МАТЕМАТИЧКИ ВЕСНИК
62, 1 (2010), 47–50
March 2010

оригинални научни рад
research paper

ON VARIATION TOPOLOGY

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Abstract. Let I be a real interval and X be a Banach space. It is observed that spaces $\Lambda BV^{(p)}([a,b],R)$, $LBV(I,X)$ (locally bounded variation), $BV_0(I,X)$ and $LBV_0(I,X)$ share many properties of the space $BV([a,b],R)$. Here we have proved that the space $\Lambda BV_0^{(p)}(I,X)$ is a Banach space with respect to the variation norm and the variation topology makes $L\Lambda BV_0^{(p)}(I,X)$ a complete metrizable locally convex vector space (i.e. a Fréchet space).

Introduction. Looking to the features of the Jordan class the space $BV([a,b],R)$ of real functions of bounded variation over $[a,b]$ is generalized in many ways and many generalized spaces are obtained [1–5]. Many mathematicians have studied different properties for these generalized classes. Recently we have proved that the class $\Lambda BV^{(p)}([a,b],R)$ is a Banach space [5]. Also, the concept of bounded variation is extended, from real valued, to function with values in R^n. Many properties of such functions hold for functions in an arbitrary Banach space X. In the present paper we have studied properties of the classes $\Lambda BV_0^{(p)}(I,X)$ and $L\Lambda BV_0^{(p)}(I,X)$.

Definition. Given a real interval I (neither empty nor reduced to a singleton), a Banach space X, a non-decreasing sequence of positive real numbers $\Lambda = \{\lambda_n\}$ $(n = 1,2,\dots)$ such that $\sum_n \frac{1}{\lambda_n}$ diverges, $1 \le p < \infty$ and a function $f : I \to X$, we say that $f \in \Lambda BV^{(p)}(I,X)$ (that is f is a function of $p-\Lambda$-bounded variation over I) if

$$V_{\Lambda_p}(f,I) = \sup_S V_{\Lambda_p}(f,S,I) < \infty,$$

where $V_{\Lambda_p}(f,S,I) = \left(\sum_{i=1}^n \frac{\|f(u_i)-f(u_{i-1})\|_X^p}{\lambda_i}\right)^{1/p}$, S: $u_0 < u_1 < \cdots < u_n$ is a finite ordered set of points of I and $\|\cdot\|_X$ denotes the Banach norm in X.

Note that, if $p = 1$, one gets the class $\Lambda BV(I,X)$ and the variation V_{Λ_p} is replaced by V_Λ; if $\lambda_m \equiv 1$ for all m, one gets the class $BV^{(p)}(I,X)$ and the

AMS Subject Classification: 26A45, 46A04.

Keywords and phrases: $\Lambda BV^{(p)}$; Banach space; complete metrizable locally convex vector space; Fréchet space.

variation V_{Λ_p} is replaced by V_p; if $p = 1$ and $\lambda_m \equiv 1$ for all m, one gets the class $BV(I,X)$ and the variation V_{Λ_p} is replaced by V.

$f \in L\Lambda BV^{(p)}(I,X)$ means that f is a function of I to X with locally p-Λ bounded variation, i.e. it has finite p-Λ bounded variation on every compact subinterval of I. It is observed that the class $LBV(I,X)$ is a vector space and for any $f \in LBV(I,X)$, for any compact subinterval $[a,b]$ of I, the mapping $f \mapsto V(f,[a,b])$ is a semi-norm in this space. If $[a,b]$ ranges through the totality of the compact subintervals of I, or equivalently through some increasing sequence of such subintervals with union equal to I, the collection of the corresponding semi-norms defines on $LBV(I,X)$ a (non Hausdorff) locally convex topology which is called the variation topology.

We shall choose once for all a reference point t in I and consider the space $L\Lambda BV_0^{(p)}(I,X)$ consisting of all those functions in $L\Lambda BV^{(p)}(I,X)$ which are vanishing at the point t. Moreau [1] proved that the space $BV_0(I,X)$ is a Banach space in the norm $\|f\|_{var} = V(f,I)$ and the variation topology makes $LBV_0(I,X)$ a Fréchet space. Here we have extended these two results for $\Lambda BV^{(p)}$.

In the first stage, let us consider class of functions whose total p-Λ-variation is finite.

Theorem 1. *The vector space $\Lambda BV_0^{(p)}(I,X)$ is a Banach space in the norm $\|f\|_{var} = V_{\Lambda_p}(f,I)$.*

Note that the above mentioned $\|.\|_{var}$ is a semi-norm on the space $\Lambda BV^{(p)}(I,X)$. For $\lambda_n = 1$ for all n and $p = 1$ Theorem 1 gives Moreau's result [1, Proposition 2.1] as a particular case.

We need the following lemma to prove the theorem.

Lemma. *If $f \in \Lambda BV_0^{(p)}(I,X)$ then f is bounded.*

Proof. For any $u \in I$, observe that

$$\|f(u)\|_X = \lambda_1 \left(\frac{\|f(u) - f(t)\|_X}{\lambda_1} \right) \le (\lambda_1)^{(1/p)} V_{\Lambda_p}(f,I).$$

Hence

$$\|f\|_\infty = \sup_{u \in I} \|f(u)\|_X \le (\lambda_1)^{(1/p)}\ V_{\Lambda_p}(f,I).$$

Similarly, for any $f \in L\Lambda BV_0^{(p)}(I,X)$ and for any $[a,b] \subset I$ containing the point t, we get

$$\sup_{u \in [a,b]} \|f(u)\|_X \le (\lambda_1)^{(1/p)} V_{\Lambda_p}(f,[a,b]).$$

Therefore, in the space $L\Lambda BV_0^{(p)}(I,X)$ the variation topology is stronger than the topology of uniform convergence on compact subsets of I. ■

Proof of Theorem 1. Consider a Cauchy sequence $\{f_n\}$ in the given normed linear space. Then there exists a constant $C > 0$ such that

$$\|f_n\| \ \le\ C, \quad \forall n \in N. \tag{1.1}$$

In view of the Lemma, $\{f_n\}$ is also Cauchy sequence in the sup norm $\|.\|_\infty$ so it converges in the latter norm to some function $f_\infty : I \to X$, with $f_\infty(t) = 0$.

For every finite ordered set of points of I, say S: $u_0 < u_1 < \cdots < u_m$, and every $f : I \to X$, let us denote

$$V_{\Lambda_p}(f, S) = \left(\sum_{i=1}^{m} \frac{(\|f(u_i) - f(u_{i-1})\|_X)^p}{\lambda_i} \right)^{1/p}.$$

Since, at every point u_i of S, the element $f_\infty(u_i)$ of X equals the limit of $f_n(u_i)$ in the $\|.\|_X$ norm, one has

$$V_{\Lambda_p}(f_\infty, S) = \lim_{n\to\infty} \left(\sum_{i=1}^{m} \frac{(\|f(u_i) - f(u_{i-1})\|_X)^p}{\lambda_i} \right)^{1/p}.$$

Due to (1.1), this is majorized by C whatever is S, hence $f_\infty \in \Lambda BV_0^{(p)}(I, X)$.

Now, let us prove that f_n converges to f_∞ in the norm $\|.\|_{var}$. In view of Cauchy property, for any $\epsilon > 0$ there exists $n \in N$ such that

$$l \geq n \;\Rightarrow\; \|f_j - f_n\|_{var} \leq \epsilon.$$

Hence, for every $l \geq n$,

$$V_{\Lambda_p}(f_l - f_n, S) \leq V_{\Lambda_p}(f_l - f_n, I) \leq \epsilon.$$

Thus

$$V_{\Lambda_p}(f_\infty - f_n, S) \leq V_{\Lambda_p}(f_\infty - f_l, S) + V_{\Lambda_p}(f_l - f_n, S) \leq \epsilon + V_{\Lambda_p}(f_l - f_\infty, S).$$

By letting l tending to $+\infty$, one concludes that $V(f_n - f_\infty, S) \leq \epsilon$ for every finite sequence S, hence $\|f_n - f_\infty\|_{var} \leq \epsilon$ for every finite sequence S. Hence the result follows. ∎

Let us drop the assumption of finite total variation on I. The variation topology on $L\Lambda BV_0^{(p)}(I, X)$ is defined by the collection of norms $f \mapsto N_k(f) = \|f\|_{var,K_k} = V_{\Lambda_p}(f, K_k)$, where $\{K_k\}$ denotes a nondecreasing sequence of compact subintervals whose union equals I. Additionally assume that all intervals K_k are large enough to contain t. Therefore the resulting topology is metrizable and Hausdorff.

Theorem 2. *The variation topology makes $L\Lambda BV_0^{(p)}(I, X)$ a complete metrizable locally convex vector space (i.e. Fréchet space).*

Note that for $\lambda_n = 1$ for all n and $p = 1$ Theorem 2 gives Moreau's result [1, Proposition 2.2] as a particular case.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $L\Lambda BV_0^{(p)}(I, X)$. By definition, for every neighborhood U of the origin in this space there exists $n \in N$ such that

$$l \geq n \text{ and } q \geq n \Rightarrow f_l - f_q \in U.$$

For $k \in N$ and for any $\epsilon > 0$ define the semi-ball,

$$U_{k,\epsilon} = \{\, u \in L\Lambda BV_0^{(p)}(I, X) : N_k(u) < \epsilon \,\}.$$

Thus

$$l \geq n \text{ and } q \geq n \;\Rightarrow\; N_k(f_l - f_q) < \epsilon.$$

Therefore the restriction of the functions $\{f_n\}$ to K_k make a Cauchy sequence in $\Lambda BV_0^{(p)}(K_k, X)$. In view of Theorem 1, this sequence converges to some element f^k in the latter space. If the same construction is effected for another compact subinterval $K_{k'}$, with $k' > k$, the resulting function $f^{k'} : K_{k'} \to X$ is an extension of f^k. Inductively, a function f is constructed on the whole I, which constitutes the limit of the sequence $\{f_n\}$ in the variation topology. Hence the result follows. ■

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(received 05.11.2008, in revised form 14.06.2009)

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