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A NOTE ON DISCRETELY ABSOLUTELY STAR-LINDELÖF SPACES

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Abstract. In this paper, we prove the following statements:

(1) If X is a normal discretely absolutely star-Lindelöf space with $e(X) < \omega_1$, then the Alexandroff duplicate $A(X)$ of X is discretely absolutely star-Lindelöf.

(2) If X is a space with $e(X) \geq \omega_1$, then $A(X)$ is not discretely absolutely star-Lindelöf.

The two statements answer a question raised by Song.

1. Introduction

By a space, we mean a T_1 topological space. Recall that a space X is *star-Lindelöf* (see [2, 4] under different names) (*discretely star-Lindelöf*) (see [7, 9]) if for every open cover $\mathcal{U}$ of X, there exists a countable subset (a countable discrete closed subset, respectively) F of X such that $St(F,\mathcal{U}) = X$, where $St(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$. It is clear that every separable space and every discretely star-Lindelöf space is star-Lindelöf as well as every space of countable extent, in particular, every countably compact space or every Lindelöf space.

A space X is *absolutely star-Lindelöf* (see [1, 5]) (*discretely absolutely star-Lindelöf*) (see [6, 8]) if for every open cover $\mathcal{U}$ of X and every dense subspace D of X, there exists a countable subset F of D such that $St(F,\mathcal{U}) = X$ (F is discrete and closed in X such that $St(F,\mathcal{U}) = X$, respectively).

From the above definitions, it is not difficult to see that every absolutely star-Lindelöf space is star-Lindelöf, every discretely absolutely star-Lindelöf space is absolutely star-Lindelöf and every discretely absolutely star-Lindelöf space is discretely star-Lindelöf.

Song [8] constructed an example showing that there exists a Tychonoff discretely absolutely star-Lindelöf space X with $e(X) = \mathfrak{c}$ such that the Alexandroff

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duplicate $A(X)$ of X is not discretely absolutely star-Lindelöf, and asked the following question:

Question. Does there exist a normal discretely absolutely star-Lindelöf space X such that $A(X)$ is not discretely absolutely star-Lindelöf?

The purpose of this paper is to answer the question by showing two statements stated in the abstract.

Moreover, the cardinality of a set A is denoted by $|A|$. The extent $e(X)$ of a space X is the smallest infinite cardinal κ such that every discrete closed subset of a space X has cardinality at most κ. Let ω denote the first infinite cardinal and ω_1 the first uncountable cardinal. Other terms and symbols that we do not define will be used as in [3].

2. Some results on discretely absolutely star-Lindelöf spaces

For a space X, recall that the Alexandroff duplicate $A(X)$ of a space X, denoted by $A(X)$, is constructed in the following way: The underlying set of $A(X)$ is $X \times \{0,1\}$ and each point of $X \times \{1\}$ is isolated; a basic neighbourhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is a set of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x of X. It is well-known that $A(X)$ is compact (countably compact, Lindelöf) if and only if so is X and $A(X)$ is Hausdorff (regular, Tychonoff, normal) if and only if so is X.

Theorem 2.1. *If X is a normal discretely absolutely star-Lindelöf space X with $e(X) < \omega_1$, then $A(X)$ is discretely absolutely star-Lindelöf.*

Proof. We prove that $A(X)$ is discretely absolutely star-Lindelöf. To this end, let $\mathcal{U}$ be an open cover of $A(X)$. Clearly every point of $X \times \{1\}$ is isolated. Let E be the set of all isolated points of X, and let

$$D = (X \times \{1\}) \cup (E \times \{0\}).$$

Then D is a dense subspace of $A(X)$ and every dense subset of $A(X)$ contains D. Thus it is sufficient to show that there exists a countable subset $F \subseteq D$ such that F is discrete closed in $A(X)$ and $St(F, \mathcal{U}) = A(X)$. For every $x \in X$, we pick an open neighborhood $(U_x \times \{0,1\}) \setminus \{\langle x, 1\rangle\}$ of $\langle x, 0\rangle$ such that $(U_x \times \{0,1\}) \setminus \{\langle x, 1\rangle\} \subseteq U$ for some $U \in \mathcal{U}$, where U_x is an open subset of X containing x. If we put

$$\mathcal{V} = \{U_x : x \in X\}.$$

Then $\mathcal{V}$ is an open cover of X. Hence there exists a countable subset $F_0 \subseteq X$ such that F_0 is discrete closed in X and $St(F_0, \mathcal{V}) = X$, since X is discretely absolutely star-Lindelöf and every discretely absolutely star-Lindelöf space is discretely star-Lindelöf. For the collection $\{U_x : x \in F_0\}$ of X, since F_0 is a discrete closed subset of a normal space X, then there exists a pairwise disjoint open family $\{V_x : x \in F_0\}$ in X such that $x \in V_x \subseteq U_x$ for each $x \in F_0$. By normality of X there exists an open subset V of X such that

$$F_0 \subseteq V \subseteq \overline{V} \subseteq \bigcup_{x \in F_0} V_x.$$

Obviously, $\{V \cap V_x : x \in F_0\}$ is a discrete family of nonempty open subsets of X.

Let

$$F_1' = \{x \in F_0 : x \text{ is not isolated in } X\}.$$

For each $x \in F_1'$, we pick $y_x \in V \cap V_x$ such that $x \neq y_x$. Then

$$\{x : x \in F_0\} \cup \{y_x : x \in F_1'\}$$

is discrete closed in X and

$$\langle y_x, 1\rangle, \langle x, 0\rangle \in (U_x \times \{0,1\}) \setminus \{\langle x, 1\rangle\} \text{ for each } x \in F_1'.$$

Let $F_1 = F_0 \times \{1\}$. For each $x \in X \setminus (F_0 \cup \{U_x : x \in F_1'\})$, there exists $x' \in X$ such that $x \in U_{x'}$ and $U_{x'} \cap F_0 \neq \emptyset$, hence $(U_{x'} \times \{0,1\} \setminus \{\langle x', 1\rangle\} \cap F_1 \neq \emptyset$. Let

$$F_2 = F_1 \cup \{\langle y_x, 1\rangle : x \in F_1'\} \cup \{(F_0 \setminus F_1') \times \{0\}).$$

Then F_2 is a countable discrete closed (in $A(X)$) subset of D and $X \times \{0\} \subseteq St(F_2, \mathcal{U})$. Let $F_3 = A(X) \setminus St(F_1, \mathcal{U})$. Then F_3 is a discrete and closed subset of $A(X)$ and $F_3 \subseteq D$. Since $e(X) < \omega_1$, then $e(A(X)) < \omega_1$. Thus F_3 is countable.

If we put $F = F_2 \cup F_3$, then F is a countable discrete closed (in $A(X)$) subset of D such that $A(X) = St(F, \mathcal{U})$, which completes the proof. ∎

THEOREM 2.2. *If X is a space with $e(X) \geq \omega_1$, then $A(X)$ is not discretely absolutely star-Lindelöf.*

Proof. Since $e(X) \geq \omega_1$, then there exists a discrete closed subset B of X such that $|B| \geq \omega_1$, hence $B \times \{1\}$ is a closed and open subset of $A(X)$ and every point of $B \times \{1\}$ is an isolated point of $A(X)$. To show that $A(X)$ is not discretely absolutely star-Lindelöf. Let C be the set all isolated points of X. Let us consider the open cover

$$\mathcal{U} = \{A(X) \setminus (B \times \{1\})\} \cup \{\langle x, 1\rangle : x \in B\}$$

and the dense subset

$$D = (C \times \{0\}) \cup (X \times \{1\})$$

of $A(X)$. Then, for any countable subset F of D there exists a point $x \in B$ such that $\langle x, 1\rangle \notin F$, since $|B| \geq \omega_1$. Hence $\langle x, 1\rangle \notin St(F, \mathcal{U})$, since $\{\langle x, 1\rangle\}$ is the only element of $\mathcal{U}$ containing $\langle x, 1\rangle$ for each $x \in B$, which completes the proof. ∎

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REFERENCES

[**1**] M. Bonanzinga, *Star-Lindelöf and absolutely star-Lindelöf spaces*, Questions Answers Gen. Topology, **16** (1998), 79–104.

[**2**] E.K. van Douwen, G.M. Reed, A.W. Roscoe, I.J. Tree, *Star covering properties*, Topology Appl. **39** (1991), 71–103.

[**3**] R. Engelking, *General Topology*, Revised and completed edition, Heldermann Verlag, Berlin, 1989.

[**4**] M.V. Matveev, *A survey on star covering properties*, Topology Atlas, preprint No. 330, 1998.

[**5**] M.V. Matveev, O.I. Pavlov, J.K. Tartir, *On relatively normal spaces, relatively regular space and on relative property (a)*, Topology Appl. **93** (1999), 121–122.

[**6**] Y-K. Song, *Remarks on star-Lindelöf spaces*, Quest. Answers Gen. Topology. **20** (2002), 49–51.

[**7**] Y-K Song, *Discretely star-Lindelöf spaces*, Tsukuba J. Math. **25** (2001), 371–382.

[**8**] Y-K. Song, *Remarks on discretely absolutely star-Lindelöf spaces*, Czech. Math. J. **58**(133) (2008), 823–831.

[**9**] Y. Yasui, Z.M. Gao, *Spaces in countable web*, Houston J. Math. **25** (1999), 327–335.

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