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GENERAL INTEGRAL OPERATOR DEFINED BY HADAMARD PRODUCT

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Abstract. In this paper, we introduce a new general integral operator defined by Hadamard product. Some properties involving this operator on a class of functions of complex order are determined . Furthermore, we obtained new sufficient conditions for this operator to be univalent in the open unit disc. Finally, we prove several subordination results involving starlike and convex functions of complex order. Several corollaries and consequences of the main results are also considered.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. For two functions f and g analytic in $\mathcal{U}$, we say that the function $f(z)$ is subordinate to $g(z)$, and we write $f \prec g$ or $f(z) \prec g(z)$, $(z \in \mathcal{U})$, if there exists a Schwarz function $w(z)$, analytic in $\mathcal{U}$ with $w(0) = 0$, $|w(z)| < 1$, $(z \in \mathcal{U})$, such that $f(z) = g(w(z))$, $(z \in \mathcal{U})$. For two functions $f(z) \in \mathcal{A}$ and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.1}$$

their Hadamard product (or convolution) is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.2}$$

For a function $g \in \mathcal{A}$ defined by (1.1), where $b_n \geq 0$ $(n \geq 2)$, Prajapat [17] defined the family $\mathcal{S}_\gamma(g, b)$ so that it consists of functions $f \in \mathcal{A}$ satisfying the condition

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{z(f*g)'(z)}{(f*g)(z)} - 1\right)\right\} > \gamma \qquad (z \in \mathcal{U};\ b \in \mathbb{C} \setminus \{0\}; 0 \leq \gamma < 1). \tag{1.3}$$

provided that $(f * g)(z) \neq 0$.

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Several well-known subclasses of analytic functions are special cases of our class $\mathcal{S}_\gamma(g,b)$ for suitable choices of $g(z)$. For example,

$$\mathcal{S}_\gamma\left(z+\sum_{n=2}^{\infty}z^n,b\right)=\mathcal{S}^{\star}{}_\gamma(b)=\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>\gamma$$

and

$$\mathcal{S}_\gamma\left(z+\sum_{n=2}^{\infty}nz^n,b\right)=\mathcal{C}_\gamma(b)=\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right\}>\gamma$$

where the classes $\mathcal{S}^{\star}{}_\gamma(b)$ and $\mathcal{C}_\gamma(b)$ are, respectively, the classes of starlike and convex functions of order b and type γ introduced and studied by Frasin [11]. Also, we have

$$\mathcal{S}_\gamma\left(z+\sum_{n=2}^{\infty}C_{k+n-1}^{k}z^n,b\right)=\mathcal{SR}_\gamma^k(b)=\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{z(R^kf(z))'}{R^kf(z)}-1\right)\right\}>\gamma$$

and

$$\mathcal{S}_\gamma\left(z+\sum_{n=2}^{\infty}n^kz^n,b\right)=\mathcal{SD}_\gamma^k(b)=\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{z(D^kf(z))'}{D^kf(z)}-1\right)\right\}>\gamma$$

where $R^kf(z)=z+\sum_{n=2}^{\infty}C_{k+n-1}^{k}a_nz^n$, $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ is Ruscheweyh derivative [14] and $D^kf(z)=z+\sum_{n=2}^{\infty}n^ka_nz^n$, $k\in\mathbb{N}_0$ is Salagean derivative [18]. Finally the class $\mathcal{S}_\gamma(g,b)$ reduces to the subclasses

$$\mathcal{S}_\gamma\left(z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}z^n,b\right)=\mathcal{S}_\gamma^{\star}(a,c,b)=\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{z(L(a,c)f(z))'}{L(a,c)f(z)}-1\right)\right\}>\gamma$$

$$\mathcal{S}_\gamma\left(z+\sum_{n=2}^{\infty}[1+(n-1)\lambda]^kz^n,b\right)=\mathcal{SD}_\gamma^k(\lambda,b)=\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{z(D_\lambda^kf(z))'}{D_\lambda^kf(z)}-1\right)\right\}>\gamma$$

and

$$\begin{aligned}\mathcal{S}_\gamma\left(z+\sum_{n=2}^{\infty}\frac{(\delta_1)_{n-1}\dots(\delta_q)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_s)_{n-1}(n-1)!}z^n,b\right)&=\mathcal{SH}_\gamma^{\star}(q,s,b)\\&=\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{z(H_s^q[\delta_1]f(z))'}{H_s^q[\delta_1]f(z)}-1\right)\right\}>\gamma.\end{aligned}$$

The subclass $\mathcal{S}_\gamma^{\star}(a,c,b)$ introduced and studied by Selvaraj and Karthikeyan [19] and this class defined by the Carlson-Shaffer [9] linear operator $L(a,c)f(z):=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}a_nz^n$. The subclass $\mathcal{SH}_\gamma^{\star}(q,s,b)$ introduced and studied by Prajapat [17] and defined by the Dziok-Srivastava operator [10]

$$H_s^q[\delta_1]=z+\sum_{n=2}^{\infty}\frac{(\delta_1)_{n-1}\dots(\delta_q)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_s)_{n-1}}\frac{a_nz^n}{(n-1)!}$$

and the subclass $\mathcal{SD}_\gamma^k(\lambda,b)$; $0\le\lambda\le 1$, defined by Al-Oboudi operator [2] $D_\lambda^kf(z)=z+\sum_{n=2}^{\infty}[1+(n-1)\lambda]^ka_nz^n$.

Remark 1.1. The classes $\mathcal{S}_0^\star(b)$ and $\mathcal{C}_0(b)$ of starlike and convex functions of complex order b in $\mathcal{U}$ were introduced and investigated earlier by Nasr and Aouf [16] and Wiatrowski [21]. Also, we note that $\mathcal{SD}_\gamma^k(1,b) = \mathcal{SD}_\gamma^k(b)$ and $\mathcal{SH}_\gamma^\star(1,1,b) = \mathcal{S}_\gamma^\star(\delta_1,\beta_1,b)$.

Using the Hadamard product defined by (1.2), we introduce the following general integral operator.

Definition 1.2. Given $f_i, g_i \in \mathcal{A}$, $\alpha_i \in \mathbb{C}$ for all $i = 1,\dots,n$, $n \in \mathbb{N}$. We let $I\colon \mathcal{A}^n \to \mathcal{A}$ be the integral operator defined by

$$I(f_1,\dots,f_n;g_1,\dots,g_n)(z) = \mathcal{F}(z) = \int_0^z \left(\frac{(f_1 * g_1)(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{(f_n * g_n)(t)}{t}\right)^{\alpha_n} dt \tag{1.4}$$

where $(f*g)(z)/z \neq 0$, $z \in \mathcal{U}$.

Remark 1.3. Note that the integral operator $\mathcal{F}(z)$ generalize many operators introduced and studied by several authors, for example:

(1) For $g_1 = \cdots = g_n = \frac{z}{1-z}$, we obtain the integral operator

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{1.5}$$

introduced and studied by Breaz and Breaz [4].

(2) For $g_1 = \cdots = g_n = \frac{z}{(1-z)^2}$, we obtain the integral operator

$$F_{\alpha_1,\dots,\alpha_n}(z) = \int_0^z (f_1'(t))^{\alpha_1} \dots (f_n'(t))^{\alpha_n}\, dt \tag{1.6}$$

introduced and studied by Breaz et al. [6].

(3) For $g_1 = \cdots = g_n = z + \sum_{n=2}^\infty C_{k+n-1}^k z^n$, we obtain the integral operator

$$I(f_1,\dots,f_n)(z) = \int_0^z \left(\frac{R^k f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{R^k f_n(t)}{t}\right)^{\alpha_n} dt \tag{1.7}$$

introduced in [13].

(4) For $g_1 = \cdots = g_n = z + \sum_{n=2}^\infty n^k z^n$, we obtain the integral operator

$$I^k F(z) = \int_0^z \left(\frac{D^k f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{D^k f_n(t)}{t}\right)^{\alpha_n} dt$$

introduced and studied by Breaz et al. [5].

(5) For $g_1 = \cdots = g_n = z + \sum_{n=2}^\infty [1+(n-1)\lambda]^k z^n$, we obtain the integral operator

$$I_n(f_1,\dots,f_n)(z) = \int_0^z \left(\frac{D_\lambda^k f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{D_\lambda^k f_n(t)}{t}\right)^{\alpha_n} dt$$

introduced and studied by Bulut [8].

(6) For $g_1 = \cdots = g_n = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n$, we obtain the integral operator

$$F_\alpha(a,c;z) = \int_0^z \left(\frac{L(a,c)f_1(t)}{t} \right)^{\alpha_1} \cdots \left(\frac{L(a,c)f_n(t)}{t} \right)^{\alpha_n} dt \tag{1.8}$$

introduced and studied by Selvaraj and Karthikeyan [19].

(7) For $g_1 = \frac{z}{1-z}$ and $\alpha_1 = 1$, $n = 1$, we obtain Alexander integral operator introduced in [1]

$$I(z) = \int_0^z \frac{f_1(t)}{t}\, dt$$

(8) For $g_1 = \frac{z}{1-z}$ and $\alpha_1 = 1$; $n = 1$, we obtain the integral operator

$$F_\alpha(z) = \int_0^z \left(\frac{f(t)}{t} \right)^{\alpha} dt$$

studied in [15].

In order to derive our main results, we have to recall here the following univalence criteria.

Lemma 1.4. [3] *If $f \in \mathcal{A}$ satisfies*

$$(1-|z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \le 1 \qquad (z \in \mathcal{U}),$$

then the function f is univalent in $\mathcal{U}$.

Lemma 1.5. [12] *Let $f \in \mathcal{C}_0(b)$, $b \in \mathbb{C} \setminus \{0\}$, and let $a \neq 0$ be a complex number and either $|2ab+1| \le 1$ or $|2ab-1| \le 1$. Then*

$$(f'(z))^a \prec (1-z)^{-2ab}$$

and this is the best dominant.

2. Convexity of the operator $I(f_1, \ldots, f_n; g_1, \ldots, g_n)$

We first prove

Theorem 2.1. *Let $\alpha_i > 0$, $0 \le \gamma_i < 1$ for all $i = 1, \ldots, n$ and $0 \le 1 + \sum_{i=1}^n \alpha_i(\gamma_i - 1) < 1$. If $f_i \in \mathcal{S}_{\gamma_i}(g_i, b)$ for $i = 1, \ldots, n$, $b \in \mathbb{C} \setminus \{0\}$ then the integral operator $\mathcal{F}$ given by (1.4) belongs to $\mathcal{C}_\delta(b)$, where $\delta = 1 + \sum_{i=1}^n \alpha_i(\gamma_i - 1)$.*

Proof. From the definition (1.4), we observe that $\mathcal{F} \in \mathcal{A}$, i.e. $\mathcal{F}(0) = \mathcal{F}'(0) - 1 = 0$. On the other hand, it is easy to see that

$$\mathcal{F}'(z) = \prod_{i=1}^n \left(\frac{(f_i * g_i)(z)}{z} \right)^{\alpha_i}$$

and

$$\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right) = \sum_{i=1}^{n} \alpha_i \left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1\right) \tag{2.1}$$

thus we have

$$\begin{aligned}\frac{1}{b}\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right) &= \frac{1}{b}\sum_{i=1}^{n} \alpha_i \left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1\right)\\ &= \sum_{i=1}^{n} \alpha_i \left[1 + \frac{1}{b}\left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1\right)\right] - \sum_{i=1}^{n} \alpha_i\end{aligned}$$

or, equivalently,

$$1 + \frac{1}{b}\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right) = \sum_{i=1}^{n} \alpha_i \left[1 + \frac{1}{b}\left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1\right)\right] - \sum_{i=1}^{n} \alpha_i + 1 \tag{2.2}$$

Taking the real part of both terms of (2.2), we have

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right)\right\} = \sum_{i=1}^{n} \alpha_i \operatorname{Re}\left[1 + \frac{1}{b}\left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1\right)\right] - \sum_{i=1}^{n} \alpha_i + 1. \tag{2.3}$$

Since $f_i \in \mathcal{S}_{\gamma_i}(g_i, b)$ for all $i = 1, \dots, n$, from ((1.3) and (2.3) we obtain

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right)\right\} > 1 + \sum_{i=1}^{n} \alpha_i(\gamma_i - 1),$$

but by the hypothesis $0 \leq 1 + \sum_{i=1}^{n} \alpha_i(\gamma_i - 1) < 1$,we have $\mathcal{F} \in \mathcal{C}_\delta(b)$, where $\delta = 1 + \sum_{i=1}^{n} \alpha_i(\gamma_i - 1)$. ∎

Letting $g_1 = \dots = g_n = \frac{z}{1-z}$ in Theorem 2.1, we have

Corollary 2.2. [7] *Let $\alpha_i > 0$, $0 \leq \gamma_i < 1$ for all $i = 1, \dots, n$ and $0 \leq 1 + \sum_{i=1}^{n} \alpha_i(\gamma_i - 1) < 1$. If $f_i \in \mathcal{S}^\star_{\gamma_i}(b)$ for $i = 1, \dots, n$, $b \in \mathbb{C} \setminus \{0\}$, then the integral operator F_n given by (1.5) belongs to $\mathcal{C}_\delta(b)$, where $\delta = 1 + \sum_{i=1}^{n} \alpha_i(\gamma_i - 1)$.*

Letting $g_1 = \dots = g_n = \frac{z}{(1-z)^2}$ in Theorem 2.1, we have

Corollary 2.3. [7] *Let $\alpha_i > 0$, $0 \leq \gamma_i < 1$ for all $i = 1, \dots, n$ and $0 \leq 1 + \sum_{i=1}^{n} \alpha_i(\gamma_i - 1) < 1$. If $f_i \in \mathcal{C}_{\gamma_i}(b)$ for $i = 1, \dots, n$, $b \in \mathbb{C} \setminus \{0\}$, then the integral operator $F_{\alpha_1,\dots,\alpha_n}$ given by (1.6) belongs to $\mathcal{C}_\delta(b)$, where $\delta = 1 + \sum_{i=1}^{n} \alpha_i(\gamma_i - 1)$.*

Letting $g_1 = \dots = g_n = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n$ in Theorem 2.1, we have

Corollary 2.4. [19] *Let $\alpha_i > 0$, $0 \leq \gamma < 1$ for all $i = 1, \dots, n$ and $0 \leq 1 + (\gamma - 1)\sum_{i=1}^{n} \alpha_i < 1$. If $f_i \in \mathcal{S}^\star_{\gamma_i}(a, c, b)$ for $i = 1, \dots, n$, $b \in \mathbb{C} \setminus \{0\}$, then the integral operator $F_\alpha(a, c; z)$ given by (1.8) belongs to $\mathcal{C}_\delta(b)$, $\delta = 1 + (\gamma - 1)\sum_{i=1}^{n} \alpha_i$.*

Remark 2.5. Taking different choices of $g_1 = \dots = g_n$ as stated in Section 1, Theorem 2.1 leads to new sufficient conditions for the integral operators $I(f_1, \dots, f_n)(z)$, $I^k F(z)$, and $I_n(f_1, \dots, f_n)(z)$ to be in $\mathcal{C}_\delta(b)$.

3. Univalence conditions

Applying Lemma 1.4, we prove

THEOREM 3.1. *Let $f_i, g_i \in \mathcal{A}$, $\alpha_i \in \mathbb{C}$ for all $i = 1, \dots, n$. If*

$$\left| \frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right| \le 1 \qquad (z \in \mathcal{U}),$$

and $\sum_{i=1}^{n} |\alpha_i| \le 1$, then the integral operator $\mathcal{F}$ given by (1.4) is univalent.

Proof. It follows from (2.1) that

$$\left| \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} \right| \le \sum_{i=1}^{n} |\alpha_i| \left| \frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right| \tag{3.1}$$

On multiplying the inequality (3.1) by $(1 - |z|^2)$, we obtain

$$\begin{aligned} (1 - |z|^2) \left| \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} \right| &\le (1 - |z|^2) \sum_{i=1}^{n} |\alpha_i| \left| \frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right| \\ &\le (1 - |z|^2) \sum_{i=1}^{n} |\alpha_i| \le \sum_{i=1}^{n} |\alpha_i| \le 1. \end{aligned} \tag{3.2}$$

From Lemma 1.4, we have $\mathcal{F} \in \mathcal{S}$. ∎

Letting $g_1 = \cdots = g_n = \frac{z}{1-z}$ in Theorem 3.1, we have

COROLLARY 3.2. [4] *Let $f_i \in \mathcal{A}$, $\alpha_i \in \mathbb{C}$ for all $i = 1, \dots, n$. If*

$$\left| \frac{z{f_i}'(z)}{f_i(z)} - 1 \right| \le 1 \qquad (z \in \mathcal{U}),$$

and $\sum_{i=1}^{n} |\alpha_i| \le 1$, then the integral operator F_n given by (1.5) is univalent.

Letting $g_1 = \cdots = g_n = z + \sum_{n=2}^{\infty} C_{k+n-1}^{k} z^n$ in Theorem 3.1, we have

COROLLARY 3.3. [13] *Let $f_i \in \mathcal{A}$, $\alpha_i \in \mathbb{C}$ for all $i = 1, \dots, n$. If*

$$\left| \frac{z(R^k f_i(z))'}{R^k f_i(z)} - 1 \right| \le 1 \qquad (z \in \mathcal{U}),$$

and $\sum_{i=1}^{n} |\alpha_i| \le 1$, then the integral operator $I(f_1, \dots, f_n)(z)$ given by (1.7) is univalent.

Letting $g_1 = \cdots = g_n = z + \sum\limits_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n$ in Theorem 3.1, we have

COROLLARY 3.4. [19] *Let $f_i \in \mathcal{A}$, $\alpha_i \in \mathbb{C}$ for all $i = 1, \dots, n$. If*

$$\left| \frac{z(L(a,c)f_i(z))'}{L(a,c)f_i(z)} - 1 \right| \le 1 \qquad (z \in \mathcal{U}),$$

and $\sum_{i=1}^{n} |\alpha_i| \le 1$, then the integral operator $F_\alpha(a, c; z)$ given by (1.8) is univalent.

Remark 3.5. Taking different choices of $g_1 = \cdots = g_n$ as stated in Section 1, Theorem 3.1 leads to new sufficient conditions for the integral operators $F_{\alpha_1,\dots,\alpha_n}(z)$, $I^k F(z)$ and $I_n(f_1,\dots,f_n)(z)$ to be univalent in $\mathcal{U}$.

Now, we prove

Theorem 3.6. *Let $f_i, g_i \in \mathcal{A}$, $\alpha_i \in \mathbb{C}$ for all $i = 1,\dots,n$. If f_i, g_i satisfy the conditions*

(*i*) $|(f_i * g_i)(z)| \le 1$,

(*ii*) $\left|\frac{z^2(f_i*g_i)'(z)}{[(f_i*g_i)(z)]^2} - 1\right| \le 1$,

(*iii*) $3\sum_{i=1}^{n}|\alpha_i| \le 1$

for all $z \in \mathcal{U}$, then the integral operator $\mathcal{F}$ given by (1.4) is univalent.

Proof. From (3.2), we get

$$(1-|z|^2)\left|\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right| \le (1-|z|^2)\sum_{i=1}^{n}|\alpha_i|\left|\frac{z(f_i*g_i)'(z)}{(f_i*g_i)(z)}\right| + (1-|z|^2)\sum_{i=1}^{n}|\alpha_i|$$
$$= (1-|z|^2)\sum_{i=1}^{n}|\alpha_i|\left|\frac{z^2(f_i*g_i)'(z)}{[(f_i*g_i)(z)]^2}\right|\left|\frac{(f_i*g_i)(z)}{z}\right| + (1-|z|^2)\sum_{i=1}^{n}|\alpha_i|.$$

Using Schwarz's lemma and the conditions (*i*), (*ii*) and (*iii*) we obtain

$$(1-|z|^2)\left|\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right| \le (1-|z|^2)\sum_{i=1}^{n}|\alpha_i|\left|\frac{z^2(f_i*g_i)'(z)}{[(f_i*g_i)(z)]^2}\right| + (1-|z|^2)\sum_{i=1}^{n}|\alpha_i|$$
$$= (1-|z|^2)\sum_{i=1}^{n}|\alpha_i|\left|\frac{z^2(f_i*g_i)'(z)}{[(f_i*g_i)(z)]^2} - 1 + 1\right| + (1-|z|^2)\sum_{i=1}^{n}|\alpha_i|$$
$$\le (1-|z|^2)\sum_{i=1}^{n}|\alpha_i|\left|\frac{z^2(f_i*g_i)'(z)}{[(f_i*g_i)(z)]^2} - 1\right| + 2(1-|z|^2)\sum_{i=1}^{n}|\alpha_i|$$
$$\le 3(1-|z|^2)\sum_{i=1}^{n}|\alpha_i| \le 1.$$

Hence by Lemma 1.4, we have $\mathcal{F} \in \mathcal{S}$. ■

Letting $g_1 = \cdots = g_n = z + \sum_{n=2}^{\infty} C_{k+n-1}^{k} z^n$ in Theorem 3.6, we have

Corollary 3.7. [13] *Let $f_i \in \mathcal{A}$, $\alpha_i \in \mathbb{C}$ for all $i = 1,\dots,n$, satisfy*

(*i*) $\left|R^k f_i(z)\right| \le 1$,

(*ii*) $\left|\frac{z^2(R^k f_i(z))'}{(R^k f_i(z))^2} - 1\right| \le 1$,

(*iii*) $3\sum_{i=1}^{n}|\alpha_i| \le 1$

then the integral operator $I(f_1,\dots,f_n)(z)$ given by (1.7) is univalent.

Remark 3.8. Taking different choices of $g_1 = \cdots = g_n$ as stated in Section 1, Theorem 3.6 leads to new sufficient conditions for the integral operators $F_n(z)$, $F_{\alpha_1,\dots,\alpha_n}(z)$, $I^k F(z)$, $I_n(f_1,\dots,f_n)(z)$ and $F_\alpha(a,c;z)$ to be univalent in $\mathcal{U}$.

4. Subordination results

At last, we prove the following subordination result

THEOREM 4.1. *Let $f_i \in \mathcal{S}_\gamma(g_i, b)$ for $i = 1, \dots, n$, $b \in \mathbb{C} \setminus \{0\}$, $\gamma = 1/\alpha_i$; $\alpha_i > 1$ with $\sum_{i=1}^n \alpha_i < n+1$ and let $a \neq 0$ be a complex number and either $|2ab+1| \leq 1$ or $|2ab-1| \leq 1$. Then*

$$(\mathcal{F}'(z))^a = \prod_{i=1}^n \left(\frac{(f_i * g_i)(z)}{z} \right)^{a\alpha_i} \prec (1-z)^{-2ab} \qquad (z \in \mathcal{U}),$$

and this is the best dominant.

Proof. Let $f_i \in \mathcal{S}_\gamma(g_i, b)$ for $i = 1, \dots, n$, $b \in \mathbb{C} \setminus \{0\}$ where $\gamma = 1 - (1/\sum_{i=1}^n \alpha_i)$ and $\sum_{i=1}^n \alpha_i \geq 1$, then from (2.3), we obtain

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right)\right\} = \sum_{i=1}^n \alpha_i \operatorname{Re}\left[1 + \tfrac{1}{b}\left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1\right)\right] - \sum_{i=1}^n \alpha_i + 1 > 0$$

and thus $\mathcal{F} \in \mathcal{C}_0(b)$. Applying Lemma 1.5, we have $(\mathcal{F}'(z))^a \prec (1-z)^{-2ab}$ that is,

$$\prod_{i=1}^n \left(\frac{(f_i * g_i)(z)}{z} \right)^{a\alpha_i} \prec (1-z)^{-2ab} \quad \blacksquare$$

Letting $g_1 = \cdots = g_n = \frac{z}{1-z}$ in Theorem 4.1, we have

COROLLARY 4.2. *Let $f_i \in \mathcal{S}^\star{}_\gamma(b)$ for $i = 1, \dots, n$, $b \in \mathbb{C} \setminus \{0\}$, $\gamma = 1/\alpha_i$; $\alpha_i > 1$ with $\sum_{i=1}^n \alpha_i < n+1$ and let $a \neq 0$ be a complex number and either $|2ab+1| \leq 1$ or $|2ab-1| \leq 1$. Then*

$$\prod_{i=1}^n \left(\frac{f_i(z)}{z} \right)^{a\alpha_i} \prec (1-z)^{-2ab} \qquad (z \in \mathcal{U}),$$

and this is the best dominant.

Letting $g_1 = \cdots = g_n = \frac{z}{(1-z)^2}$ in Theorem 4.1, we have

COROLLARY 4.3. *Let $f_i \in \mathcal{C}_\gamma(b)$ for $i = 1, \dots, n$, $b \in \mathbb{C} \setminus \{0\}$, $\gamma = 1/\alpha_i$; $\alpha_i > 1$ with $\sum_{i=1}^n \alpha_i < n+1$ and let $a \neq 0$ be a complex number and either $|2ab+1| \leq 1$ or $|2ab-1| \leq 1$. Then*

$$\prod_{i=1}^n (f_i'(z))^{a\alpha_i} \prec (1-z)^{-2ab} \qquad (z \in \mathcal{U}),$$

and this is the best dominant.

Letting $n = 1$, $\alpha_1 = a = 1$ and $f_1 = f$ in Theorem 4.1, then we have the following result obtained by Srivastava and Lashin [20].

Corollary 4.4. [20] *Let* $f \in \mathcal{C}_0(b)$, $b \in \mathbb{C} \setminus \{0\}$. *Then*

$$f'(z) \prec \frac{1}{(1-z)^{2b}} \qquad (z \in \mathcal{U}),$$

and this is the best dominant.

Remark 4.5. In view of Theorem 4.1 and by taking different choices of $g_1 = \cdots = g_n$ as mentioned in Section 1, we obtain new subordination results for functions in the classes $\mathcal{SR}_\gamma^k(b)$, $\mathcal{SD}_\gamma^k(b)$, $\mathcal{S}_\gamma^\star(a,c,b)$, $\mathcal{SH}_\gamma^\star(q,s,b)$ and $\mathcal{SD}_\gamma^k(\lambda,b)$.

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