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β-CONNECTEDNESS AND $\mathcal{S}$-CONNECTEDNESS OF TOPOLOGICAL SPACES

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Abstract. Characterizations of β-connectedness and $\mathcal{S}$-connectedness of topological spaces are investigated. Further results concerning preservation of these connectedness-like properties under surjections are obtained. The paper completes our previous study [Z. Duszyński, *On some concepts of weak connectedness of topological spaces*, Acta Math. Hungar. **110** (2006), 81–90].

1. Preliminaries

Throughout the present paper, (X,τ) denotes a topological space. Let S be a subset of (X,τ). By $\operatorname{int}(S)$ (or $\operatorname{int}_\tau(S)$) and $\operatorname{cl}(S)$ (or $\operatorname{cl}_\tau(S)$) we denote the interior of S and the closure of S, respectively. An S is said to be *α-open* [18] (resp. *semi-open* [15], *preopen* [17], b-*open* [3] (equiv. *γ-open* [4] or sp-*open* [7]), *β-open* [1] (equiv. *semi-preopen* [2])) in (X,τ), if $S\subset\operatorname{int}(\operatorname{cl}(\operatorname{int}(S)))$ (resp. $S\subset\operatorname{cl}(\operatorname{int}(S))$, $S\subset\operatorname{int}(\operatorname{cl}(S))$, $S\subset\operatorname{int}(\operatorname{cl}(S))\cup\operatorname{cl}(\operatorname{int}(S))$, $S\subset\operatorname{cl}(\operatorname{int}(\operatorname{cl}(S)))$). An S is said to be *semi-closed* [5] (resp. b-*closed* [3], *β-closed* [1] (equiv. *semi-preclosed* [2])) in (X,τ), if $S\supset\operatorname{int}(\operatorname{cl}(S))$ (resp. $S\supset\operatorname{int}(\operatorname{cl}(S))\cap\operatorname{cl}(\operatorname{int}(S))$, $S\supset\operatorname{int}(\operatorname{cl}(\operatorname{int}(S)))$). The family of all α-open (resp. semi-open, semi-closed, preopen, b-open, b-closed, β-open, β-closed) subsets of (X,τ) is denoted by τ^α (resp. $\operatorname{SO}(X,\tau)$, $\operatorname{SC}(X,\tau)$, $\operatorname{PO}(X,\tau)$, $\operatorname{BO}(X,\tau)$, $\operatorname{BC}(X,\tau)$, $\operatorname{SPO}(X,\tau)$, $\operatorname{SPC}(X,\tau)$). The sets in $\operatorname{SO}(X,\tau)\cap\operatorname{SC}(X,\tau)=\operatorname{SR}(X,\tau)$ are called semi-regular (in (X,τ)) [6]. The family τ^α forms a topology on (X,τ) such that $\tau\subset\tau^\alpha$ [18]. The following inclusions hold in any space (X,τ): $\tau^\alpha=\operatorname{SO}(X,\tau)\cap\operatorname{PO}(X,\tau)$ [21], $\operatorname{SO}(X,\tau)\cup\operatorname{PO}(X,\tau)\subset\operatorname{BO}(X,\tau)\subset\operatorname{SPO}(X,\tau)$ [3]. The intersection of any family $\{S_i\}_{i\in\mathcal{I}}\subset\operatorname{SC}(X,\tau)$ (resp. $\{S_i\}_{i\in\mathcal{I}}\subset\operatorname{SPC}(X,\tau)$) is a member of $\operatorname{SC}(X,\tau)$ (resp. $\operatorname{SPC}(X,\tau)$). The union of any family $\{S_i\}_{i\in\mathcal{I}}\subset\operatorname{SPO}(X,\tau)$ is a member of $\operatorname{SPO}(X,\tau)$. The operators of *semi-closure* [5] (briefly: $\operatorname{scl}(.)$), *preclosure* ($\operatorname{pcl}(.)$), b-*closure* ($\operatorname{bcl}(.)$), *semi-preclosure*, *semi-preinterior* ($\operatorname{spcl}(.)$, $\operatorname{spint}(.)$ resp.) are defined in a manner similar to that of definitions of ordinary closure and interior (compare [16]). The following properties will be useful in the sequel:

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1° $\operatorname{spcl}(\operatorname{spint}(S)) = \operatorname{spint}(\operatorname{spcl}(S))$ for every S [2, Theorem 3.18]; 2° if either $S_1 \in \mathrm{SO}(X,\tau)$ or $S_2 \in \mathrm{SO}(X,\tau)$, then $\operatorname{int}(\operatorname{cl}(S_1 \cap S_2)) = \operatorname{int}(\operatorname{cl}(S_1)) \cap \operatorname{int}(\operatorname{cl}(S_2))$ [21, Lemma 3.5] (see also [10, Lemma 3]); 3° $\operatorname{cl}(S) \in \mathrm{SO}(X,\tau)$ for any $S \in \mathrm{SO}(X,\tau)$; 4° every non-empty semi-open set S has non-empty interior [5, Remark 1.2]. A space (X,τ) is said to be *semi-connected* [24] (or $\mathcal{S}$-connected) (resp. *preconnected* [25] (or $\mathcal{P}$-connected)) if X cannot be split into two nonempty members of $\mathrm{SO}(X,\tau)$ (resp. $\mathrm{PO}(X,\tau)$).

2. β-connectedness

Definition 1. A topological space (X,τ) is said to be *β-connected* [26] (resp. *γ-connected* [12]), if X cannot be expressed as a union of two non-empty and disjoint semi-open (resp. b-open) subsets of (X,τ).

We will need the following lemma

Lemma 1. *Let S be an arbitrary subset of (X,τ). Then:*

1° $\operatorname{cl}(\operatorname{int}(S)) \cap \operatorname{int}(\operatorname{cl}(S)) \in \mathrm{SR}(X,\tau)$,

2° $\operatorname{cl}(\operatorname{int}(S)) \cup \operatorname{int}(\operatorname{cl}(S)) \in \mathrm{SR}(X,\tau)$.

Proof. 1° Clearly, the sets $\operatorname{cl}(\operatorname{int}(S)), \operatorname{int}(\operatorname{cl}(S)) \in \mathrm{SC}(X,\tau)$. Hence $\operatorname{cl}(\operatorname{int}(S)) \cap \operatorname{int}(\operatorname{cl}(S)) \in \mathrm{SC}(X,\tau)$ [5, Remark 1.4]. It is enough to show that this set is a member of $\mathrm{SO}(X,\tau)$. Indeed, we calculate as follows:

$$\operatorname{cl}\big(\operatorname{int}\big(\operatorname{cl}(\operatorname{int}(S)) \cap \operatorname{int}(\operatorname{cl}(S))\big)\big) = \operatorname{cl}\big(\operatorname{int}(\operatorname{cl}(\operatorname{int}(S))) \cap \operatorname{int}(\operatorname{cl}(S))\big) \supset \supset \operatorname{cl}(\operatorname{int}(S)) \cap \operatorname{int}(\operatorname{cl}(S)).$$

2° Follows immediately by 1°. ■

Theorem 1. *If a space (X,τ) is $\mathcal{S}$-connected and $\mathcal{P}$-connected, then it is γ-connected.*

Proof. Assume (X,τ) is not γ-connected. Then $X = S_1 \cup S_2$, where $S_1 \neq \varnothing \neq S_2$, $S_1 \cap S_2 = \varnothing$, and $S_1, S_2 \in \mathrm{BO}(X,\tau)$. We consider two cases.

1° Let $\operatorname{int}(S_1) = \varnothing = \operatorname{int}(S_2)$. By the definition of b-openness we directly get $S_1 \subset \operatorname{int}(\operatorname{cl}(S_1))$ and $S_2 \subset \operatorname{int}(\operatorname{cl}(S_2))$. So, $S_1, S_2 \in \mathrm{PO}(X,\tau)$ and (X,τ) is not $\mathcal{P}$-connected.

2° Let $\operatorname{int}(S_1) \neq \varnothing$. It is enough to show that

$$\varnothing \neq \operatorname{cl}(\operatorname{int}(S_1)) \cap \operatorname{int}(\operatorname{cl}(S_1)) \neq X \tag{1}$$

(compare Lemma 1). By the inclusions $\operatorname{int}(S_1) \subset \operatorname{int}(\operatorname{cl}(S_1))$ and $\operatorname{int}(S_1) \subset \operatorname{cl}(\operatorname{int}(S_1))$ we have $\operatorname{int}(S_1) \subset \operatorname{cl}(\operatorname{int}(S_1)) \cap \operatorname{int}(\operatorname{cl}(S_1))$ and so $\operatorname{cl}(\operatorname{int}(S_1)) \cap \operatorname{int}(\operatorname{cl}(S_1)) \neq \varnothing$. On the other hand, if, suppose, $\operatorname{cl}(\operatorname{int}(S_1)) \cap \operatorname{int}(\operatorname{cl}(S_1)) = X$, then since $S_1 \in \mathrm{BC}(X,\tau)$ one gets $X \subset S_1$. This shows that (1) holds, that is, (X,τ) is not $\mathcal{S}$-connected. ■

Corollary 1. *A space* (X,τ) *is* γ*-connected if and only if it is* $\mathcal{S}$*-connected and* $\mathcal{P}$*-connected.*

Lemma 2. *Let* (X,τ) *be a space. If there exists disjoint sets* $S_1, S_2 \subset X$ *such that* $S_1 \cup S_2 = X$ *and* $\mathrm{cl}\,(S_1) = X = \mathrm{cl}\,(S_2)$*, then* (X,τ) *is not* $\mathcal{P}$*-connected.*

Proof. Clear. ■

Theorem 2. *If a space* (X,τ) *is* γ*-connected then it is* β*-connected.*

Proof. Suppose (X,τ) is not β-connected. Then for some disjoint sets $S_1, S_2 \in \mathrm{SPO}\,(X,\tau)$ with $S_1 \neq \varnothing \neq S_2$ we have $S_1 \cup S_2 = X$.

1° Let $\mathrm{cl}\,(S_1) = X = \mathrm{cl}\,(S_2)$. From Lemma 2 and the inclusion $\mathrm{PO}\,(X,\tau) \subset \mathrm{BO}\,(X,\tau)$ we infer that (X,τ) is not γ-connected.

2° Let $\mathrm{cl}\,(S_1) \neq X$. It is not difficult to check that in this case we have $\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S_1))) \neq X$. We get the following split of X: $X = \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S_1))) \cup \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S_2)))$, where $\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S_1))) \neq \varnothing$ because $S_1 \in \mathrm{SPO}\,(X,\tau)$. But the sets $\mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S_1))), \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S_2))) \in \mathrm{SO}\,(X,\tau)$ and by the inclusion $\mathrm{SO}\,(X,\tau) \subset \mathrm{BO}\,(X,\tau)$, (X,τ) is not γ-connected. ■

Corollary 2. *A space* (X,τ) *is* γ*-connected if and only if it is* β*-connected.*

Remark 1. If (X,τ) is $\mathcal{S}$-connected and $\mathcal{P}$-connected, then (X,τ) is connected ($\mathcal{S}$-connectedness and $\mathcal{P}$-connectedness are independent notions – see [14, Examples 2.1&2.2]. The problem arises, does the reverse implication hold?

Definition 2. A space (X,τ) is said to be *B-SP-connected* (resp. *P-SP-connected*) if X cannot be written as a union of two non-empty disjoint sets $S_1, S_2 \subset X$ such that $S_1 \in \mathrm{BO}\,(X,\tau)$, $S_2 \in \mathrm{SPO}\,(X,\tau)$ (resp. $S_1 \in \mathrm{PO}\,(X,\tau)$, $S_2 \in \mathrm{SPO}\,(X,\tau)$).

Theorem 3. *For every topological space* (X,τ) *the following statements are equivalent:*

1° (X,τ) *is* β*-connected,*

2° (X,τ) *is B-SP-connected,*

3° (X,τ) *is P-SP-connected,*

4° (X,τ) *is* $\mathcal{S}$*-connected and* $\mathcal{P}$*-connected.*

Proof. The implications 1° ⇒ 2° and 2° ⇒ 3° are obvious. For 3° ⇒ 4° see [9, Theorem 9]. 4° ⇒ 1° follows by Corollaries 1 and 2. ■

A proof for the following lemma is clear (see [14, Theorem 3.1(6)]).

Lemma 3. *A space* (X,τ) *is* β*-connected if and only if there is no set* $S \in \mathrm{SPR}\,(X,\tau) = \mathrm{SPO}\,(X,\tau) \cap \mathrm{SPC}\,(X,\tau)$ *such that* $\varnothing \neq S \neq X$.

Lemma 4. [14, Theorem 3.1] *In every* (X,τ) *the following properties are equivalent:*

1° (X, τ) *is* β*-connected,*

2° $\operatorname{pcl}(S) = X$ *for each non-empty* $S \in \operatorname{PO}(X, \tau)$*,*

3° $\operatorname{pcl}(S) = X$ *for each non-empty* $S \in \operatorname{SPO}(X, \tau)$*,*

4° $\operatorname{spcl}(S) = X$ *for each non-empty* $S \in \operatorname{PO}(X, \tau)$*,*

5° $\operatorname{spcl}(S) = X$ *for each non-empty* $S \in \operatorname{SPO}(X, \tau)$*.*

Theorem 4. *In every topological space* (X, τ) *the following statements are equivalent:*

1° (X, τ) *is* β*-connected,*

2° $\operatorname{bcl}(S) = X$ *for each non-empty* $S \in \operatorname{PO}(X, \tau)$*,*

3° $\operatorname{bcl}(S) = X$ *for each non-empty* $S \in \operatorname{SPO}(X, \tau)$*,*

4° $\operatorname{pcl}(S) = X$ *for each non-empty* $S \in \operatorname{BO}(X, \tau)$*,*

5° $\operatorname{bcl}(S) = X$ *for each non-empty* $S \in \operatorname{BO}(X, \tau)$ (see [12, Theorem 3]),

6° $\operatorname{spcl}(S) = X$ *for each non-empty* $S \in \operatorname{BO}(X, \tau)$*.*

Proof. 1° ⇒ 3°. Let arbitrary $S \in \operatorname{SPO}(X, \tau)$ be non-empty. By Lemma 4 we have $X = \operatorname{spcl}(S) \subset \operatorname{bcl}(S) \subset \operatorname{pcl}(S) = X$. Thus $\operatorname{bcl}(S) = X$.

3° ⇒ 2° is obvious.

2° ⇒ 1°. Let for $\varnothing \neq S \in \operatorname{PO}(X, \tau)$, $\operatorname{bcl}(S) = X$. So, $\operatorname{pcl}(S) = X$ (since $\operatorname{bcl}(S) \subset \operatorname{pcl}(S)$) and consequently by Lemma 4, (X, τ) is β-connected.

1° ⇒ 6°. Let for some S, $\varnothing \neq S \in \operatorname{BO}(X, \tau)$, $\operatorname{spcl}(S) \neq X$. Since $S \in \operatorname{SPO}(X, \tau)$ we have $S = \operatorname{spint}(S)$. Hence $\varnothing \neq \operatorname{spcl}(\operatorname{spint}(S)) \neq X$. By [2, Theorem 3.18], $\operatorname{spcl}(\operatorname{spint}(S)) = \operatorname{spint}(\operatorname{spcl}(S)) = S_1 \in \operatorname{SPR}(X, \tau)$. Thus it follows from Lemma 3 that (X, τ) is not β-connected.

6° ⇒ 5° ⇒ 4°. $X = \operatorname{spcl}(S) \subset \operatorname{bcl}(S) \subset \operatorname{pcl}(S)$.

4° ⇒ 1°. Let (X, τ) be not β-connected. Then by Lemma 4 there exists a non-empty set $S \in \operatorname{PO}(X, \tau) \subset \operatorname{BO}(X, \tau)$ with $\operatorname{pcl}(S) \neq X$. ■

3. $\mathcal{S}$-connectedness

Definition 3. A topological space (X, τ) is said to be *α-B-connected* (resp. *α-SP-connected*, *α-S-connected* [9], if X cannot be expressed as a union of two non-empty disjoint sets $S_1, S_2 \subset X$ such that $S_1 \in \tau^\alpha$ and $S_2 \in \operatorname{BO}(X, \tau)$ (resp. $S_2 \in \operatorname{SPO}(X, \tau)$, $S_2 \in \operatorname{SO}(X, \tau)$).

Theorem 5. *For every topological space* (X, τ) *the following are equivalent:*

1° (X, τ) *is* $\mathcal{S}$*-connected,*

2° (X, τ) *is* α*-S-connected,*

3° (X, τ) *is* α*-SP-connected,*

4° (X, τ) *is* α*-B-connected.*

Proof. $1° \Leftrightarrow 2°$. [9, Corollary 1]. $1° \Rightarrow 3°$. Let $X = S_1 \cup S_2$, where $S_1 \cap S_2 = \varnothing$, $S_1 \neq \varnothing \neq S_2$, $S_1 \in \tau^\alpha$ and $S_2 \in \mathrm{SPO}\,(X,\tau)$. Obviously we have $X = \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S_1))) \cup \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S_2)))$. It is enough to show that $\mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S_1))) \cap \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S_2))) = \varnothing$. Using [21, Lemma 3.5] we calculate as follows: $\mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S_1))) \cap \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S_2))) \subset \mathrm{cl}\,\big(\mathrm{int}\,(\mathrm{cl}\,(S_2)) \cap \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S_1)))\big) \subset \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S_1 \cap S_2))) = \varnothing$. $3° \Rightarrow 4°$ and $4° \Rightarrow 2°$ are obvious. ∎

Definition 4. A space (X,τ) is said to be *S-P-connected* (resp. *S-B-connected*; *S-SP-connected* [9]), if X cannot be expressed as a union of two non-empty disjoint sets $S_1, S_2 \subset X$ such that $S_1 \in \mathrm{SO}\,(X,\tau)$ and $S_2 \in \mathrm{PO}\,(X,\tau)$ (resp. $S_2 \in \mathrm{BO}\,(X,\tau)$, $S_2 \in \mathrm{SPO}\,(X,\tau)$).

Theorem 6. *For every topological space (X,τ) the following are equivalent:*

1° *(X,τ) is S-P-connected,*

2° *(X,τ) is S-B-connected,*

3° *(X,τ) is S-SP-connected,*

4° *(X,τ) is $\mathcal{S}$-connected.*

Proof. $1° \Leftrightarrow 2°$. [9, Corollary 3]. Implications $1° \Rightarrow 3°$ and $3° \Rightarrow 4°$ are clear. $4° \Rightarrow 1°$. Suppose (X,τ) is not $\mathcal{S}$-connected; i.e., equivalently, (X,τ) is not α-S-connected [9, Corollary 1]. Let $X = S_1 \cup S_2$, where $S_1 \cap S_2 = \varnothing$, $S_1 \neq \varnothing \neq S_2$, $S_1 \subset \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S_1)))$ and $S_2 \subset \mathrm{cl}\,(\mathrm{int}\,(S_2))$. Obviously $X = \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S_1))) \cup \mathrm{cl}\,(\mathrm{int}\,(S_2))$. On the other hand, by [21, Lemma 3.5] one has what follows:

$$\mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S_1))) \cap \mathrm{cl}\,(\mathrm{int}\,(S_2)) \subset$$
$$\subset \mathrm{cl}\,\big(\mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S_1))) \cap \mathrm{int}\,(\mathrm{cl}\,(S_2))\big) \subset \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S_1 \cap S_2))) = \varnothing.$$

Therefore (X,τ) is not S-P-connected, because $\varnothing \neq \mathrm{cl}\,(\mathrm{int}\,(S_2)) \in \mathrm{SO}\,(X,\tau)$ and $\varnothing \neq \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S_1))) \in \tau \subset \mathrm{PO}\,(X,\tau)$. ∎

Definition 5. A space (X,τ) is said to be *τ-S-connected* (resp. *τ-B-connected*; *τ-SP-connected*), if X cannot be written as a union of two non-empty disjoint sets $S_1, S_2 \subset X$ such that $S_1 \in \tau$ and $S_2 \in \mathrm{SO}\,(X,\tau)$ (resp. $S_2 \in \mathrm{BO}\,(X,\tau)$, $S_2 \in \mathrm{SPO}\,(X,\tau)$).

Theorem 7. *For every topological space (X,τ) the following are equivalent:*

1° *(X,τ) is $\mathcal{S}$-connected,*

2° *(X,τ) is τ-SP-connected,*

3° *(X,τ) is τ-B-connected,*

4° *(X,τ) is τ-S-connected.*

Proof. $2° \Rightarrow 3°$ and $3° \Rightarrow 4°$ are obvious. $1° \Rightarrow 2°$. Suppose (X,τ) is not τ-S-connected. Then it is not S-SP-connected; i.e., not $\mathcal{S}$-connected. $4° \Rightarrow 1°$. Suppose (X,τ) is not $\mathcal{S}$-connected. By [9, Corollary 1] (X,τ) is not α-S-connected. The rest is the same as in the proof of Theorem 6, case $4° \Rightarrow 1°$. ∎

Definition 6. A space (X,τ) is called *B(*int*)-connected* if X cannot be split into two non-empty disjoint b-open sets $S_1, S_2 \subset X$ with $\operatorname{int}(S_1) \neq \varnothing \neq \operatorname{int}(S_2)$.

Theorem 8. *A space (X,τ) is $\mathcal{S}$-connected if and only if it is B(*int*)-connected.*

Proof. Let (X,τ) be not $\mathcal{S}$-connected; i.e., $X = S_1 \cup S_2$, $S_1 \cap S_2 = \varnothing$, $S_1 \neq \varnothing \neq S_2$ for certain $S_1, S_2 \in \mathrm{SO}(X,\tau)$. But, each non-empty semi-open set has non-empty interior [5, Remark 1.2]. Thus (X,τ) is not B(int)-connected, since $\mathrm{SO}(X,\tau) \subset \mathrm{BO}(X,\tau)$. For the converse, if (X,τ) is not $B($int$)$-connected, then it is not *SP*(int)-connected (see [9, Definition 3] or in the sequel). The latter is equivalent that (X,τ) is not $\mathcal{S}$-connected (by [9, Corollary 2]). ∎

Remark 2. In the proof of the case 2° of Theorem 1 we have applied relations (1) and Lemma 1.1°. Generally, one can express the following characterization of $\mathcal{S}$-connectedness (analogously to Lemma 3): a space (X,τ) is $\mathcal{S}$-connected if and only if there is no set $S \in \mathrm{SR}(X,\tau) = \mathrm{SO}(X,\tau) \cap \mathrm{SC}(X,\tau^\alpha)$ such that $\varnothing \neq S \neq X$.

Lemma 5. *Let (X,τ) be any space. Then:*

1° $\mathrm{cl}_{\tau^\alpha}(S) = \mathrm{cl}_\tau(S)$ *for every set* $S \in \mathrm{SO}(X,\tau)$ [13, Lemma 1(i)],

2° $\mathrm{scl}(S) = \mathrm{bcl}(S) = \mathrm{spcl}(S)$ *for every set* $S \in \mathrm{SO}(X,\tau)$ [11],

3° $\mathrm{cl}(S) = \mathrm{pcl}(S)$ *for every set* $S \in \mathrm{SO}(X,\tau)$,

4° $\mathrm{int}_\tau\big(\mathrm{cl}_{\tau^\alpha}(S)\big) = \mathrm{int}_\tau\big(\mathrm{cl}_\tau(S)\big)$ *for every* $S \subset X$.

Proof. 3° By [2, Theorem 1.5(e)] we have $\mathrm{pcl}(S) = S \cup \mathrm{cl}(\mathrm{int}(S))$ for any $S \subset X$. But $S \in \mathrm{SO}(X,\tau)$ if and only if $\mathrm{cl}(S) = \mathrm{cl}(\mathrm{int}(S))$ [19, Lemma 2]. So, the result follows.

4° The inclusion $\mathrm{int}_\tau\big(\mathrm{cl}_{\tau^\alpha}(S)\big) \subset \mathrm{int}_\tau\big(\mathrm{cl}_\tau(S)\big)$ holds for any $S \subset X$. For a proof of the opposite inclusion we use [2, Theorem 1.5(c)]. We calculate as follows: $\mathrm{int}(\mathrm{cl}(S)) \subset \mathrm{int}(S) \cup \mathrm{int}(\mathrm{cl}(S)) \subset \mathrm{int}\big(S \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(S)))\big) = \mathrm{int}\big(\mathrm{cl}_{\tau^\alpha}(S)\big)$. ∎

Theorem 9. *For every topological space (X,τ) the following are equivalent:*

1° (X,τ) *is $\mathcal{S}$-connected;*

2° $\mathrm{cl}(S) = X$ *for every non-empty* $S \in \tau$;

3° $\mathrm{cl}_{\tau^\alpha}(S) = X$ *for every non-empty* $S \in \tau$;

4° $\mathrm{scl}(S) = X$ *for every non-empty* $S \in \tau$;

5° $\mathrm{pcl}(S) = X$ *for every non-empty* $S \in \tau$;

6° $\mathrm{bcl}(S) = X$ *for every non-empty* $S \in \tau$;

7° $\mathrm{spcl}(S) = X$ *for every non-empty* $S \in \tau$.

Proof. 1° ⇒ 2°. Let $\varnothing \neq S \in \tau$ be such a set that $\mathrm{cl}(S) \neq X$. We have $\varnothing \neq \mathrm{cl}(S) \in \mathrm{SR}(X,\tau)$ $(\mathrm{cl}(S) \in \mathrm{SO}(X,\tau))$. So, by Remark 2, (X,τ) is not $\mathcal{S}$-connected.

2° ⇒ 3°. Use Lemma 5.1°.

$3° \Rightarrow 4°$. By hypothesis and Lemma 5.4° we obtain that $\operatorname{int}(\operatorname{cl}(S)) = X$ for each non-empty $S \in \tau$. However, by [2, Theorem 1.5(a)], $\operatorname{scl}(A) = A \cup \operatorname{int}(\operatorname{cl}(A))$ for every $A \subset X$. So, for our $S \in \tau$ we get $\operatorname{scl}(S) = X$.

$4° \Rightarrow 1°$. Suppose (X,τ) is not $\mathcal{S}$-connected. Then by Remark 2 there exists a set $S \in \mathrm{SR}(X,\tau)$ with $\varnothing \neq S \neq X$. Since $S \in \mathrm{SC}(X,\tau)$, $S = \operatorname{scl}(S)$ [5, Theorem 1.4(2)]. Obviously, $\operatorname{scl}(\operatorname{int}(S)) \neq X$, where $\operatorname{int}(S) \neq \varnothing$ [5, Remark 1.2].

$4° \Leftrightarrow 6° \Leftrightarrow 7°$ follow directly by Lemma 5.2°.

$2° \Leftrightarrow 5°$. Apply Lemma 5.3°. ∎

Another characterizations of $\mathcal{S}$-connectedness may be obtained if we take into consideration the classes τ^α or $\mathrm{SO}(X,\tau)$ instead of τ (in Theorem 9). The proofs in these cases are completely analogous to the proof of Theorem 9. These characterizations are the content of the next two theorems.

Theorem 10. *For every topological space (X,τ) the following are equivalent:*

1° *(X,τ) is $\mathcal{S}$-connected;*

2° *$\operatorname{cl}(S) = X$ for every non-empty $S \in \tau^\alpha$;*

3° *$\operatorname{cl}_{\tau^\alpha}(S) = X$ for every non-empty $S \in \tau^\alpha$;*

4° *$\operatorname{scl}(S) = X$ for every non-empty $S \in \tau^\alpha$;*

5° *$\operatorname{pcl}(S) = X$ for every non-empty $S \in \tau^\alpha$;*

6° *$\operatorname{bcl}(S) = X$ for every non-empty $S \in \tau^\alpha$;*

7° *$\operatorname{spcl}(S) = X$ for every non-empty $S \in \tau^\alpha$.*

Theorem 11. *For every (X,τ) the following are equivalent:*

1° *(X,τ) is $\mathcal{S}$-connected;*

2° *$\operatorname{cl}(S) = X$ for every non-empty $S \in \mathrm{SO}(X,\tau)$* [8, Theorem 12(e)]*;*

3° *$\operatorname{cl}_{\tau^\alpha}(S) = X$ for every non-empty $S \in \mathrm{SO}(X,\tau)$* [8, Theorem 12(e')]*;*

4° *$\operatorname{scl}(S) = X$ for every non-empty $S \in \mathrm{SO}(X,\tau)$* [20, Theorem 3.1(b)]*;*

5° *$\operatorname{pcl}(S) = X$ for every non-empty $S \in \mathrm{SO}(X,\tau)$* [22, Theorem 3.1(d)]*;*

6° *$\operatorname{bcl}(S) = X$ for every non-empty $S \in \mathrm{SO}(X,\tau)$;*

7° *$\operatorname{spcl}(S) = X$ for every non-empty $S \in \mathrm{SO}(X,\tau)$.*

Theorem 12. *The following statements are equivalent for every (X,τ):*

1° *(X,τ) is $\mathcal{S}$-connected;*

2° *$\operatorname{cl}(S) = X$ for every non-empty $S \in \mathrm{BO}(X,\tau)$;*

3° *$\operatorname{cl}_{\tau^\alpha}(S) = X$ for every non-empty $S \in \mathrm{BO}(X,\tau)$;*

4° *$\operatorname{scl}(S) = X$ for every non-empty $S \in \mathrm{BO}(X,\tau)$.*

Proof. $1° \Rightarrow 2°$. Suppose a non-empty $S \in \mathrm{BO}(X,\tau)$ is a set such that $\operatorname{cl}(S) \neq X$. Then the set $S_1 = \operatorname{int}(\operatorname{cl}(S)) \cup \operatorname{cl}(\operatorname{int}(S))$ is non-empty and moreover $S_1 \neq X$. Indeed, in the opposite case we would have $X = \operatorname{cl}(S_1) = \operatorname{cl}(\operatorname{int}(\operatorname{cl}(S)))\cup$

$\mathrm{cl}(\mathrm{int}(S)) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(S))) \subset \mathrm{cl}(S) \subset X$. Hence $\mathrm{cl}(S) = X$, a contradiction. Finally, by Lemma 1.2° and Remark 2 we infer that (X,τ) is not $\mathcal{S}$-connected.

2° ⇒ 3°. Let $\varnothing \neq S \in \mathrm{BO}(X,\tau)$ and $\mathrm{cl}(S) = X$. Then $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(S))) = X$ and since $\mathrm{cl}_{\tau^\alpha}(S) = S \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(S)))$ [2, Theorem 1.5(c)] the result follows.

3° ⇒ 4°. Similar to the proof of 3° ⇒ 4° of Theorem 9.

4° ⇒ 1°. Suppose (X,τ) is not $\mathcal{S}$-connected. Then by Remark 2 there exists a set $S \in \mathrm{SR}(X,\tau) \subset \mathrm{BO}(X,\tau)$ with $\varnothing \neq S \neq X$. But as $S \in \mathrm{SC}(X,\tau)$, we have $S = \mathrm{scl}(S)$ [5, Theorem 1.4(2)]. Thus, $\mathrm{scl}(S) \neq X$ and the proof is complete. ■

Lemma 6. [23, proof of Theorem 3.1]. *In any space* (X,τ), $S \in \mathrm{SPO}(X,\tau)$ *if and only if* $\mathrm{cl}(S) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(S)))$.

Theorem 13. *The following statements are equivalent for every* (X,τ)*:*

1° (X,τ) *is* $\mathcal{S}$*-connected;*

2° $\mathrm{cl}(S) = X$ *for every non-empty* $S \in \mathrm{SPO}(X,\tau)$*;*

3° $\mathrm{cl}_{\tau^\alpha}(S) = X$ *for every non-empty* $S \in \mathrm{SPO}(X,\tau)$*;*

4° $\mathrm{scl}(S) = X$ *for every non-empty* $S \in \mathrm{SPO}(X,\tau)$.

Proof. 1° ⇒ 2°. Suppose a non-empty $S \in \mathrm{SPO}(X,\tau)$ is a set such that $\mathrm{cl}(S) \neq X$. By Lemma 6 the set $S_1 = \mathrm{cl}(S) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(S))) \in \mathrm{SR}(X,\tau)$. Moreover, $\varnothing \neq S_1 \neq X$. Thus by Remark 2 the space (X,τ) is not $\mathcal{S}$-connected. Proofs for the chain 2° ⇒ 3° ⇒ 4° ⇒ 1° are similar to the corresponding ones in the proof of Theorem 12. ■

4. Summarizing conclusions

In order to complete our knowledge on various types of connectedness, including the 'mixed' ones, there are yet some cases we have to look at.

Definition 7. A space (X,τ) is said to be τ-τ^α*-connected* (resp. α*-connected* [28]) if X cannot be split into two non-empty disjoint sets $S_1 \in \tau$ and $S_2 \in \tau^\alpha$ (resp. $S_1, S_2 \in \tau^\alpha$).

α-connectedness and connectedness turn out to be equivalent notions [28, Theorem 2]

Theorem 14. *The following statements are equivalent for every* (X,τ)*:*

1° (X,τ) *is connected;*

2° (X,τ) *is* τ-τ^α*-connected.*

Proof. 1° ⇒ 2°. If (X,τ) is not τ-τ^α-connected, then it is not α-connected. Thus by [28, Theorem 2], (X,τ) is disconnected. 2° ⇒ 1°. Suppose (X,τ) is disconnected. Then it is not τ-τ^α-connected. ■

Definition 8. A space (X,τ) is said to be τ-P*-connected* (resp. α-P*-connected* [9]) if it cannot be split into two non-empty disjoint sets $S_1 \in \tau$ (resp. $S_1 \in \tau^\alpha$) and $S_2 \in \mathrm{PO}(X,\tau)$.

It is known that α-P-connectedness and connectedness are equivalent [9, Corollary 4].

THEOREM 15. *The following statements are equivalent for every* (X, τ)*:*

1° (X, τ) *is connected;*

2° (X, τ) *is* τ-P-*connected.*

Proof. 1° $\Rightarrow$ 2°. Suppose (X, τ) is not τ-P-connected. Hence it is not α-P-connected and thus disconnected. 2° $\Rightarrow$ 1° is obvious. ∎

DEFINITION 9. A space (X, τ) is said to be *P-B-connected* if it cannot be split into two non-empty disjoint sets $S_1 \in \mathrm{PO}\,(X, \tau)$ and $S_2 \in \mathrm{BO}\,(X, \tau)$.

PROBLEM 1. It is unknown what type of non-mixed connectedness is P-B-connectedness. That is, is it connectedness (briefly: C), $\mathcal{S}$-connectedness ($\mathcal{S}$), P-connectedness (P), or β-connectedness (β)?

Recall the following definitions.

DEFINITION 10. A space (X, τ) is called *SP*(int)-*connected* (resp. *P*(int)-*connected*) if it cannot be split into two non-empty disjoint sets $S_1, S_2 \in \mathrm{SPO}\,(X, \tau)$ (resp. $S_1, S_2 \in \mathrm{PO}\,(X, \tau)$) with $\mathrm{int}\,(S_1) \neq \varnothing \neq \mathrm{int}\,(S_2)$.

The following results are known:

(a) [9, Corollary 2] (X, τ) is *SP*(int)-connected if and only if it is $\mathcal{S}$-connected;

(b) [9, Theorem 7] (X, τ) is *P*(int)-connected if and only if it is $\mathcal{S}$-connected;

The results concerning various types of connectedness of topological spaces obtained in this article and in [9], we recollect in Table 1. Here, for instance, '$\mathcal{S}$' in the column with '$\mathcal{S}$' atop and in the row with 'B' ahead means *$\mathcal{S}$-B-connectedness is equivalent $\mathcal{S}$-connectedness.*

	τ	α	S	P	B	SP
τ	C	C	S	C	S	S
α		C	S	C	S	S
S			S	S	S	S
P	$P(\mathrm{int}) = S$			P	?	β
B	$B(\mathrm{int}) = S$				β	β
SP	$SP(\mathrm{int}) = S$					β

Table 1

Before we recollect (in Table 2) results concerning characterizations of forms of connectedness by using suitably generalized closure operators of suitably generalized open sets, we should complete them with the following ones:

Theorem 16. *The following statements are equivalent for every* (X,τ)*:*

1° (X,τ) *is* β*-connected;*

2° $\mathrm{bcl}\,(S) = X$ *for every non-empty* $S \in \mathrm{PO}\,(X,\tau)$*;*

3° $\mathrm{bcl}\,(S) = X$ *for every non-empty* $S \in \mathrm{SPO}\,(X,\tau)$*.*

Proof. 1° $\Rightarrow$ 2°. Let (X,τ) be β-connected and $S \in \mathrm{PO}\,(X,\tau)$. Applying Lemma 4 we obtain

$$X = \mathrm{spcl}\,(S) \subset \mathrm{bcl}\,(S) \subset \mathrm{pcl}\,(S) = X.$$

2° $\Rightarrow$ 1°. Suppose 2° holds and let a non-empty $S \in \mathrm{PO}\,(X,\tau)$ be arbitrary. Then $X = \mathrm{bcl}\,(S) \subset \mathrm{pcl}\,(S)$. Thus $\mathrm{pcl}\,(S) = X$ and by Lemma 4.2°, (X,τ) is β-connected. 1° $\Leftrightarrow$ 3° is analogous to the proof of 1° $\Leftrightarrow$ 2°. ∎

Recall a known result.

Lemma 7. [8, Theorem 12] *The following statements are equivalent for every* (X,τ)*:*

1° (X,τ) *is* $\mathcal{S}$*-connected;*

2° $\mathrm{cl}\,(S) = X$ *for every non-empty* $S \in \mathrm{PO}\,(X,\tau)$*;*

3° $\mathrm{cl}_{\tau^\alpha}(S) = X$ *for every non-empty* $S \in \mathrm{PO}\,(X,\tau)$*;*

4° $\mathrm{scl}\,(S) = X$ *for every non-empty* $S \in \mathrm{PO}\,(X,\tau)$*.*

We are ready now to display Table 2, where for instance, 'β' in the column with 'pcl' atop and in the row with '$\mathrm{BO}\,(X,\tau)$' ahead means (X,τ) *is* β*-connected if and only if* $\mathrm{pcl}\,(S) = X$ *for every non-empty* $S \in \mathrm{BO}\,(X,\tau)$.

	cl	**cl**$_{\tau^\alpha}$	**scl**	**pcl**	**bcl**	**spcl**
τ	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$
τ^α	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$
SO (X,τ)	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$
PO (X,τ)	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	β	β	β
BO (X,τ)	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	β	β	β
SPO (X,τ)	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	β	β	β

Table 2

5. Surjections

In [27] the notion of M-continuity have been introduced and studied. Recall that a subfamily $\mathfrak{m}_X$ of the power set $P(X)$ of a non-empty set X is said to be a *minimal structure* on X if $\varnothing, X \in \mathfrak{m}_X$. The families $\mathrm{SO}\,(X,\tau)$, $\mathrm{PO}\,(X,\tau)$, $\mathrm{BO}\,(X,\tau)$, and $\mathrm{SPO}\,(X,\tau)$ are minimal structures with the property of closedness under the unions of any family of subsets belong to SO$\,(X,\tau)$, PO$\,(X,\tau)$, BO$\,(X,\tau)$, and SPO$\,(X,\tau)$, respectively. The $\mathfrak{m}_X$-*closure* operator [16] (with respect to $\mathfrak{m}_X$) is defined in a usual manner, that is

$$\mathfrak{m}_X\text{-}\mathrm{cl}\,(S) = \bigcap \big\{F\colon S \subset F \text{ and } X \setminus F \in \mathfrak{m}_X\big\}.$$

So, scl, pcl, bcl, and spcl are $\mathfrak{m}_X$-closure operators for cases SO$\,(X,\tau)$, PO$\,(X,\tau)$, BO$\,(X,\tau)$, and SPO$\,(X,\tau)$, respectively.

Definition 11. [27, Definition 3.3] A function $f : (X,\mathfrak{m}_X) \to (Y,\mathfrak{m}_Y)$, where $\mathfrak{m}_X$ and $\mathfrak{m}_Y$ are minimal structures on X and Y, respectively, is said to be *M-continuous* if for each $x \in X$ and each $V \in \mathfrak{m}_Y$ containing $f(x)$, there is $U \in \mathfrak{m}_X$ containing x with $f(U) \subset V$.

By [27, Theorem 3.1] and [27, Corollary 3.1] the following holds.

Lemma 8. *Let X be a non-empty set with a minimal structure $\mathfrak{m}_X$ closed under any union of members of $\mathfrak{m}_X$, and let $\mathfrak{m}_Y$ be a minimal structure on a non-empty set Y. Then for a function $f : (X,\mathfrak{m}_X) \to (Y,\mathfrak{m}_Y)$ we have what follows.*

(I) *the next three statements are equivalent:*

1° *f is M-continuous;*

2° *$f\big(\mathfrak{m}_X\text{-}\mathrm{cl}\,(S)\big) \subset \mathfrak{m}_Y\text{-}\mathrm{cl}\,(f(S))$ for every subset S of X;*

3° *$f^{-1}(V) \in \mathfrak{m}_X$ for every $V \in \mathfrak{m}_Y$.*

(II) *(by the above* **(I)***) If f is M-continuous, then $f\big(\mathfrak{m}_X\text{-}\mathrm{cl}\big(f^{-1}(V)\big)\big) \subset \mathfrak{m}_Y\text{-}\mathrm{cl}\,(V)$ for every $V \in \mathfrak{m}_Y$.*

Several results from Table 2 and Lemma 8(II) allow to collect in Table 3 below, all possible cases in which $\mathcal{S}$-connectedness and β-connectedness by respective generalized types of continuity of surjections $f : (X,\tau) \to (Y,\sigma)$. In the table, all generalized continuities are represented by properties of preimages $f^{-1}(V)$ for each set V from families σ, σ^α, SO$\,(Y,\sigma)$, PO$\,(Y,\sigma)$, BO$\,(Y,\sigma)$, SPO$\,(Y,\sigma)$, respectively. For instance, '$\mathcal{S} \to \beta$' in the row with 'SO$\,(X,\tau)$' ahead and in the column with 'BO$\,(X,\tau)$' atop stands for: *given a surjection $f : (X,\tau) \to (Y,\sigma)$, if (X,τ) is $\mathcal{S}$-connected then (Y,σ) is β-connected.*

	σ	σ^α	**SO** (Y,σ)	**PO** (Y,σ)	**BO** (Y,σ)	**SPO** (Y,σ)
τ	$S \to S$	$S \to S$	$S \to S$	$S \to \beta$	$S \to \beta$	$S \to \beta$
τ^α	$S \to S$	$S \to S$	$S \to S$	$S \to \beta$	$S \to \beta$	$S \to \beta$
SO (X,τ)	$S \to S$	$S \to S$	$S \to S$	$S \to \beta$	$S \to \beta$	$S \to \beta$
PO (X,τ)	$\beta \to S$	$\beta \to S$	$\beta \to S$	$\beta \to \beta$	$\beta \to \beta$	$\beta \to \beta$
BO (X,τ)	$\beta \to S$	$\beta \to S$	$\beta \to S$	$\beta \to \beta$	$\beta \to \beta$	$\beta \to \beta$
SPO (X,τ)	$\beta \to S$	$\beta \to S$	$\beta \to S$	$\beta \to \beta$	$\beta \to \beta$	$\beta \to \beta$

Table 3

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