\left(1+\frac{D_y}{D_x}\right)^{n}x^{0}y^{-n-1}
$$

$$
P_n^{(\alpha,\beta)}\left(1-\frac{2x}{y}\right)=\frac{x^{-\alpha}y^{-n}}{n!}D_x^{n}\left(1+\frac{D_y}{D_x}\right)^{-n-\alpha-\beta-1}x^{n+\alpha}y^{n}
$$

$$P_n^{(\alpha,\beta)}\left(1-\frac{2x}{y}\right)=\frac{(1+\alpha)_n x^{-\alpha}y^{1+\alpha+\beta+n}}{n!}\left(1+\frac{D_y}{D_x}\right)^n x^{\alpha}y^{-1-\alpha-\beta-n}$$

$$P_n^{(\alpha,\beta)}\left(1-\frac{2x}{y}\right)=\frac{(1+\alpha)_n x^{-\alpha}y^{-n}}{n!}\left(1+\frac{D_y}{D_x}\right)^{-n-\alpha-\beta-1} x^{\alpha}y^{n}$$

$$P_n^{(\alpha,\alpha)}\left(1-\frac{2x}{y}\right)=\frac{x^{-\alpha}y^{-n}}{n!}D_x^n\left(1+\frac{D_y}{D_x}\right)^{-n-2\alpha-1} x^{n+\alpha}y^{n}$$

$$P_n^{(\alpha,\alpha)}\left(1-\frac{2x}{y}\right)=\frac{(1+\alpha)_n x^{-\alpha}y^{-n}}{n!}\left(1+\frac{D_y}{D_x}\right)^{-n-2\alpha-1} x^{\alpha}y^{n}$$

$$P_n^{(\alpha,\alpha)}\left(1-\frac{2x}{y}\right)=\frac{(1+\alpha)_n x^{-\alpha}y^{1+2\alpha+n}}{n!}\left(1+\frac{D_y}{D_x}\right)^{n} x^{\alpha}y^{-1-2\alpha-n}$$

$$C_n^{(\nu)}\left(1-\frac{2x}{y}\right)=\frac{(2\nu)_n x^{-\nu+\frac{1}{2}}y^{-n}}{n!}\left(1+\frac{D_y}{D_x}\right)^{-n-2\nu} x^{\nu-\frac{1}{2}}y^{n}$$

$$C_n^{(\nu)}\left(1-\frac{2x}{y}\right)=\frac{(2\nu)_n x^{-\nu+\frac{1}{2}}y^{2\nu+n}}{n!}\left(1+\frac{D_y}{D_x}\right)^{n} x^{\nu-\frac{1}{2}}y^{-2\nu-n}$$

$$y_n\left(\frac{1}{x}\right)=x^{-n}\left(1-\frac{D_x}{2}\right)^{-n-1}x^n$$

$$y_n\left(\frac{1}{x}\right)=x^{n+1}\left(1-\frac{D_x}{2}\right)^{n}x^{-n-1}$$

$$y_n\left(a,b,\frac{1}{x}\right)=x^{-n}\left(1-\frac{D_x}{b}\right)^{-n-a+1}x^n$$

$$y_n\left(a,b,\frac{1}{x}\right)=x^{n+a-1}\left(1-\frac{D_x}{b}\right)^{-n}x^{-n-a+1}$$

$$g_n^{(\alpha,\beta)}(x,y)=\frac{(\alpha)_n}{n!}y^{n+\alpha}x^{-n}(1+\frac{D_x}{D_y})^{-\beta}y^{-n-\alpha}x^n$$

$$g_n^{(\alpha,\beta)}\left(\frac{1}{x},\frac{1}{y}\right)=\frac{(\alpha)_n}{n!}x^{n+\alpha}y^{\beta}\left(1+\frac{D_y}{D_x}\right)^n x^{-n-\alpha}y^{\beta}$$

$$\Phi_n(x)=\frac{1}{n!}\left(1+D_x\right)^{-x}x^n$$

$$R_n(a,x)=\frac{(a)_{2n}}{(a)_n n!}x^{-n-a+1}\left(1+\frac{1}{D_x}\right)^n x^{n+a-1}$$

$$R_n\left(a,\frac{x}{y}\right)=\frac{(a)_{2n}}{(a)_n n!}x^{-n-a+1}y^{-n}\exp\left(\frac{-D_y}{D_x}\right)x^{n+a-1}y^n$$

$$H_n\left(\frac{x}{2}\right)=e^{-D_x^2}x^n$$

$$H_n\left(\sqrt{x}\right)=2^n\left(1-D_x\right)^{\frac{n}{2}-\frac{1}{2}}x^{\frac{n}{2}}$$

$$H_n\left(\sqrt{x}\right)=2^n x^{\frac{1}{2}}\left(1-D_x\right)^{\frac{n}{2}}x^{\frac{n}{2}-\frac{1}{2}}$$

$$Z_n\left(x\right)=\frac{1}{n!}D_x^n\left\{x^n\left(1-\frac{1}{D_x}\right)^n x^0\right\}$$

$$Z_n(x)=\frac{1}{(n!)^2}D_x^n\left\{x^n(D_x-1)^n x^n\right\}$$

$$Z_n^\nu\left(b,\frac{xy}{z}\right)=z^{2\nu+n}y^{-b}x^{-\nu+\frac{1}{2}}\left(1+\frac{D_z}{D_xD_y}\right)^n z^{-2\nu-n}y^bx^{\nu-\frac{1}{2}}$$

$$Z_n^\nu\left(b,\frac{xy}{z}\right)=x^{-b}y^{-\nu+\frac{1}{2}}z^{-n}\left(1+\frac{D_z}{D_xD_y}\right)^{-2\nu-n}x^by^{\nu-\frac{1}{2}}z^n$$

$$Z_n^\nu\left(b,\frac{xy}{zw}\right)=z^{-n}y^{-b}x^{-\nu+\frac{1}{2}}w^{2\nu+n}\exp\left(\frac{D_zD_w}{D_xD_y}\right)z^ny^bx^{\nu-\frac{1}{2}}w^{-2\nu-n}$$

$$Z_n^{(\alpha,\beta)}\left(b,\frac{xy}{z}\right)=x^{-\alpha}y^{-b}z^{-n}\left(1+\frac{D_z}{D_xD_y}\right)^{-n-\alpha-\beta-1}x^\alpha y^bz^n$$

$$Z_n^{(\alpha,\beta)}\left(b,\frac{xy}{z}\right)=x^{-\alpha}y^{-b}z^{n+\alpha+\beta+1}\left(1+\frac{D_z}{D_xD_y}\right)^n x^\alpha y^bz^{-n-\alpha-\beta-1}$$

$$Z_n^{(\alpha,\beta)}\left(b,\frac{xy}{zw}\right)=x^{-\alpha}y^{-b}z^{n+\alpha+\beta+1}w^{-n}\exp\left(\frac{D_zD_w}{D_xD_y}\right)x^\alpha y^bz^{-n-\alpha-\beta-1}w^n$$

$$T_n\left(1-\frac{2x}{y}\right)=x^{\frac{1}{2}}y^{-n}\left(1+\frac{D_y}{D_x}\right)^{-n}x^{-\frac{1}{2}}y^n$$

$$U_n\left(1-\frac{2x}{y}\right)=(n+1)x^{-\frac{1}{2}}y^{-n}\left(1+\frac{D_y}{D_x}\right)^{-n-2}x^{\frac{1}{2}}y^n$$

$$g_n^{(s)}\left(\frac{x}{y}\right)=\binom{s+n}{n}x^{s+n+1}y^{-n}\left(1+\frac{D_y}{D_x}\right)^{-1}x^{-s-n-1}y^n$$

$$H_n\left(\xi,p,\frac{y}{x}\right)=\frac{x^\xi}{n!}D_y^n\left(y^{n-p+1}\left(1+\frac{D_x}{D_y}\right)^n x^{-\xi}y^{-p-1}\right)$$

$$H_n^{(\alpha,\beta)}\left(\xi,p,\frac{xy}{zw}\right)=x^{-\alpha}y^{-p+1}z^{-n}w^\xi\left(1-\frac{D_zD_w}{D_xD_y}\right)^{-\alpha-\beta-n-1}x^\alpha y^{p-1}z^nw^{-\xi}$$

$$G_n\left(\alpha,\beta;\frac{x}{2\sqrt{yz}}\right)=\frac{(\alpha)_n(\beta)_n}{n!(\alpha+\beta)_n}z^{\alpha+\frac{n}{2}}y^{\beta+\frac{n}{2}}\left(1-\frac{D_x^2}{D_yD_z}\right)^{\beta+\alpha+n-1}x^ny^{-\beta-n}z^{-\alpha-n}$$

$$R_n\left(\beta,\gamma;\frac{x}{2\sqrt{yz}}\right)=\frac{(\beta)_n}{n!}z^{-\gamma+1-\frac{n}{2}}y^{\beta+\frac{n}{2}}\left(1-\frac{D_x^2}{D_yD_z}\right)^{\beta-\gamma}x^ny^{-\beta-n}z^{\gamma-1}.$$

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