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CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY A FAMILY OF LINEAR OPERATORS

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Abstract. In this paper, we obtain some applications of first order differential subordination and superordination results involving Dziok-Srivastava operator and other linear operators for certain normalized analytic functions in the open unit disc.

1. Introduction

Let $H(U)$ be the class of analytic functions in the unit disk $U = \{z \in C : |z| < 1\}$ and let $H[a, k]$ be the subclass of $H(U)$ consisting of functions of the form

$$f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \cdots \quad (a \in C). \tag{1.1}$$

Also, let A be the subclass of $H(U)$ consisting of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

If f, $g \in H(U)$, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g. ,[3], [10]; see also [11]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $p, h \in H(U)$ and let $\varphi(r, s, t; z) : C^3 \times U \to C$. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then p is a solution of the differential superordination (1.3). Note that if f is subordinate to g, then g is superordinate to f. An analytic function q is called a

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subordinant if $q(z) \prec p(z)$ for all p satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants of (1.3) is called the best subordinant. Recently Miller and Mocanu [12] obtained conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z). \tag{1.4}$$

Using the results of Miller and Mocanu [12], Bulboacă considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [4]. Ali et al. [1], have used the results of Bulboacă to obtain sufficient conditions for normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

here q_1 and q_2 are given univalent functions in U. Also, Tuneski [17] obtained a sufficient condition for starlikeness of f in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [15] obtained sufficient conditions for the normalized analytic function f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

They also obtained results for functions defined by using Carlson-Shaffer operator.

For complex numbers $\alpha_1, \alpha_2, \dots, \alpha_q$ and $\beta_1, \beta_2, \dots, \beta_s$ ($\beta_j \notin Z_0^- = \{0, -1, -2, \dots\}$, $j = 1, 2, \dots, s$), we define the generalized hypergeometric function ${}_qF_s(\alpha_1, \dots, \alpha_q;\ \beta_1, \dots, \beta_s; z)$ by (see, for example, [16]) the following infinite series

$$\begin{gathered}{}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_q)_k}{(\beta_1)_k \dots (\beta_s)_k (1)_k} z^k \\ (q \le s+1;\ s, q \in N_0 = N \cup \{0\};\ z \in U),\end{gathered} \tag{1.5}$$

where

$$(d)_k = \begin{cases} 1, & (k = 0;\ d \in C\backslash\{0\}), \\ d(d+1)\dots(d+k-1), & (k \in N;\ d \in C). \end{cases}$$

Dziok and Srivastava [8] considered a linear operator $H_{q,s}(\alpha_1, \dots, \alpha_q;$ $\beta_1, \dots, \beta_s) : A \to A$, defined by the following Hadamard product

$$\begin{gathered}H_{q,s}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)f(z) = [z\ {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)] * f(z), \\ (q \le s+1;\ s, q \in N_0;\ z \in U).\end{gathered} \tag{1.6}$$

We observe that for a function f of the form (1.2), we have

$$H_{q,s}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)f(z) = z + \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1} \dots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \dots (\beta_s)_{k-1}(1)_{k-1}} a_k z^k. \tag{1.7}$$

If, for convenience, we write

$$H_{q,s}(\alpha_1) = H_{q,s}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s), \tag{1.8}$$

then one can easily verify from the definition (1.7) that

$$z(H_{q,s}(\alpha_1)f(z))' = \alpha_1 H_{q,s}(\alpha_1 + 1)f(z) - (\alpha_1 - 1)H_{q,s}(\alpha_1)f(z) \quad (f \in A). \tag{1.9}$$

It should be remarked that the linear operator $H_{q,s}(\alpha_1)f(z)$ is a generalization of many other linear operators considered earlier. In particular, for $f \in A$, we have:

(i) $H_{q,s}(a, 1; c)f(z) = L(a, c)f(z)$ $(a > 0, c > 0)$, where $L(a, c)$ is the Carlson-Shaffer operator (see [5]);

(ii) $H_{q,s}(\lambda + 1, c; a)f(z) = I^{\lambda}(a, c)f(z)$ $(a, c \in R\backslash Z_0^-; \lambda > -1)$, where $I^{\lambda}(a, c)$ is the Cho-Kwon-Srivastava operator (see [6]);

(iii) $H_{q,s}(\eta, 1; \lambda + 1)f(z) = I_{\lambda,\eta}f(z)$ $(\lambda > -1; \eta > 0)$, where $I_{\lambda,\eta}$ is the Choi-Saigo-Srivastava operator (see [7]);

(iv) $H_{q,s}(\eta + 1, 1; \eta + 2)f(z) = F_{\eta}(f)(z) = \frac{\eta+1}{z^{\eta}} \int_0^z t^{\eta-1} f(t)dt$ $(\eta > -1)$ where F_{η} is the Libera operator (see [9]);

(v) $H_{q,s}(\delta+1, 1; 1)f(z) = D^{\delta}f(z)$ $(\delta > -1)$, where $D^{\delta}f(z)$ is the δ-Ruscheweyh derivative of $f(z)$ (see [13]).

In this paper, we obtain sufficient conditions for the normalized analytic function f defined by using Dziok-Srivastava operator to satisfy

$$q_1(z) \prec \left(\frac{z}{H_{q,s}(\alpha_1)f(z)} \right)^{\mu} \prec q_2(z)$$

and q_1 and q_2 are given univalent functions in U.

2. Definitions and preliminaries

In order to prove our results, we shall make use of the following known results.

Definition 1. [12] Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \backslash E(f)$, where

$$E(f) = \{ \xi \in \partial U : \lim_{z \to \xi} f(z) = \infty \},$$

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial U \setminus E(f)$.

Lemma 1. [11] *Let q be univalent in the unit disk U and θ and φ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set*

$$\psi(z) = zq'(z)\varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + \psi(z). \tag{2.1}$$

Suppose that:

(i) $\psi(z)$ is starlike univalent in U,

(ii) $\operatorname{Re}\left\{\frac{zh'(z)}{\psi(z)}\right\} > 0$ *for* $z \in U$.

If p *is analytic with* $p(0) = q(0)$, $p(U) \subseteq D$ *and*

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.2}$$

then $p(z) \prec q(z)$ *and* q *is the best dominant.*

Lemma 2. [2] *Let* q *be convex univalent in* U *and* ϑ *and* ϕ *be analytic in a domain* D *containing* $q(U)$. *Suppose that:*

(i) $\operatorname{Re}\{\vartheta'(q(z))/\phi(q(z))\} > 0$ *for* $z \in U$,

(ii) $Q(z) = zq'(z)\phi(q(z))$ *is starlike univalent in* U.

If $p(z) \in H[q(0), 1] \cap Q$, *with* $p(U) \subseteq D$, *and* $\vartheta(p(z)) + zp'(z)\phi(p(z))$ *is univalent in* U *and*

$$\vartheta(q(z)) + zq'(z)\phi(q(z)) \prec \vartheta(p(z)) + zp'(z)\phi(p(z)), \tag{2.3}$$

then $q(z) \prec p(z)$ *and* q *is the best subordinant.*

3. Applications to Dziok-Srivastava operator and sandwich theorems

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $\gamma, \xi, \delta \in C$ and β, $\mu \in C^* = C \setminus \{0\}$.

Theorem 1. *Let* q *be analytic univalent in* U *with* $q(z) \neq 0$. *Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U. *Let* $\gamma, \xi, \delta \in C$; $\beta, \mu \in C^*$ *satisfy:*

$$\operatorname{Re}\left\{1 + \frac{\xi}{\beta}q(z) + \frac{2\delta}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0, \tag{3.1}$$

and

$$\Psi(\alpha_1, \gamma, \xi, \delta, \beta, \mu, f) =$$

$$\gamma + \xi\left(\frac{z}{H_{q,s}(\alpha_1)f(z)}\right)^{\mu} + \delta\left(\frac{z}{H_{q,s}(\alpha_1)f(z)}\right)^{2\mu} + \beta\mu\alpha_1\left[1 - \frac{H_{q,s}(\alpha_1+1)f(z)}{H_{q,s}(\alpha_1)f(z)}\right]. \tag{3.2}$$

If q *satisfies the following subordination:*

$$\Psi(\alpha_1, \gamma, \xi, \delta, \beta, \mu, f) \prec \gamma + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)}, \tag{3.3}$$

then

$$\left(\frac{z}{H_{q,s}(\alpha_1)f(z)}\right)^{\mu} \prec q(z) \quad (\mu \in C^*) \tag{3.4}$$

and q *is the best dominant.*

Proof. Define a function p by

$$p(z) = \left(\frac{z}{H_{q,s}(\alpha_1)f(z)}\right)^{\mu} \quad (z \in U; \mu \in C^*). \tag{3.5}$$

Then the function p is analytic in U and $p(0) = 1$. Therefore, differentiating (3.5) logarithmically with respect to z and using the identity (1.9) in the resulting equation, we have

$$\gamma + \xi(\frac{z}{H_{q,s}(\alpha_1)f(z)})^{\mu} + \delta(\frac{z}{H_{q,s}(\alpha_1)f(z)})^{2\mu} + \beta\mu\alpha_1[1 - \frac{H_{q,s}(\alpha_1+1)f(z)}{H_{q,s}(\alpha_1)f(z)}] = \gamma + \xi p(z) + \delta(p(z))^2 + \beta\frac{zp'(z)}{p(z)}. \quad (3.6)$$

Using (3.6) and (3.3), we have

$$\gamma + \xi p(z) + \delta(p(z))^2 + \beta\frac{zp'(z)}{p(z)} \prec \gamma + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)}. \quad (3.7)$$

Setting

$$\theta(w) = \gamma + \xi w + \delta w^2 \quad \textit{and} \quad \varphi(w) = \frac{\beta}{w},$$

it can be easily observed that θ is analytic in C, φ is analytic in C^* and $\varphi(w) \neq 0$ $(w \in C^*)$. Hence, the result now follows by using Lemma 1. ■

Putting $q(z) = (1+Az)/(1+Bz)$ $(-1 \leq B < A \leq 1)$ in Theorem 1, we have the following corollary.

Corollary 1. *Let* $-1 \leq B < A \leq 1$ *and*

$$\mathrm{Re}\left\{1 + \frac{\xi}{\beta}\left(\frac{1+Az}{1+Bz}\right) + \frac{2\delta}{\beta}\left(\frac{1+Az}{1+Bz}\right)^2 - \frac{(A+B+3AB)z}{(1+Az)(1+Bz)}\right\} > 0$$

holds. If $f(z) \in A$, *and*

$$\gamma + \xi(\frac{z}{H_{q,s}(\alpha_1)f(z)})^{\mu} + \delta(\frac{z}{H_{q,s}(\alpha_1)f(z)})^{2\mu} + \beta\mu\alpha_1[1 - \frac{H_{q,s}(\alpha_1+1)f(z)}{H_{q,s}(\alpha_1)f(z)}] \prec \gamma + \xi\frac{1+Az}{1+Bz} + \delta(\frac{1+Az}{1+Bz})^2 + \beta\frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\left(\frac{z}{H_{q,s}(\alpha_1)f(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz} \quad (\mu \in C^*)$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Putting $q(z) = \left(\frac{1+z}{1-z}\right)^{\nu}$ $(0 < \nu \leq 1)$ in Theorem 1, we obtain the following corollary.

Corollary 2. *Assume that* (3.1) *holds. If* $f \in A$, *and*

$$\gamma + \xi(\frac{z}{H_{q,s}(\alpha_1)f(z)})^{\mu} + \delta(\frac{z}{H_{q,s}(\alpha_1)f(z)})^{2\mu} + \beta\mu\alpha_1[1 - \frac{H_{q,s}(\alpha_1+1)f(z)}{H_{q,s}(\alpha_1)f(z)}] \prec \gamma + \xi\left(\frac{1+z}{1-z}\right)^{\nu} + \delta\left(\frac{1+z}{1-z}\right)^{2\nu} + \beta\frac{2\nu z}{(1-z)^2},$$

then

$$\left(\frac{z}{H_{q,s}(\alpha_1)f(z)}\right)^{\mu} \prec \left(\frac{1+z}{1-z}\right)^{\nu} \quad (\mu \in C^*;\ \ 0 < \nu \leq 1)$$

and $\left(\frac{1+z}{1-z}\right)^{\nu}$ *is the best dominant.*

Taking $\alpha_1 = a > 0$, $\beta_1 = c > 0$, $\alpha_j = 1$ $(j = 2, \ldots, s+1)$ and $\beta_j = 1$ $(j = 2, \ldots, s)$, in Theorem 1, we have the following corollary which improves the result obtained by Shanmugam et al. [14, Theorem 3.1].

Corollary 3. *Let q be analytic univalent in U with $q(z) \neq 0$ and condition* (3.1) *holds. Suppose also that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U and*

$$\zeta(\gamma,\xi,\delta,\beta,\mu) = \gamma + \xi(\frac{z}{L(a,c)f(z)})^{\mu} + \delta(\frac{z}{L(a,c)f(z)})^{2\mu} + \beta\mu a[1 - \frac{L(a+1,c)f(z)}{L(a,c)f(z)}]. \tag{3.8}$$

If q satisfies the following subordination:

$$\zeta(\gamma,\xi,\delta,\beta,\mu) \prec \gamma + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)},$$

then

$$(\frac{z}{L(a,c)f(z)})^{\mu} \prec q(z) \quad (\mu \in C^*)$$

and q is the best dominant.

Taking $\alpha_1 = \lambda + 1$, $\alpha_2 = c$, $\beta_1 = a$ $(a, c \in R \setminus Z_0^-;\ \lambda > -1)$, $\alpha_j = 1$ $(j = 3, \ldots, s+1)$ and $\beta_j = 1 (j = 2, \ldots, s)$, in Theorem 1, we have

Corollary 4. *Let q be analytic univalent in U with $q(z) \neq 0$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Further, assume that* (3.1) *holds. If $f \in A$, and*

$$\gamma + \xi(\frac{z}{I^{\lambda}(a,c)f(z)})^{\mu} + \delta(\frac{z}{I^{\lambda}(a,c)f(z)})^{2\mu} + \beta\mu(\lambda+1)[1 - \frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)f(z)}] \prec \gamma + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{z}{I^{\lambda}(a,c)f(z)}\right)^{\mu} \prec q(z) \quad (\mu \in C^*)$$

and q is the best dominant.

Taking $\alpha_1 = \eta, \beta_1 = \lambda + 1$ $(\lambda > -1; \eta > 0)$, $\alpha_j = 1$ $(j = 2, \ldots, s+1)$ and $\beta_j = 1 (j = 2, \ldots, s)$ in Theorem 1, we have

Corollary 5. *Let q be analytic univalent in U with $q(z) \neq 0$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Further, assume that* (3.1) *holds. If $f \in A$, and*

$$\gamma + \xi(\frac{z}{I_{\lambda,\eta}f(z)})^{\mu} + \delta(\frac{z}{I_{\lambda,\eta}f(z)})^{2\mu} + \beta\mu\eta[1 - \frac{I_{\lambda,\eta+1}f(z)}{I_{\lambda,\eta}f(z)}] \prec \gamma + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{z}{I_{\lambda,\eta}f(z)}\right)^{\mu} \prec q(z) \quad (\mu \in C^*)$$

and q is the best dominant.

Taking $\alpha_1 = \eta + 1, \beta_1 = \eta + 2 \quad (\eta > -1),\ \alpha_j = 1\ (j = 2, \ldots, s+1)$ and $\beta_j = 1 (j = 2, \ldots, s)$ in Theorem 1, we have

Corollary 6. *Let q be analytic univalent in U with $q(z) \neq 0$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Further, assume that* (3.1) *holds. If $f \in A$, and*

$$\gamma + \xi(\frac{z}{F_{\eta}f(z)})^{\mu} + \delta(\frac{z}{F_{\eta}f(z)})^{2\mu} + \beta\mu(1+\eta)[1 - \frac{f(z)}{F_{\eta}f(z)}] \prec \gamma + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)},$$

then

$$(\frac{z}{F_{\eta}f(z)})^{\mu} \prec q(z) \quad (\mu \in C^*)$$

and q is the best dominant.

Now, by appealing to Lemma 2 it can be easily prove the following theorem.

Theorem 2. *Let q be convex univalent in U, $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Assume that*

$$\operatorname{Re}\left\{\frac{2\delta}{\beta}(q(z))^2 + \frac{\xi}{\beta}q(z)\right\} > 0 \quad (z \in U). \tag{3.9}$$

If $f \in A$, $0 \neq \left(\frac{z}{H_{q,s}(\alpha_1)f(z)}\right)^{\mu} \in H[q(0), 1] \cap Q$, $\Psi(\alpha_1, \gamma, \xi, \delta, \beta, \mu, f)$ is univalent in U, and

$$\gamma + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)} \prec \Psi(\alpha_1, \gamma, \xi, \delta, \beta, \mu, f),$$

where $\Psi(\alpha_1, \gamma, \xi, \delta, \beta, \mu, f)$ is given by (3.2), *then*

$$q(z) \prec \left(\frac{z}{H_{q,s}(\alpha_1)f(z)}\right)^{\mu} \quad (\mu \in C^*) \tag{3.10}$$

and q is the best subordinant.

Proof. Taking

$$\vartheta(w) = \gamma + \xi w + \delta w^2 \ \ and \ \ \varphi(w) = \frac{\beta}{w},$$

it is easily observed that ϑ is analytic in C, φ is analytic in C^* and $\varphi(w) \neq 0$ ($w \in C^*$). Since q is convex (univalent) function it follows that

$$\mathrm{Re}\left\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\right\} = \mathrm{Re}\left\{\frac{2\delta}{\beta}(q(z))^2 + \frac{\xi}{\beta}q(z)\right\} q'(z) > 0 \quad (z \in U).$$

Thus the assertion (3.10) of Theorem 2 follows by an application of Lemma 2. ■

Taking $\alpha_1 = a > 0$, $\beta_1 = c > 0$, $\alpha_j = 1$ ($j = 2, \ldots, s+1$) and $\beta_j = 1$ ($j = 2, \ldots, s$) in Theorem 2, we have the following corollary which improves the result of Shanmugam et. al. [14, Theorem 3.6].

Corollary 7. *Let q be convex univalent in U, $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Assume that* (3.9) *holds. If $f \in A$, $0 \neq (\frac{z}{L(a,c)f(z)})^{\mu} \in H[q(0),1] \cap Q$, $\zeta(\gamma,\xi,\delta,\beta,\mu)$ is univalent in U and*

$$\gamma + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)} \prec \zeta(\gamma,\xi,\delta,\beta,\mu),$$

where $\zeta(\gamma,\xi,\delta,\beta,\mu)$ is given by (3.8)*, then*

$$q(z) \prec \left(\frac{z}{L(a,c)f(z)}\right)^{\mu} \ (\mu \in C^*)$$

and q is the best subordinant.

Taking $\alpha_1 = \lambda + 1$, $\alpha_2 = c$, $\beta_1 = a$ ($a, c \in R \setminus Z_0^-$; $\lambda > -1$), $\alpha_j = 1$ ($j = 3, \ldots, s+1$) and $\beta_j = 1$ ($j = 2, \ldots, s$), in Theorem 2, we have

Corollary 8. *Let q be convex univalent in U, $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Assume that* (3.9) *holds. If $f \in A$, $0 \neq (\frac{z}{I^{\lambda}(a,c)f(z)})^{\mu} \in H[q(0),1] \cap Q$,*

$$\gamma + \xi(\frac{z}{I^{\lambda}(a,c)f(z)})^{\mu} + \delta(\frac{z}{I^{\lambda}(a,c)f(z)})^{2\mu} + \beta\mu(\lambda+1)[1 - \frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)f(z)}]$$

is univalent in U, and

$$\gamma + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)}$$
$$\prec \gamma + \xi(\frac{z}{I^{\lambda}(a,c)f(z)})^{\mu} + \delta(\frac{z}{I^{\lambda}(a,c)f(z)})^{2\mu} + \beta\mu(\lambda+1)[1 - \frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)f(z)}]$$

then

$$q(z) \prec \left(\frac{z}{I^{\lambda}(a,c)f(z)}\right)^{\mu} \quad (\mu \in C^*)$$

and q is the best subordinant.

Letting $\alpha_1 = \eta$, $\beta_1 = \lambda + 1$ $(\lambda > -1;\ \eta > 0)$, $\alpha_j = 1$ $(j = 2, \ldots, s+1)$ and $\beta_j = 1$ $(j = 2, \ldots, s)$, in Theorem 2, we have

Corollary 9. *Let q be convex univalent in U, $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Assume that (3.9) holds. If $f \in A$, $0 \neq (\frac{z}{I_{\lambda,\eta}f(z)})^{\mu} \in H[q(0),1] \cap Q$,*

$$\gamma + \xi(\frac{z}{I_{\lambda,\eta}f(z)})^{\mu} + \delta(\frac{z}{I_{\lambda,\eta}f(z)})^{2\mu} + \beta\mu\eta[1 - \frac{I_{\lambda,\eta+1}f(z)}{I_{\lambda,\eta}f(z)}]$$

is univalent in U, and

$$\gamma + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)} \prec \gamma + \xi(\frac{z}{I_{\lambda,\eta}f(z)})^{\mu} + \delta(\frac{z}{I_{\lambda,\eta}f(z)})^{2\mu} + \beta\mu\eta[1 - \frac{I_{\lambda,\eta+1}f(z)}{I_{\lambda,\eta}f(z)}]$$

then

$$q(z) \prec \left(\frac{z}{I_{\lambda,\eta}f(z)}\right)^{\mu} \quad (\mu \in C^*)$$

and q is the best subordinant.

Taking $\alpha_1 = \eta + 1$, $\beta_1 = \eta + 2$ $(\eta > -1)$, $\alpha_j = 1$ $(j = 2, \ldots, s+1)$ and $\beta_j = 1$ $(j = 2, \ldots, s)$, in Theorem 2, we have

Corollary 10. *Let q be convex univalent in U, $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Assume that (3.9) holds. If $f \in A$, $0 \neq \left(\frac{z}{F_{\mu}f(z)}\right)^{\mu} \in H[q(0),1] \cap Q$,*

$$\gamma + \xi(\frac{z}{F_{\eta}f(z)})^{\mu} + \delta(\frac{z}{F_{\eta}f(z)})^{2\mu} + \beta\mu(1+\eta)[1 - \frac{f(z)}{F_{\eta}f(z)}]$$

is univalent in U, and

$$\gamma+\xi q(z)+\delta(q(z))^2+\beta\frac{zq'(z)}{q(z)} \prec \gamma+\xi(\frac{z}{F_{\eta}f(z)})^{\mu}+\delta(\frac{z}{F_{\eta}f(z)})^{2\mu}+\beta\mu(1+\eta)[1-\frac{f(z)}{F_{\eta}f(z)}]$$

then

$$q(z) \prec \left(\frac{z}{F_{\eta}f(z)}\right)^{\mu} \quad (\mu \in C^*)$$

and q is the best dominant.

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem.

THEOREM 3. *Let q_1 be convex univalent in U and q_2 be univalent in $U, q_1(z) \neq 0$ and $q_2(z) \neq 0$ in U. Suppose that q_2 and q_1 satisfy* (3.1) *and* (3.9)*, respectively. If $f \in A$, $0 \neq \left(\frac{z}{H_{q,s}(\alpha_1)f(z)}\right)^{\mu} \in H[q(0),1] \cap Q$ and*

$$\gamma + \xi(\frac{z}{H_{q,s}(\alpha_1)f(z)})^{\mu} + \delta(\frac{z}{H_{q,s}(\alpha_1)f(z)})^{2\mu} + \beta\mu\alpha_1[1 - \frac{H_{q,s}(\alpha_1+1)f(z)}{H_{q,s}(\alpha_1)f(z)}]$$

is univalent in U. Then

$$\gamma + \xi q_1(z) + \delta(q_1(z))^2 + \beta\frac{zq_1'(z)}{q_1(z)}$$
$$\prec \gamma + \xi(\frac{z}{H_{q,s}(\alpha_1)f(z)})^{\mu} + \delta(\frac{z}{H_{q,s}(\alpha_1)f(z)})^{2\mu} + \beta\mu\alpha_1[1 - \frac{H_{q,s}(\alpha_1+1)f(z)}{H_{q,s}(\alpha_1)f(z)}]$$
$$\prec \gamma + \xi q_2(z) + \delta(q_2(z))^2 + \beta\frac{zq_2'(z)}{q_2(z)}$$

implies

$$q_1(z) \prec \left(\frac{z}{H_{q,s}(\alpha_1)f(z)}\right)^{\mu} \prec q_2(z) \quad (\mu \in C^*)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Taking $\alpha_1 = a > 0$, $\beta_1 = c > 0$, $\alpha_j = 1$ $(j = 2, \ldots, s+1)$ and $\beta_j = 1$ $(j = 2, \ldots, s)$ in Theorem 3, we have the following corollary which improves the result of Shanmugam et al. [14, Theorem 3.7].

COROLLARY 11. *Let q_1 be convex univalent in U and q_2 be univalent in $U, q_1(z) \neq 0$ and $q_2(z) \neq 0$ in U. Suppose that q_2 and q_1 satisfy* (3.1) *and* (3.9)*, respectively. If $f \in A$, $0 \neq \left(\frac{z}{L(a,c)f(z)}\right)^{\mu} \in H[q(0),1] \cap Q$ and*

$$\gamma + \xi(\frac{z}{L(a,c)f(z)})^{\mu} + \delta(\frac{z}{L(a,c)f(z)})^{2\mu} + \beta\mu a[1 - \frac{L(a+1,c)f(z)}{L(a,c)f(z)}]$$

is univalent in U. Then

$$\gamma + \xi q_1(z) + \delta(q_1(z))^2 + \beta\frac{zq_1'(z)}{q_1(z)}$$
$$\prec \gamma + \xi(\frac{z}{L(a,c)f(z)})^{\mu} + \delta(\frac{z}{L(a,c)f(z)})^{2\mu} + \beta\mu a[1 - \frac{L(a+1,c)f(z)}{L(a,c)f(z)}]$$
$$\prec \gamma + \xi q_2(z) + \delta(q_2(z))^2 + \beta\frac{zq_2'(z)}{q_2(z)}$$

implies

$$q_1(z) \prec \left(\frac{z}{L(a,c)f(z)}\right)^{\mu} \prec q_2(z) \ (\mu \in C^*)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

REMARKS. Combining: (i) Corollary 4 and Corollary 8; (ii) Corollary 5 and Corollary 9; (iii) Corollary 6 and Corollary 10, we obtain similar sandwich theorems for the corresponding operators.

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