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UNIQUENESS OF MEROMORPHIC FUNCTIONS WHEN TWO DIFFERENTIAL POLYNOMIALS SHARE ONE VALUE IM

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Abstract. In the paper, we prove two uniqueness theorems concerning nonlinear differential polynomials, one of which generalizes a recent result in [1], and the other supplements a recent result in [10].

1. Introduction, definitions and results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [7, 14, 15]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$ $(r \to \infty, r \notin E)$.

Let f and g be two nonconstant meromorphic functions, and let a be a finite value. We say that f and g share the value a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, and we say that f and g share ∞ IM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM (see [15]). We say that α is a small function of f, if α is a meromorphic function satisfying $T(r,\alpha) = S(r,f)$ (see [15]). Throughout this paper, we need the following definition.

$$\Theta(b,f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r,b;f)}{T(r,f)},$$

where b is a value in the extended complex plane.

In 1959, W.K. Hayman proved that if f is a transcendental meromorphic function and $n(\geq 3)$ is a positive integer, then $f^n f' = 1$ has infinitely many solutions

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(see [6, Corollary of Theorem 9]). Corresponding to which, the following result was obtained by Fang and Hua [4] and by Yang and Hua [13] respectively.

Theorem A. *Let f and g be two non-constant entire functions, $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

Considering k-th derivative instead of 1st derivative Fang [5] proved the following theorems.

Theorem B. [5] *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k+4$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem C. [5] *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n \geq 2k+8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f \equiv g$.*

Recently S.S. Bhoosnurmath and R.S. Dyavanal [3] considered the uniqueness of meromorphic functions corresponding to the k th derivative of a linear polynomial expression. It is worth mentioning that in the above area some investigations has already been carried out by A. Banerjee [2]. Banerjee proved the following result.

Theorem D. [2] *Let f and g be two transcendental meromorphic functions, and let n, k be two positive integers with $n > 9k+14$. Suppose that $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share a non-zero constant b IM. Then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = b^2$ or $f \equiv tg$ for some nth root of unity* 1.

Recently in [10] Lahiri and Sahoo proved the following result.

Theorem E. [10] *Let f and g be two non-constant meromorphic functions and $\alpha(\not\equiv 0, \infty)$ be a small function of f and g. Let n and $m(\geq 2)$ be two positive integers with $n > \max\{4, 4m+22-5\Theta(\infty, f)-5\Theta(\infty, g)\} - \min\{\Theta(\infty, f), \Theta(\infty, g)\}$. If $f^n(f^m - a)f'$ and $g^n(g^m - a)g'$ share α IM for a non-zero constant a, then either $f \equiv g$ or $f \equiv -g$.*

Also the possibility $f \equiv -g$ does not arise if n and m are both even, both odd or n is even and m is odd.

Regarding Theorem D and Theorem E, it is natural to ask the following two questions.

Question 1. What can be said about the relationship between two nonconstant meromorphic functions f and g, if $\{f^n(f-1)^m\}^{(k)}$ and $\{g^n(g-1)^m\}^{(k)}$ share the value 1 IM?

Question 2. What can be said about the relationship between two nonconstant meromorphic functions f and g, if $\{f^n(f^m-a)\}^{(k)}$ and $\{g^n(g^m-a)\}^{(k)}$ share the value 1 IM?

In this paper, we will prove the following two theorems, which generalize Theorems A–E. Moreover, Theorem 1 and Theorem 2 deal with Question 1 and Question 2 respectively.

Theorem 1. *Let f and g be two transcendental meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 0)$ be three integers. Let $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share the value 1 IM. Then one of the following holds:*

(i) when $m=0$, if $f(z)\neq\infty$, $g(z)\neq\infty$ and $n>9k+14$, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=1$ or $f\equiv tg$ for a constant t such that $t^n=1$;

(ii) when $m=1$, $n>9k+20$ and $\Theta(\infty,f)>\frac{2}{n}$, then either $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)}\equiv 1$ or $f\equiv g$;

(iii) when $m\geq 2$ and $n>9k+4m+16$, then either $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)}\equiv 1$ or $f\equiv g$ or f and g satisfy the algebraic equation $R(f,g)=0$, where

$$R(x,y)=x^n(x-1)^m-y^n(y-1)^m.$$

The possibility $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)}\equiv 1$ does not arise for $k=1$.

Remark 1. Clearly Theorem 1 improves Theorem D.

Theorem 2. *Let f and g be two transcendental meromorphic functions, and let n, $m(\geq 2)$ and k be three positive integers such that $n>9k+6m+14$. If $[f^n(f^m-a)]^{(k)}$ and $[g^n(g^m-a)]^{(k)}$ share 1 IM, where $a(\neq 0)$ is a finite complex number, then either $[f^n(f^m-a)]^{(k)}[g^n(g^m-a)]^{(k)}\equiv 1$ or $f\equiv g$ or $f\equiv -g$.*

The possibility $[f^n(f^m-a)]^{(k)}[g^n(g^m-a)]^{(k)}\equiv 1$ does not arise for $k=1$ and the possibility $f\equiv -g$ does not arise if n and m are both odd or if n is even and m is odd or if n is odd and m is even.

We now explain some definitions and notations which are used in the paper.

Definition 1. [8] Let p be a positive integer and $b\in\mathbb{C}\cup\{\infty\}$. Then by $N(r,b;f\mid\leq p)$ we denote the counting function of those b-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r,b;f\mid\leq p)$ we denote the corresponding reduced counting function. In an analogous manner we define $N(r,b;f\mid\geq p)$ and $\overline{N}(r,b;f\mid\geq p)$.

Definition 2. [9] Let k be a positive integer or infinity. We denote by $N_k(r,b;f)$ the counting function of b-points of f, where a b-point of multiplicity m is counted m times if $m\leq k$ and k times if $m>k$. That is

$$N_k(r,b;f)=\overline{N}(r,b;f)+\overline{N}(r,b;f\mid\geq 2)+\cdots+\overline{N}(r,b;f\mid\geq k).$$

Definition 3. For $b \in \mathbb{C} \cup \{\infty\}$ we put

$$\delta_k(b,f) = 1 - \limsup_{r\to\infty} \frac{N_k(r,b;f)}{T(r,f)}.$$

Definition 4. Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. We denote by $N_{11}(r,1;f)$ the reduced counting function of the common simple 1-points of f and g.

Definition 5. Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a zero of $f-1$ with multiplicity p, and a zero of $g-1$ with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the reduced counting function of those common zeros of $f-1$ and $g-1$ satisfying $p > q$. Similarly we define $\overline{N}_L(r,1;g)$.

2. Lemmas

In this section we present some lemmas which will be needed to prove the theorems.

Lemma 1. [11, 12] *Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where a_0, a_1, a_2, $\ldots$, a_n are constants and $a_n \neq 0$. Then*

$$T(r,P(f)) = nT(r,f) + S(r,f).$$

Lemma 2. [7] *Let f be a non-constant meromorphic function, k be a positive integer, and let c be a non-zero finite complex number. Then*

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,\infty;f) + N(r,0;f) + N\left(r,c;f^{(k)}\right) - N\left(r,0;f^{(k+1)}\right) + S(r,f) \\ &\leq \overline{N}(r,\infty;f) + N_{k+1}(r,0;f) + \overline{N}\left(r,c;f^{(k)}\right) - N_0\left(r,0;f^{(k+1)}\right) + S(r,f), \end{aligned}$$

where $N_0\left(r,0;f^{(k+1)}\right)$ denotes the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f\left(f^{(k)} - c\right) \neq 0$.

Lemma 3. [1] *For two positive integers p and k*

$$N_p\left(r,0;f^{(k)}\right) \leq N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) - \sum_{m=p+1}^{\infty} \overline{N}\left(r,0;\frac{f^{(k)}}{f} \mid\geq m\right) + S(r,f).$$

Lemma 4. [14] *Let f be a transcendental meromorphic function, and let a_1, a_2 be two meromorphic functions such that $T(r,a_i) = S(r,f)$, $i = 1, 2$. Then*

$$T(r,f) \leq \overline{N}(r,\infty;f) + \overline{N}(r,a_1;f) + \overline{N}(r,a_2;f) + S(r,f).$$

Lemma 5. *Let f and g be two transcendental meromorphic functions, and let k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 IM and*

$$\Delta = (2k+3)\Theta(\infty, f) + (2k+4)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + 2\delta_{k+1}(0, f) + 3\delta_{k+1}(0, g) > 4k + 13, \quad (2.1)$$

then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

Proof. Let

$$h(z) = \frac{f^{(k+2)}(z)}{f^{(k+1)}(z)} - \frac{2f^{(k+1)}(z)}{f^{(k)}(z) - 1} - \frac{g^{(k+2)}(z)}{g^{(k+1)}(z)} + \frac{2g^{(k+1)}(z)}{g^{(k)}(z) - 1}. \quad (2.2)$$

Let z_0 be a common simple 1-point of $f^{(k)}$ and $g^{(k)}$. Then from (2.2) we see that z_0 is a zero of $h(z)$. Thus

$$\begin{aligned} N_{11}\left(r, 1; f^{(k)}\right) = N_{11}\left(r, 1; g^{(k)}\right) \le \overline{N}(r, 0; h) \le T(r, h) + O(1) \\ \le N(r, \infty; h) + S(r, f) + S(r, g). \end{aligned} \quad (2.3)$$

From (2.2) we know that the poles of $h(z)$ possibly result from those zeros of $f^{(k+1)}$ and $g^{(k+1)}$ which are not the common 1-points of $f^{(k)}$ and $g^{(k)}$, from the poles of f and g, and from those common 1-points of $f^{(k)}$ and $g^{(k)}$ such that each such point has different multiplicity related to $f^{(k)}$ and $g^{(k)}$. Thus

$$\begin{aligned} N(r, \infty; h) \le \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}_L\left(r, 1; f^{(k)}\right) \\ + \overline{N}_L\left(r, 1; g^{(k)}\right) + N_0\left(r, 0; f^{(k+1)}\right) + N_0\left(r, 0; g^{(k+1)}\right), \end{aligned} \quad (2.4)$$

where $N_0\left(r, 0; f^{(k+1)}\right)$ denotes the counting function of those zeros of $f^{(k+1)}$ which are not the zeros of $f\left(f^{(k)} - 1\right)$. By Lemma 2 we have

$$T(r, f) \le \overline{N}(r, \infty; f) + N_{k+1}(r, 0; f) + \overline{N}\left(r, 1; f^{(k)}\right) - N_0\left(r, 0; f^{(k+1)}\right) + S(r, f), \quad (2.5)$$

$$T(r, g) \le \overline{N}(r, \infty; g) + N_{k+1}(r, 0; g) + \overline{N}\left(r, 1; g^{(k)}\right) - N_0\left(r, 0; g^{(k+1)}\right) + S(r, g). \quad (2.6)$$

Since $f^{(k)}$ and $g^{(k)}$ share 1 IM, we obtain

$$\begin{aligned} \overline{N}\left(r, 1; f^{(k)}\right) + \overline{N}\left(r, 1; g^{(k)}\right) &\le N_{11}\left(r, 1; f^{(k)}\right) + \overline{N}_L\left(r, 1; g^{(k)}\right) + N\left(r, 1; f^{(k)}\right) \\ &\le N_{11}\left(r, 1; f^{(k)}\right) + \overline{N}_L\left(r, 1; g^{(k)}\right) + T\left(r, f^{(k)}\right) + O(1) \\ &\le N_{11}\left(r, 1; f^{(k)}\right) + \overline{N}_L\left(r, 1; g^{(k)}\right) + T(r, f) + k\overline{N}(r, \infty; f) + S(r, f). \end{aligned} \quad (2.7)$$

Using Lemma 3 we obtain

$$\begin{aligned}
\overline{N}_L\left(r,1;f^{(k)}\right) &\le N\left(r,1;f^{(k)}\right)-\overline{N}\left(r,1;f^{(k)}\right)\le N\left(r,\infty;\frac{f^{(k)}}{f^{(k+1)}}\right)\\
&\le N\left(r,\infty;\frac{f^{(k+1)}}{f^{(k)}}\right)+S(r,f)\\
&\le \overline{N}(r,\infty;f)+\overline{N}\left(r,0;f^{(k)}\right)+S(r,f)\\
&\le (k+1)\overline{N}(r,\infty;f)+N_{k+1}(r,0;f)+S(r,f). \qquad (2.8)
\end{aligned}$$

Similarly

$$\overline{N}_L\left(r,1;g^{(k)}\right)\le (k+1)\overline{N}(r,\infty;g)+N_{k+1}(r,0;g)+S(r,g). \qquad (2.9)$$

So we get from (2.3)–(2.9) that

$$\begin{aligned}
T(r,g)\le{}&(2k+3)\overline{N}(r,\infty;f)+(2k+4)\overline{N}(r,\infty;g)+\overline{N}(r,0;f)+\overline{N}(r,0;g)\\
&+2N_{k+1}(r,0;f)+3N_{k+1}(r,0;g)+S(r,f)+S(r,g).
\end{aligned}$$

Similarly

$$\begin{aligned}
T(r,f)\le{}&(2k+3)\overline{N}(r,\infty;g)+(2k+4)\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,0;f)\\
&+2N_{k+1}(r,0;g)+3N_{k+1}(r,0;f)+S(r,f)+S(r,g).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set of infinite measure such that $T(r,f)\le T(r,g)$ for $r\in I$. Hence we have

$$\begin{aligned}
T(r,g)\le{}&\{[4k+14-(2k+3)\Theta(\infty,f)-(2k+4)\Theta(\infty,g)-\Theta(0,f)-\Theta(0,g)\\
&-2\delta_{k+1}(0,f)-3\delta_{k+1}(0,g)]+\epsilon\}T(r,g)+S(r,g), \qquad (2.10)
\end{aligned}$$

for $r\in I$ and $0<\epsilon<\Delta-(4k+13)$. Thus we obtain $T(r,g)\le S(r,g)$ for $r\in I$, which is a contradiction. Hence $h(z)\equiv 0$. That is

$$\frac{f^{(k+2)}(z)}{f^{(k+1)}(z)}-\frac{2f^{(k+1)}(z)}{f^{(k)}(z)-1}\equiv\frac{g^{(k+2)}(z)}{g^{(k+1)}(z)}-\frac{2g^{(k+1)}(z)}{g^{(k)}(z)-1}.$$

By integrating two sides of the above equality we get

$$\frac{1}{f^{(k)}(z)-1}\equiv\frac{bg^{(k)}(z)+a-b}{g^{(k)}(z)-1}, \qquad (2.11)$$

where $a(\neq 0)$ and b are constants. We consider the following three cases.

Case I. Let $b\neq 0$ and $a=b$. If $b=-1$, from (2.11) we obtain $f^{(k)}g^{(k)}\equiv 1$.

If $b\neq -1$, from (2.11) we get

$$\frac{1}{f^{(k)}(z)}\equiv\frac{bg^{(k)}(z)}{(1+b)g^{(k)}(z)-1}.$$

This together with Lemma 3 gives

$$\overline{N}\left(r, \frac{1}{1+b}; g^{(k)}(z)\right) \leq k\overline{N}(r,\infty;f) + N_{k+1}(r,0;f). \tag{2.12}$$

From (2.12) and Lemma 2 we obtain

$$\begin{aligned}T(r,g) &\leq \overline{N}(r,\infty;g) + N_{k+1}(r,0;g) + \overline{N}\left(r, \frac{1}{1+b}; g^{(k)}\right) - N_0\left(r,0;g^{(k+1)}\right) + S(r,g)\\ &\leq \overline{N}(r,\infty;g) + N_{k+1}(r,0;g) + k\overline{N}(r,\infty;f) + N_{k+1}(r,0;f) + S(r,f) + S(r,g)\\ &\leq (2k+3)\overline{N}(r,\infty;f) + (2k+4)\overline{N}(r,\infty;g) + \overline{N}(r,0;f) + \overline{N}(r,0;g)\\ &\quad + 2N_{k+1}(r,0;f) + 3N_{k+1}(r,0;g) + S(r,f) + S(r,g).\end{aligned}$$

Thus we obtain

$$[\Delta - (4k+13)]T(r,g) \leq S(r,g),$$

a contradiction.

Case II. Let $b \neq 0$ and $a \neq b$.

If $b = -1$, we have from (2.11) that

$$f^{(k)}(z) \equiv \frac{a}{-g^{(k)}(z) + a + 1}.$$

Therefore

$$\overline{N}\left(r,\infty; \frac{a}{-g^{(k)}(z) + a + 1}\right) = \overline{N}(r,\infty;f).$$

So by Lemma 2 and using the same argument as in case I, we get a contradiction.

If $b \neq -1$, we have from (2.11)

$$f^{(k)}(z) - \left(1 + \frac{1}{b}\right) \equiv \frac{-a}{b^2\left(g^{(k)}(z) + \frac{a-b}{b}\right)}.$$

Therefore

$$\overline{N}\left(r,0; g^{(k)}(z) + \frac{a-b}{b}\right) = \overline{N}(r,\infty;f).$$

Next by Lemma 2 and in the same manner as in case I we get a contradiction.

Case III. Let $b = 0$. Then we obtain from (2.11) that

$$f^{(k)} = \frac{1}{a}g^{(k)} + 1 - \frac{1}{a}, \tag{2.13}$$

i.e.,

$$f = \frac{1}{a}g + Q(z), \tag{2.14}$$

where $Q(z)$ is a polynomial with its degree $\leq k$. Let $Q(z) \not\equiv 0$. Then by Lemma 4 we have

$$\begin{aligned}T(r,f) &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,Q;f) + S(r,f)\\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,0;g) + S(r,f).\end{aligned} \tag{2.15}$$

From (2.14) we obtain
$$T(r,f)=T(r,g)+S(r,f).$$
This together with (2.15) gives
$$(2k+2)\Theta(\infty,f)+(2k+4)\Theta(\infty,g)+2\delta_{k+1}(0,f)+3\delta_{k+1}(0,g)>4k+11,$$
which is impossible. Hence $Q(z)\equiv 0$. So from (2.13) and (2.14) we obtain $a=1$ and so $f\equiv g$. This proves the lemma. ■

LEMMA 6. [2] *Let f, g be two non-constant entire functions and $k\geq 1$ and $n>3k+8$ be two integers. If $[f^n]^{(k)}[g^n]^{(k)}\equiv b^2$, where $b(\neq 0)$, be a constant, then $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=b^2$.*

LEMMA 7. *Let f and g be two non-constant meromorphic functions, and let $n(\geq 1)$, $m(\geq 1)$, $k(\geq 1)$ be three integers. Then*
$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)}\not\equiv 1,$$
for $k=1$ and $n\geq m+3$.

Proof. If possible, let
$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)}\equiv 1,$$
for $k=1$. That is,
$$f^{n-1}(f-1)^{m-1}(af-b)f'g^{n-1}(g-1)^{m-1}(ag-b)g'\equiv 1,$$
where $a=n+m$ and $b=n$.

Let z_0 be a 1-point of f with multiplicity $p(\geq 1)$, and a pole of g with multiplicity $q(\geq 1)$ such that $mp-1=(n+m)q+1$, i.e., $mp=(n+m)q+2\geq n+m+2$, i.e.,
$$p\geq\frac{n+m+2}{m}.$$
Let z_1 be a zero of $af-b$ with multiplicity $p_1(\geq 1)$, and a pole of g with multiplicity $q_1(\geq 1)$ such that $2p_1-1=(n+m)q_1+1$, i.e.,
$$p_1\geq\frac{n+m+2}{2}.$$
Let z_2 be a zero of f with multiplicity $p_2(\geq 1)$, and a pole of g with multiplicity $q_2(\geq 1)$. Then
$$np_2-1=(n+m)q_2+1. \tag{2.16}$$
From (2.16) we get $mq_2+2=n(p_2-q_2)\geq n$, i.e., $q_2\geq\frac{n-2}{m}$. Thus from (2.16) we get
$$np_2=(n+m)q_2+2\geq\frac{(n+m)(n-2)}{m}+2,$$
i.e., $p_2\geq\dfrac{n+m-2}{m}$.

Since a pole of f is either a zero of $g(g-1)(ag-b)$ or a zero of g', we have

$$\begin{aligned}\overline{N}(r,\infty;f) &\le \overline{N}(r,0;g)+\overline{N}(r,1;g)+\overline{N}\left(r,\frac{b}{a};g\right)+\overline{N}_0(r,0;g')+S(r,f)+S(r,g)\\ &\le \left(\frac{m+2}{n+m+2}+\frac{m}{n+m-2}\right)T(r,g)+\overline{N}_0(r,0;g')+S(r,f)+S(r,g),\end{aligned}$$

where $\overline{N}_0(r,0;g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $g(g-1)(ag-b)$.

Then by the second fundamental theorem of Nevanlinna we get

$$\begin{aligned}2T(r,f) &\le \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}\left(r,\frac{b}{a};f\right)+\overline{N}(r,\infty;f)-\overline{N}_0(r,0;f')+S(r,f)\\ &\le \left(\frac{m+2}{n+m+2}+\frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\}-\overline{N}_0(r,0;f')\\ &\quad+\overline{N}_0(r,0;g')+S(r,f)+S(r,g).\end{aligned} \tag{2.17}$$

Similarly

$$\begin{aligned}2T(r,g) \le &\left(\frac{m+2}{n+m+2}+\frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\}+\overline{N}_0(r,0;f')\\ &-\overline{N}_0(r,0;g')+S(r,f)+S(r,g).\end{aligned} \tag{2.18}$$

Adding (2.17) and (2.18) we obtain

$$\left(1-\frac{m+2}{n+m+2}-\frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g),$$

which is a contradiction for $n\ge m+3$. This proves the lemma. ∎

Lemma 8. *Let f and g be two nonconstant meromorphic functions, and let $n(\ge 1)$ and $m(\ge 1)$ be two positive integers. If $n+m\ge 7$, then*

$$[f^n(f^m-a)]'[g^n(g^m-a)]'\not\equiv 1.$$

Proof. We assume that $[f^n(f^m-a)]'[g^n(g^m-a)]'\equiv 1$. Then

$$f^{n-1}\left(cf^m-d\right)f'g^{n-1}\left(cg^m-d\right)g'\equiv 1, \tag{2.19}$$

where $c=n+m$ and $d=an$. Let z_0 be a zero of f with multiplicity $p(\ge 1)$. Then z_0 is a pole of g with multiplicity $q(\ge 1)$, say. Then from (2.19) we obtain $np-1=(n+m)q+1$, and so

$$mq+2=n(p-q). \tag{2.20}$$

From (2.20) we get $q\ge\frac{n-2}{m}$ and so we have

$$p\ge\frac{1}{n}\left[\frac{(n+m)(n-2)}{m}+2\right]=\frac{n+m-2}{m}.$$

Let z_1 be a zero of $cf^m - d$ with multiplicity $p_1(\geq 1)$. Then z_1 is a pole of g with multiplicity $q_1(\geq 1)$, say. So from (2.19) we obtain $2p_1 - 1 = (n+m)q_1 + 1$, i.e.,

$$p_1 \geq \frac{n+m+2}{2}.$$

Since a pole of f is either a zero of $g(cg^m - d)$ or a zero of g', we have

$$\begin{aligned}\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g) + \overline{N}\left(r,\frac{d}{c};g^m\right) + \overline{N}_0(r,0;g') + S(r,f) + S(r,g)\\ &\leq \left(\frac{m}{n+m-2} + \frac{2m}{n+m+2}\right)T(r,g) + \overline{N}_0(r,0;g') + S(r,f) + S(r,g),\end{aligned}$$

where $\overline{N}_0(r,0;g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $g(cg^m - d)$.

Let

$$cf^m - d = c(f-a_1)(f-a_2)\cdots(f-a_m),$$

where a_1, a_2, $\ldots$ a_m are m distinct complex numbers. Then by the second fundamental theorem of Nevanlinna we get

$$\begin{aligned}mT(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \sum_{j=1}^{m}\overline{N}(r,a_j;f) - \overline{N}_0(r,0;f') + S(r,f)\\ &\leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}\left(r,\frac{d}{c};f^m\right) - \overline{N}_0(r,0;f') + S(r,f)\\ &\leq \left(\frac{m}{n+m-2} + \frac{2m}{n+m+2}\right)\{T(r,f)+T(r,g)\} - \overline{N}_0(r,0;f')\\ &\quad + \overline{N}_0(r,0;g') + S(r,f) + S(r,g). \qquad (2.21)\end{aligned}$$

Similarly

$$\begin{aligned}mT(r,g) &\leq \left(\frac{m}{n+m-2} + \frac{2m}{n+m+2}\right)\{T(r,f)+T(r,g)\} + \overline{N}_0(r,0;f')\\ &\quad - \overline{N}_0(r,0;g') + S(r,f) + S(r,g). \qquad (2.22)\end{aligned}$$

Adding (2.21) and (2.22) we obtain

$$\left(1 - \frac{2}{n+m-2} - \frac{4}{n+m+2}\right)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

which is a contradiction as $n+m \geq 7$. This proves the lemma. ■

3. Proofs of Theorems

Proof of Theorem 1. Consider $F(z) = f^n(f-1)^m$ and $G(z) = g^n(g-1)^m$. Then $[F(z)]^{(k)}$ and $[G(z)]^{(k)}$ share 1 IM. Let

$$\begin{aligned}\Delta &= (2k+3)\Theta(\infty,F) + (2k+4)\Theta(\infty,G) + \Theta(0,F)\\ &\quad + \Theta(0,G) + 2\delta_{k+1}(0,F) + 3\delta_{k+1}(0,G).\end{aligned}$$

By Lemma 1 we have

$$\begin{aligned}\Theta(\infty,F)&=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,\infty;F)}{T(r,F)}=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,\infty;f^n(f-1)^m)}{(m+n)T(r,f)}\\&\geq 1-\limsup_{r\longrightarrow\infty}\frac{T(r,f)}{(m+n)T(r,f)}\geq\frac{n+m-1}{m+n}.\end{aligned}\tag{3.1}$$

Similarly

$$\Theta(\infty,G)\geq\frac{n+m-1}{m+n}.\tag{3.2}$$

$$\begin{aligned}\Theta(0,F)&=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,0;F)}{T(r,F)}=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,0;f^n(f-1)^m)}{(m+n)T(r,f)}\\&\geq 1-\limsup_{r\longrightarrow\infty}\frac{(1+m^*)T(r,f)}{(m+n)T(r,f)}\geq\frac{n+m-1-m^*}{m+n},\end{aligned}\tag{3.3}$$

where $m^*=\begin{cases}0 & \text{if } m=0\\ 1 & \text{if } m\geq 1.\end{cases}$ Similarly

$$\Theta(0,G)\geq\frac{n+m-1-m^*}{n+m}.\tag{3.4}$$

$$\begin{aligned}\delta_{k+1}(0,F)&=1-\limsup_{r\longrightarrow\infty}\frac{N_{k+1}(r,0;F)}{T(r,F)}=1-\limsup_{r\longrightarrow\infty}\frac{N_{k+1}(r,0;f^n(f-1)^m)}{(m+n)T(r,f)}\\&\geq 1-\limsup_{r\longrightarrow\infty}\frac{(k+m+1)T(r,f)}{(m+n)T(r,f)}\geq\frac{n-k-1}{m+n}.\end{aligned}\tag{3.5}$$

Similarly

$$\delta_{k+1}(0,G)\geq\frac{n-k-1}{m+n}.\tag{3.6}$$

From (3.1)–(3.6) we obtain

$$\begin{aligned}\Delta&=(4k+7)\frac{n+m-1}{m+n}+2\frac{n+m-1-m^*}{m+n}+5\frac{n-k-1}{m+n}\\&=\frac{1}{m+n}\left[(4k+7)(n+m-1)+2(n+m-1-m^*)+5(n-k-1)\right].\end{aligned}$$

It is easily verified that if $n>9k+4m+2m^*+14$, then $\Delta>4k+13$. Since

$$9k+4m+2m^*+14=\begin{cases}9k+14 & \text{if } m=0\\ 9k+20 & \text{if } m=1\\ 9k+4m+16 & \text{if } m\geq 2,\end{cases}$$

by Lemma 5 we obtain either $F^{(k)}G^{(k)}\equiv 1$ or $F\equiv G$.

Let $m=0$. Since $f(z)\neq\infty$ and $g(z)\neq\infty$, by $F^{(k)}G^{(k)}\equiv 1$ and Lemma 6 we obtain $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=1$. Also by Lemma 7 the case $F^{(k)}G^{(k)}\equiv 1$ does not arise for $k=1$ and $m\geq 1$. Let $F\equiv G$, i.e.,

$$f^n(f-1)^m\equiv g^n(g-1)^m.\tag{3.7}$$

Now we consider following three cases.

Case (i). Let $m=0$. Then from (3.7) we get $f\equiv tg$ for a constant t such that $t^n=1$.

Case (ii). Let $m=1$. Then from (3.7) we have

$$f^n(f-1)\equiv g^n(g-1). \tag{3.8}$$

Suppose $f\not\equiv g$. Let $h=\frac{f}{g}$ be a constant. Then from (3.8) it follows that $h\neq 1$, $h^n\neq 1$, $h^{n+1}\neq 1$ and $g=\frac{1-h^n}{1-h^{n+1}}=$ constant, a contradiction. So we suppose that h is not a constant. Since $f\not\equiv g$, we have $h\not\equiv 1$. From (3.8) we obtain $g=\frac{1-h^n}{1-h^{n+1}}$ and $f=\left(\frac{1-h^n}{1-h^{n+1}}\right)h$. Hence it follows that

$$T(r,f)=nT(r,h)+S(r,f).$$

Again by second fundamental theorem of Nevanlinna, we have

$$\overline{N}(r,\infty;f)=\sum_{j=1}^{n}\overline{N}(r,\alpha_j;h)\geq(n-2)T(r,h)+S(r,f),$$

where α_j $(\neq 1)$ $(j=1,2,\ldots,n)$ are distinct roots of the equation $h^{n+1}=1$. So we obtain

$$\Theta(\infty,f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,\infty;f)}{T(r,f)}\leq\frac{2}{n},$$

which contradicts the assumption $\Theta(\infty,f)>\frac{2}{n}$. Thus $f\equiv g$.

Case (iii). Let $m\geq 2$. Then from (3.7) we obtain

$$\begin{aligned}f^n[f^m+\cdots+(-1)^{i}\,{}^{m}C_{m-i}f^{m-i}+\cdots+(-1)^m]\\=g^n[g^m+\cdots+(-1)^{i}\,{}^{m}C_{m-i}g^{m-i}+\cdots+(-1)^m].\end{aligned} \tag{3.9}$$

Let $h=\frac{f}{g}$. If h is a constant, then substituting $f=gh$ in (3.9) we obtain

$$\begin{aligned}g^{n+m}(h^{n+m}-1)+\cdots+(-1)^{i}\,{}^{m}C_{m-i}g^{n+m-i}(h^{n+m-i}-1)\\+\cdots+(-1)^m g^n(h^n-1)=0,\end{aligned}$$

which imply $h=1$. Hence $f\equiv g$. If h is not a constant, then from (3.9) we can say that f and g satisfy the algebraic equation $R(f,g)=0$, where

$$R(x,y)=x^n(x-1)^m-y^n(y-1)^m.$$

This completes the proof of the theorem. ■

Proof of Theorem 2. Consider $F(z)=f^n(f^m-a)$ and $G(z)=g^n(g^m-a)$. Then $[F(z)]^{(k)}$ and $[G(z)]^{(k)}$ share 1 IM. Let

$$\begin{aligned}\Delta=(2k+3)\Theta(\infty,F)+(2k+4)\Theta(\infty,G)+\Theta(0,F)\\+\Theta(0,G)+2\delta_{k+1}(0,F)+3\delta_{k+1}(0,G).\end{aligned}$$

By Lemma 1 we have

$$\begin{aligned}\Theta(\infty, F) &= 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}(r,\infty;F)}{T(r,F)} = 1 - \limsup_{\longrightarrow\infty} \frac{\overline{N}(r,\infty;f^n(f^m-a))}{(m+n)T(r,f)} \\ &\geq 1 - \limsup_{\longrightarrow\infty} \frac{T(r,f)}{(m+n)T(r,f)} = \frac{n+m-1}{m+n}. \end{aligned} \tag{3.10}$$

Similarly

$$\Theta(\infty, G) \geq \frac{n+m-1}{m+n}. \tag{3.11}$$

$$\begin{aligned}\Theta(0, F) &= 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}(r,0;F)}{T(r,F)} = 1 - \limsup_{\longrightarrow\infty} \frac{\overline{N}(r,0;f^n(f^m-a))}{(m+n)T(r,f)} \\ &\geq 1 - \limsup_{\longrightarrow\infty} \frac{(m+1)T(r,f)}{(m+n)T(r,f)} = \frac{n-1}{m+n}. \end{aligned} \tag{3.12}$$

Similarly

$$\Theta(0, G) \geq \frac{n-1}{m+n}. \tag{3.13}$$

$$\begin{aligned}\delta_{k+1}(0, F) &= 1 - \limsup_{r\longrightarrow\infty} \frac{N_{k+1}(r,0;F)}{T(r,F)} = 1 - \limsup_{\longrightarrow\infty} \frac{N_{k+1}(r,0;f^n(f^m-a))}{(m+n)T(r,f)} \\ &\geq 1 - \limsup_{\longrightarrow\infty} \frac{(k+m+1)T(r,f)}{(m+n)T(r,f)} = \frac{n-k-1}{m+n}. \end{aligned} \tag{3.14}$$

Similarly

$$\delta_{k+1}(0, G) \geq \frac{n-k-1}{m+n}. \tag{3.15}$$

From (3.10)–(3.15) we get

$$\begin{aligned}\Delta &= (4k+7)\frac{n+m-1}{m+n} + 2\frac{n-1}{m+n} + 5\frac{n-k-1}{m+n} \\ &= \frac{1}{m+n}\left[(4k+7)(n+m-1) + 2(n-1) + 5(n-k-1)\right]. \end{aligned}$$

Since $n > 9k + 6m + 14$, we get $\Delta > 4k + 13$. So by Lemma 5 we obtain either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$. Also by Lemma 8 the case $F^{(k)}G^{(k)} \equiv 1$ does not arise for $k = 1$ and $m \geq 1$.

Let $F \equiv G$. Then

$$f^n(f^m - a) \equiv g^n(g^m - a). \tag{3.16}$$

Clearly if n and m are both odd or if n is even and m is odd or if n is odd and m is even, then $f \equiv -g$ contradicts $F \equiv G$. Let neither $f \equiv g$ or $f \equiv -g$. We put $h = \frac{f}{g}$. Then $h \not\equiv 1$ and $h \not\equiv -1$. So from (3.16) we get

$$g^m = \frac{a(1-h^n)}{1-h^{n+m}}.$$

Since g is non-constant, we see that h is not a constant. Again since g^m has no simple pole, $h-u_r$ has no simple zero, where $u_r=\exp(\frac{2\pi ir}{n+m})$ and $r=1,2,\ldots,n+m-1$. Hence $\Theta(u_r,h)\geq\frac{1}{2}$ for $r=1,2,\ldots,n+m-1$, which is impossible. Therefore either $f\equiv g$ or $f\equiv -g$. This proves the theorem. ■

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