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# A NUMERICAL METHOD FOR SOLUTION OF SEMIDIFFERENTIAL EQUATIONS

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**Abstract.** In this paper, a new algorithm for the numerical solution of semidifferential equations with constant coefficients and fractional derivative defined in the Caputo sense is presented. The algorithm is obtained by using the spline collocation method. Moreover, a new technique for calculating the fractional derivative of the spline polynomial is derived. Numerical examples are also presented to test and illustrate the method.

## 1. Introduction

In this paper, we propose a numerical method for approximating the solution of the semidifferential equation of order $v \in \mathbb{N}$;

$$\left[D_*^{v/2} + A_1 D_*^{(v-1)/2} + \cdots + A_v D_*^0\right] y(t) = f(t), \tag{1}$$

with initial conditions

$$y(0) = \beta_0, y'(0) = \beta_1, \dots, y^{(k-1)}(0) = \beta_{k-1}, \quad k \in \mathbb{N}, \quad k-1 < \frac{v}{2} \le k, \tag{2}$$

where $A_1, \dots, A_v, \beta_0, \beta_1, \dots, \beta_{k-1} \in \mathbb{R}$, and $f(t)$ is a given function defined on the interval $I = [0, T]$. Here, $y$ is the unknown function and the fractional derivatives are defined in the Caputo sense:

$$D_*^{\alpha} y(t) = \begin{cases} \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-x)^{k-\alpha-1} y^{(k)}(x)\, dx, & \text{if } \alpha \in \mathbb{R}_+ \setminus \mathbb{N}, \\ y^{(k)}(t), & \text{if } \alpha \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3}$$

where $k = \lceil \alpha \rceil$ is the smallest integer not less than than $\alpha$ and $\Gamma(\cdot)$ is the Euler gamma function. The Caputo definition of fractional derivatives has many applications when physical systems requiring inhomogeneous initial conditions are to be modeled.

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The initial value problem given in (1) and (2) has found many applications in recent studies of scaling phenomena, see for example [1, 6, 8, 9]. We mention two examples: The first one is the single degree-of-freedom spring-mass-damper system whose dynamics is described by the following fractional differential equation

$$\left[mD_*^2 + cD_*^{0.5} + kD_*^0\right] y(t) = f(t), \tag{4}$$

where $m$, $c$, and $k$ represent the mass, damping coefficient, and stiffness, respectively, and $f(t)$ is the externally applied force. The second one is the Bagley-Torvik equation

$$[D_*^2 + A_1 D_*^{3/2} + A_4 D_*^0] y\left(t\right) = f\left(t\right), \tag{5}$$

which arises, for example, in the modeling of the motion of a rigid plate immersed in a Newtonian fluid. Note that traditionally the Bagley-Torvik equation is formulated with the Riemann-Liouville definition:

$$D^\alpha y(t) = \frac{d^k}{dt^k}\left(\frac{1}{\Gamma(k-\alpha)}\int_0^t (t-x)^{k-\alpha-1} y(x)\,dx\right), \qquad t > 0, \tag{6}$$

rather than with the Caputo definition, but I. Podlubny in [10] has proved a very useful and interesting relationship between the operators $D^\alpha$ and $D_*^\alpha$

$$D_*^\alpha y(t) = D^\alpha(y(t) - T_y),$$

where $T_y$ is the Taylor's polynomial of degree $k-1$ of the function $y$ about $t = 0$. So, under homogenous conditions the two problems are equivalent.

Studying the numerical solution of (1) have been increased in the last two decades. A specific numerical method to approximate the solution of equation (1) is given by Diethelm and Ford in [4] and Ghorbani and Alavi in [7]. Also, a numerical technique for solving (4) and (5) have been studied by Diethelm and Ford [5] and Yuan and Agrawal [13], respectively.

In this paper the semidifferential equation (1) with the initial conditions (2) will be solved numerically using the polynomial spline collocation method. The polynomial spline collocation method for differential and integral equations has been extensively examined by many authors, see for example [2, 3, 11, 12].

This paper is organized as follows. In Section 2 , a new technique for finding the fractional derivative of the spline polynomial is derived, then this technique is used to solve a semidifferential equation of order $v$ numerically. Numerical illustrations are given in Section 3.

## 2. Spline collocation method

Let $t_n = nh$, where $n = 0, \ldots, N$; and $t_N = T$, define the uniform partition of the interval $I = [0, T]$ whose $N$ subintervals are $\sigma_n = [t_n, t_{n+1}]$, where $n = 0, \ldots, N-1$, and set $Z_N = \{t_1, t_2, \ldots, t_{N-1}\}$, $\bar{Z}_N = Z_N \cup \{T\}$. Moreover, denote by $\pi_{m+d}$ the set of all real polynomials of degree not exceeding $m + d$. The exact solution $y$ of (1) and (2) will be approximated on $I$ by an element $u$ in the polynomial spline space

$$S_{m+d}^{(d)}(Z_N) = \{u \in C^d(I) : u \in \pi_{m+d} \text{ on } \sigma_n(n = 0, 1, \dots, N-1)\};$$

that is, by $d$-times continuously differentiable polynomial spline function of degree $m + d$.

This approximation $u$ will be determined by collocation. Let $\{c_i\}_{i=1}^m$ be a given set of $m$ real numbers (subsequently called collocation parameters) satisfying $0 < c_1 < c_2 < \cdots < c_m < 1$, and define the collocation points by

$$X(N) = \bigcup_{n=0}^{N-1} X_n \text{ with } X_n = \{t_{n,i} = t_n + c_i h : i = 1, 2, \dots, m\} \subset \sigma_n.$$

The solution of (1) and (2) is then given by an element $u \in S_{m+d}^{(d)}(Z_N)$ satisfying for each $t_{n,i} \in X(N)$

$$\left[D_*^{v/2} + A_1 D_*^{(v-1)/2} + \cdots + A_v D_*^0\right] u(t_{n,i}) = f(t_{n,i}) \tag{7}$$

$$u(0) = \beta_0, u'(0) = \beta_1, \dots, u^{(k-1)}(0) = \beta_{k-1}, \quad k \in \mathbb{N}, \quad k - 1 < \frac{v}{2} \le k, \tag{8}$$

where $u$ is described on each subinterval $[t_j, t_{j+1}]$ by the polynomial

$$u_j(t) = \sum_{l=0}^{m+d} a_l^{(j)} t^l, \tag{9}$$

where $a_0^{(j)}, a_1^{(j)}, \dots, a_{m+d}^{(j)}$ are constants to be determined later.

The collocation equations (7) and (8) then lead to a system of linear algebraic equations for the coefficients $a_i^{(n)}$, $i = 1, 2, \dots, m$; $n = 0, 1, \dots, N-1$. Note that in order to determine the approximate solution $u$ using (7) and (8), it is necessary to require that $d = \lceil \frac{v}{2} \rceil - 1$, so from now we will assume $d = \lceil \frac{v}{2} \rceil - 1$ (see Remark 2.3 in Section 2).

To find the fractional derivative of $u \in S_{m+d}^{(d)}(Z_N)$, we need the following auxiliary notations, definitions, and lemmas: first, we define the function $S_b(t)$ by

$$S_b(t) = \begin{cases} 0, & \text{if } t \le b, \\ 1, & \text{if } t > b. \end{cases} \tag{10}$$

LEMMA 2.1. *For each $t \in I$, the spline function $u$ can be represented by the formula*

$$\begin{aligned} u(t) &= u_0(t) + S_{t_1}(t)[u_1(t) - u_0(t)] + \cdots + S_{t_{N-1}}(t)[u_{N-1}(t) - u_{N-2}(t)] \\ &= u_0(t) + \sum_{j=1}^{N-1} S_{t_j}(t)[u_j(t) - u_{j-1}(t)]. \end{aligned} \tag{11}$$

Definition 2.1. For $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and $k = \lceil \alpha \rceil$, define the operator $D^{\alpha}_{*,a}$ by

$$D^{\alpha}_{*,a} g(t) = \frac{1}{\Gamma(k-\alpha)} \int_a^t (t-\xi)^{k-\alpha-1} g^{(k)}(\xi)\, d\xi, \qquad t > a.$$

Remark 2.1. In fact, Definition 2.1 is the general definition of the Caputo fractional derivative operator. More precisely; $D^{\alpha}_{*,0} \equiv D^{\alpha}_{*}$ (see [10]).

Lemma 2.2. *Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and $g \in C^k[a,c]$ with $g^{(i)}(b) = 0$ for $i = 0, 1, \ldots, k$, where $k = \lceil \alpha \rceil$ and $a < b < c$. Then*

$$D^{\alpha}_{*,a}[S_b(t) g(t)] = S_b(t) D^{\alpha}_{*,b} g(t), \qquad a < t < b,$$

*where the function $S_b(t)$ is defined by [10].*

*Proof.* Since $g \in C^k[a,c]$ and $g^{(i)}(b) = 0$ for $i = 0, 1, \ldots, k$, $S_b g \in C^k[a,c]$. Thus, we obtain

$$\begin{aligned} D^{\alpha}_{*,a}[S_b(t) g(t)] &= \begin{cases} 0, & \text{if } t \le b, \\ \frac{1}{\Gamma(k-\alpha)} \int_a^t \frac{d^k}{d\xi^k} (S_b(\xi) g(\xi))(t-\xi)^{k-\alpha-1} d\xi, & \text{if } t > b. \end{cases} \\ &= \begin{cases} 0, & \text{if } t \le b, \\ \frac{1}{\Gamma(k-\alpha)} \int_b^t g^k(\xi)(t-\xi)^{k-\alpha-1}\, d\xi, & \text{if } t > b. \end{cases} \\ &= S_b(t) D^{\alpha}_{*,b} g(t). \qquad \blacksquare \end{aligned}$$

Remark 2.2. Lemma 2.2 can be applied to the spline function $u \in S^{(k)}_{m+k}(Z_N)$ (with $k \le d$) defined by equation (11) since $u_j - u_{j-1} \in C^k[0,T]$ and $(u_j - u_{j-1})^{(i)}(t_j) = 0$ for $i = 0, 1, \ldots, k$. In fact, Lemma 2.2 is also valid even if $k = d+1$, because in this case $u^{(k)}$ is a piecewise continuous polynomial which is integrable over $[0,T]$.

Lemma 2.3. *Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$, $l \in \mathbb{N} \cup \{0\}$ and $k = \lceil \alpha \rceil$. Then for $t > a \ge 0$, we have*

$$D^{\alpha}_{*,a} t^l = \begin{cases} 0, & \text{if } l < k, \\ C^{(l)}_{a,\alpha}(t), & \text{if } l \ge k, \end{cases}$$

*where*

$$C^{(l)}_{a,\alpha}(t) = \sum_{r=1}^{l-k} \frac{\Gamma(l+1) a^{l-k-r+1}}{\Gamma(k-\alpha+r)\Gamma(l-k-r+2)} (t-a)^{k-\alpha+r-1} + \frac{\Gamma(l+1)}{\Gamma(l-\alpha+1)} (t-a)^{l-\alpha}.$$

*Proof.* If $l < k$, then by definition

$$D^{\alpha}_{*,a} t^l = \frac{1}{\Gamma(k-\alpha)} \int_a^t \frac{d^k \xi^l}{d\xi^k} (t-\xi)^{k-\alpha-1} d\xi = \frac{1}{\Gamma(k-\alpha)} \int_a^t 0\, d\xi = 0.$$

Now, if $l \geq k$, then integration by parts yields

$$\begin{aligned} D^{\alpha}_{*,a} t^l &= \tfrac{1}{\Gamma(k-\alpha)} \int_a^t \tfrac{\Gamma(l+1)}{\Gamma(l-k+1)} \xi^{l-k} (t-\xi)^{k-\alpha-1}\, d\xi \\ &= \tfrac{\Gamma(l+1)}{\Gamma(k-\alpha)\Gamma(l-k+1)} \Big[ \tfrac{a^{l-k}(t-a)^{k-\alpha}}{(k-\alpha)} + \tfrac{(l-k)a^{l-k-1}(t-a)^{k-\alpha+1}}{(k-\alpha)(k-\alpha+1)} \\ &\quad + \tfrac{(l-k)(l-k-1)a^{l-k-2}(t-a)^{k-\alpha+2}}{(k-\alpha)(k-\alpha+1)(k-\alpha+2)} + \cdots + \tfrac{(l-k)!(t-a)^{l-\alpha}}{(k-\alpha)(k-\alpha+1)\cdots(l-\alpha)} \Big] \\ &= \sum_{r=1}^{l-k} \tfrac{\Gamma(l+1)a^{l-k-r+1}}{\Gamma(k-\alpha+r)\Gamma(l-k-r+2)} (t-a)^{k-\alpha+r-1} + \tfrac{\Gamma(l+1)}{\Gamma(l-\alpha+1)} (t-a)^{l-\alpha}. \quad \blacksquare \end{aligned}$$

The linearity of $D^{\alpha}_{*,a}$ and a direct application of Lemma 2.3 imply the following corollary.

Corollary 2.1. *Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and $k = \lceil \alpha \rceil$. Then for $j = 0, 1, \ldots, N-1$, we have*

$$D^{\alpha}_{*,a} u_j(t) = \sum_{l=\lceil \alpha \rceil}^{m+d} C^{(l)}_{a,\alpha}(t) a^{(j)}_l,$$

*where $u_j$ is defined by* (9).

Now we are in a position to find the fractional derivative of the spline polynomial $D^{\alpha}_{*} u(t)$.

Lemma 2.4. *Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and $u \in S^{(k-1)}_{m+k-1}(Z_N)$ where $k = \lceil \alpha \rceil$. Then*

$$D^{\alpha}_{*} u(t) = \sum_{l=\lceil \alpha \rceil}^{m+d} C^{(l)}_{t_0,\alpha}(t) a^{(0)}_l + \sum_{j=1}^{N-1} \sum_{l=\lceil \alpha \rceil}^{m+d} S_{t_j}(t) C^{(l)}_{t_j,\alpha}(t) \left( a^{(j)}_l - a^{(j-1)}_l \right).$$

*Proof.* Using the linearity of the operator $D^{\alpha}_{*}$, equation (11), Lemma 2.2, Corollary 2.1, and Remarks 2.1 and 2.2, the result follows immediately. $\blacksquare$

Now, the semidifferential equation of order $v$ in the Caputo sense is

$$\left[ D^{v/2}_{*} + A_1 D^{(v-1)/2}_{*} + \cdots + A_v D^{0}_{*} \right] y(t) = f(t)$$

with initial conditions

$$y(0) = \beta_0, y'(0) = \beta_1, \ldots, y^{(\lceil \frac{v}{2} \rceil - 1)}(0) = \beta_{\lceil \frac{v}{2} \rceil - 1}, \qquad \lceil \frac{v}{2} \rceil \in \mathbb{N}, \tag{12}$$

which can be written in the following form

$$\sum_{j=0}^{\lfloor \frac{v}{2} \rfloor} b_j D^{j}_{*} y(t) + \sum_{j=1}^{\lfloor \frac{v+1}{2} \rfloor} d_j D^{j-\frac{1}{2}}_{*} y(t) = f(t), \tag{13}$$

where $\lfloor p \rfloor$ is the largest integer not greater than $p$ and $b_0, \ldots, b_{\lfloor \frac{v}{2} \rfloor}, d_1, \ldots, d_{\lfloor \frac{v+1}{2} \rfloor} \in \mathbb{R}$.

The exact solution $y(t)$ of (13) and (12) will be approximate by collocation solution given by an element $u \in S_{m+d}^{(d)}(Z_N)$, where $d = \lceil \frac{v}{2} \rceil - 1$, satisfying for each $t_{n,i} \in X(N)$

$$\sum_{j=0}^{\lfloor \frac{v}{2} \rfloor} b_j D_*^j u(t_{n,i}) + \sum_{j=1}^{\lfloor \frac{v+1}{2} \rfloor} d_j D_*^{j-\frac{1}{2}} u(t_{n,i}) = f(t_{n,i}). \tag{14}$$

For the ordinary derivative $D_*^j u(t_{n,i})$, we have

$$D_*^j u(t_{n,i}) = \sum_{l=j}^{m+d} a_l^{(n)} \frac{\Gamma(l+1)}{\Gamma(l+1-j)} t_{n,i}^{l-j}, \tag{15}$$

and by using Lemma 2.4, we obtain

$$\begin{aligned} D_*^{j-\frac{1}{2}} u(t_{n,i}) = &\sum_{l=\lceil j-\frac{1}{2} \rceil}^{m+d} C_{t_0,j-\frac{1}{2}}^{(l)}(t_{n,i}) a_l^{(0)} \\ &+ \sum_{k=1}^{N-1} \sum_{l=\lceil j-\frac{1}{2} \rceil}^{m+d} S_{t_k}(t_{n,i}) C_{t_k,j-\frac{1}{2}}^{(l)}(t_{n,i}) \left( a_l^{(k)} - a_l^{(k-1)} \right). \end{aligned} \tag{16}$$

Now $\lceil j - \frac{1}{2} \rceil = j$ and $S_{t_k}(t_{n,i}) = 0$ if $n \leq k$, so (16) becomes

$$D_*^{j-\frac{1}{2}} u(t_{n,i}) = \sum_{l=j}^{m+d} C_{t_0,j-\frac{1}{2}}^{(l)}(t_{n,i}) a_l^{(0)} + \sum_{k=1}^{n} \sum_{l=j}^{m+d} C_{t_k,j-\frac{1}{2}}^{(l)}(t_{n,i}) \left( a_l^{(k)} - a_l^{(k-1)} \right). \tag{17}$$

By substituting (15) and (17) in (14), we get

$$\begin{aligned} &\sum_{j=0}^{\lfloor \frac{v}{2} \rfloor} \sum_{l=j}^{m+d} b_j \tfrac{\Gamma(l+1)}{\Gamma(l+1-j)} t_{n,i}^{l-j} a_l^{(n)} + \sum_{j=1}^{\lfloor \frac{v+1}{2} \rfloor} \sum_{l=j}^{m+d} d_j C_{t_0,j-\frac{1}{2}}^{(l)}(t_{n,i}) a_l^{(0)} \\ &\qquad + \sum_{j=1}^{\lfloor \frac{v+1}{2} \rfloor} \sum_{k=1}^{n} \sum_{l=j}^{m+d} d_j C_{t_k,j-\frac{1}{2}}^{(l)}(t_{n,i}) \left( a_l^{(k)} - a_l^{(k-1)} \right) = f(t_{n,i}). \end{aligned} \tag{18}$$

*Case 1*: $n = 0$. In this case $t_{0,i} \in [t_0, t_1)$ and $S_{t_k}(t_{0,i}) = 0$ for all $k = 1, 2, \ldots, N-1$, so

$$D_*^j u(t_{0,i}) = \sum_{l=j}^{m+d} \tfrac{\Gamma(l+1)}{\Gamma(l+1-j)} t_{0,i}^{l-j} a_l^{(0)}, \tag{19}$$

and from Lemma 2.4, we have

$$D_*^{j-\frac{1}{2}} u(t_{0,i}) = \sum_{l=1}^{m+d} C_{t_0,j-\frac{1}{2}}^{(l)}(t_{0,i}) a_l^{(0)}. \tag{20}$$

Thus from (19) and (20), equation (18) becomes

$$\sum_{j=0}^{\lfloor \frac{v}{2} \rfloor} \sum_{l=j}^{m+d} b_j \tfrac{\Gamma(l+1)}{\Gamma(l+1-j)} t_{0,i}^{l-j} a_l^{(0)} + \sum_{j=1}^{\lfloor \frac{v+1}{2} \rfloor} \sum_{l=j}^{m+d} d_j C_{t_0,j-\frac{1}{2}}^{(l)}(t_{0,i}) a_l^{(0)} = f(t_{0,i}). \tag{21}$$

Now, equation (21) can be written in the form

$$\sum_{j=0}^{\lfloor\frac{v}{2}\rfloor}\sum_{l=j}^{m+d} H_{l,j}^{(0,i)} a_l^{(0)} + \sum_{j=1}^{\lfloor\frac{v+1}{2}\rfloor}\sum_{l=j}^{m+d} R_{0,l,j-\frac{1}{2}}^{(0,i)} a_l^{(0)} = f(t_{0,i}), \tag{22}$$

where

$$H_{l,j}^{(0,i)} = b_j \frac{\Gamma(l+1)}{\Gamma(l+1-j)} t_{0,i}^{l-j}, \qquad R_{0,l,j-\frac{1}{2}}^{(0,i)} = d_j C_{t_0,j-\frac{1}{2}}^{(l)}(t_{0,i}).$$

With a simple arrangement we may write equation (22) as

$$\sum_{j=0}^{v}\sum_{l=\lceil\frac{j}{2}\rceil}^{m+d} F_{l,j}^{(0,i)} a_l^{(0)} = f(t_{0,i}), \tag{23}$$

where $F_{l,j}^{(0,i)} = \begin{cases} H_{l,j}^{(0,i)} & \text{if } j \text{ is even,} \\ R_{0,l,j-\frac{1}{2}}^{(0,i)} & \text{if } j \text{ is odd.} \end{cases}$

*Case 2*: $0 < n \leq N-1$.

With a simple arrangement, equation (18) can be written in the form

$$\sum_{j=0}^{\lfloor\frac{v}{2}\rfloor}\sum_{l=j}^{m+d} H_{l,j}^{(n,i)} a_l^{(n)} + \sum_{j=1}^{\lfloor\frac{v+1}{2}\rfloor}\sum_{l=j}^{m+d} R_{0,l,j-\frac{1}{2}}^{(n,i)} a_l^{(0)} + \sum_{j=1}^{\lfloor\frac{v+1}{2}\rfloor}\sum_{k=0}^{n}\sum_{l=j}^{m+d} T_{k,l,j-\frac{1}{2}}^{(n,i)} a_l^{(k)} = f(t_{n,i}), \tag{24}$$

where

$$H_{l,j}^{(n,i)} = b_j \frac{\Gamma(l+1)}{\Gamma(l+1-j)} t_{n,i}^{l-j}, \qquad R_{k,l,j-\frac{1}{2}}^{(n,i)} = d_j C_{t_k,j-\frac{1}{2}}^{(l)}(t_{n,i}),$$

and

$$T_{k,l,j-\frac{1}{2}}^{(n,i)} = \begin{cases} -R_{k+1,l,j-\frac{1}{2}}^{(n,i)} & \text{if } k=0, \\ R_{k,l,j-\frac{1}{2}}^{(n,i)} - R_{k+1,l,j-\frac{1}{2}}^{(n,i)} & \text{if } k=1,2,\ldots,n-1, \\ R_{k,l,j-\frac{1}{2}}^{(n,i)} & \text{if } k=n. \end{cases}$$

Now, we may write

$$\sum_{j=0}^{\lfloor\frac{v}{2}\rfloor}\sum_{l=j}^{m+d} H_{l,j}^{(n,i)} a_l^{(n)} + \sum_{j=1}^{\lfloor\frac{v+1}{2}\rfloor}\sum_{l=j}^{m+d} T_{k,l,j-\frac{1}{2}}^{(n,i)} a_l^{(n)} = \sum_{j=0}^{v}\sum_{l=\lceil\frac{j}{2}\rceil}^{m+d} F_{l,j}^{(n,i)} a_l^{(n)}, \tag{25}$$

where

$$F_{l,j}^{(n,i)} = \begin{cases} H_{l,j}^{(n,i)} & \text{if } j \text{ is even,} \\ T_{k,l,j-\frac{1}{2}}^{(n,i)} & \text{if } j \text{ is odd.} \end{cases}$$

Hence by using (25), we may write (24) as

$$\sum_{j=0}^{v}\sum_{l=\lceil\frac{j}{2}\rceil}^{m+d} F_{l,j}^{(n,i)} a_l^{(n)} = f(t_{n,i}) - \sum_{j=1}^{\lfloor\frac{v+1}{2}\rfloor}\sum_{l=j}^{m+d} R_{0,l,j-\frac{1}{2}}^{(n,i)} a_l^{(0)} - \sum_{j=1}^{\lfloor\frac{v+1}{2}\rfloor}\sum_{k=0}^{n-1}\sum_{l=j}^{m+d} T_{k,l,j-\frac{1}{2}}^{(n,i)} a_l^{(k)}. \tag{26}$$

By applying equation (23) for $t_{0,i}$ for each $i = 1, 2, \ldots, m$, and (26) for each $t_{n,i}$, $n = 1, \ldots, N-1$, $i = 1, 2, \ldots, m$, we get a system with $mN$ linear equations with $(m+d+1)N$ unknown coefficients $\{a_l^{(k)}\}_{k=0,\ldots,N-1}^{l=0,\ldots,m+d}$. Therefore, we need extra $(d+1)N$ equations, but from the conditions of smoothness

$$D^j u_n(t_n) = D^j u_{n-1}(t_n), \quad j = 0, 1, \ldots, d \text{ and } n = 1, 2, \ldots, N-1,$$

which can be written in the following form

$$\sum_{l=j}^{m+d} a_l^{(n)} \frac{\Gamma(l+1)}{\Gamma(l+1-j)} t_n^{l-j} = \sum_{l=j}^{m+d} a_l^{(n-1)} \frac{\Gamma(l+1)}{\Gamma(l+1-j)} t_n^{l-j}, \tag{27}$$

we get $(N-1)(d+1)$ equations, and from the initial conditions we get $(d+1)$ equations. Once the coefficients $\{a_l^{(k)}\}_{k=0,\ldots,N-1}^{l=0,\ldots,m}$ are known, the value of the spline solution $u$ is determined on each $\sigma_n$ by equation (9). ■

Remark 2.3. The collocation equations (23) and (26) give us $mN$ linear equations, but the number of unknown coefficients of the spline function $u(t)$ is $(m+d+1)N$, so we need extra $(d+1)N$ equations to recover the coefficient $\{a_l^{(k)}\}_{k=0,\ldots,N-1}^{l=0,\ldots,m+d}$. Thus, the choice of $S_{m+d}^{(d)}(Z_N)$ where $d = \lceil \frac{v}{2} \rceil - 1$ as the approximation space is the natural one because by the conditions of smoothness (27) we get $(d+1)(N-1)$ equations and in the first subinterval, $d+1$ additional conditions are furnished by the initial conditions (8) and so we obtain $(m+d+1)N$ linear equations.

## 3. Numerical examples

In this section, we experiment with the splines collocation method for different examples. In each example, we choose the $m$ collocation parameters $c_i = i/(m+1)$, $i = 1, 2, \ldots, m$. The computations are performed using the software `Maple` with 30-digit floating-point arithmetic.

Example 3.1. Consider the following semidifferential equation of order 1

$$D_*^{0.5} y(t) + y(t) = \frac{8}{3\sqrt{\pi}} t^{1.5} - \frac{2}{\sqrt{\pi}} t^{0.5} + t^2 - t,$$

with the initial condition $y(0) = 0$. Then the exact solution is $y(t) = t^2 - t$ . We applied the collocation method using $m = 3$. The absolute error $|y(t) - u(t)|$ at $t = 1$ and different values of $N$ are listed in Table 1.

| $N$ | $\lvert y(t) - u(t) \rvert$ when $m = 3$ |
|---|---|
| 5 | $0.15054 \times 10^{-8}$ |
| 10 | $0.43837 \times 10^{-9}$ |
| 20 | $0.28192 \times 10^{-9}$ |

Table 1: Absolute errors for Example 3.1 using the Splines Collocation method with $m = 3$ at $t = 1$

Example 3.2. Consider the following semidifferential equation of order 4

$$y''(t) - D_*^{1.5} y(t) + \frac{6}{5} y'(t) + D_*^{0.5} y(t) + \frac{1}{5} y(t) = f(t),$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$, and choose $f(t)$ so that the exact solution is $y(t) = t^2 + t^{5/2}$. We applied the collocation method for different $m$. The absolute error $|y(t) - u(t)|$ at $t = 1$ and different values of $N$ are listed in Table 2.

| $N$ | $\lvert y(t) - u(t)\rvert$ when $m = 3$ | $\lvert y(t) - u(t)\rvert$ when $m = 5$ |
|---|---|---|
| 5 | $0.21143 \times 10^{-2}$ | $0.85869 \times 10^{-3}$ |
| 10 | $0.74188 \times 10^{-3}$ | $0.30280 \times 10^{-3}$ |
| 20 | $0.26077 \times 10^{-3}$ | $0.10661 \times 10^{-3}$ |

Table 2: Absolute errors for Example 3.2 using the Splines Collocation method with $m = 3$ and $m = 5$ at $t = 1$

Example 3.3. Consider the semidifferential equation of order 6

$$y'''(t) + D_*^{0.5} y(t) + 2y(t) = 10e^{2t} + \sqrt{2}e^{2t}\,\mathrm{erf}(\sqrt{2t}),$$

with the initial conditions $y(0) = 1$, $y'(0) = 2$ and $y''(0) = 4$, where erf is the error function defined by $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$ and the exact solution is $y(t) = e^{2t}$. We applied the collocation method for different $m$. The absolute error $|y(t) - u(t)|$ at $t = 1$ and different values of $N$ are listed in Table 3.

| $N$ | $\lvert y(t) - u(t)\rvert$ when $m = 3$ | $\lvert y(t) - u(t)\rvert$ when $m = 5$ |
|---|---|---|
| 5 | $0.11308 \times 10^{-3}$ | $0.98093 \times 10^{-7}$ |
| 10 | $0.70160 \times 10^{-5}$ | $0.14043 \times 10^{-8}$ |
| 20 | $0.43789 \times 10^{-6}$ | $0.80805 \times 10^{-10}$ |

Table 3: Absolute errors for Example 3.3 using the Splines Collocation method with $m = 3$ and $m = 5$ at $t = 1$

## 4. Conclusion

In this paper an algorithm for the approximate solution of semidifferential equation has been analyzed. The algorithm is based on the spline collocation method. The approximate solution is obtained by deriving a linear system. Three numerical examples are introduced showing that the method is convergent with a good accuracy. However, the convergence and the rate of convergence of the proposed method need further investigations. These issues must wait for another paper.

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