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EXTREMAL NON-COMPACTNESS OF WEIGHTED COMPOSITION OPERATORS ON THE DISK ALGEBRA

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Abstract. Let $A(D)$ denote the disk algebra and $W_{\psi,\phi}$ be weighted composition operator on $A(D)$. In this paper we obtain a condition on ψ and ϕ for $W_{\psi,\phi}$ to exhibit extremal non-compactness. As a consequence we show that the essential norm of a composition operator on $A(D)$ is either 0 or 1.

1. Introduction

Throughout this paper $\mathbb{D}$ denotes the unit disk in the complex plane $\mathbb{C}$ and $\overline{\mathbb{D}}$ denotes its closure in $\mathbb{C}$. Let $A(\mathbb{D})$ be the Banach algebra of all continuous functions on $\overline{\mathbb{D}}$ which are analytic in $\mathbb{D}$, under the supremum norm $\|f\| = \sup\{\, |f(z)| : z \in \overline{\mathbb{D}}\,\}$. If $\phi \in A(\mathbb{D})$ and $\|\phi\| \leq 1$ then ϕ induces a linear operator given by the following equation

$$(C_\phi f)(z) = (f \circ \phi)(z) \quad \forall z \in \overline{\mathbb{D}}.$$

This operator is called the composition operator induced by ϕ (see [8]).

Let X be a Banach space and T be a bounded linear operator on X. The essential norm of T is the distance from T to the compact operators on X,

$$\|T\|_e = \inf\{\, \|T - K\| : K \text{ is a compact operator on } X\,\}.$$

Clearly T is compact if and only if its essential norm is zero. Since the zero operator is compact, $\|T\|_e \leq \|T\|$. The operators having norm equal to the essential norm are said to exhibit extremal non-compactness (see [1]).

In this paper we obtain conditions on ψ and ϕ for $W_{\psi,\phi}$ to exhibit extremal non-compactness on the disk algebra. As a corollary of our result we show that the essential norm of a composition operator on $A(\mathbb{D})$ is either 0 or 1. The relevant definitions are as follows.

Definition 1.1. If ψ, $\phi \in A(\mathbb{D})$ and $\|\phi\| \leq 1$, then the weighted composition operator $W_{\psi,\phi} : A(\mathbb{D}) \to A(\mathbb{D})$ is defined as

$$(W_{\psi,\phi} f)(z) = \psi(z) f(\phi(z)) \quad \forall z \in \overline{\mathbb{D}}.$$

It is easy to see that $W_{\psi,\phi}$ is a bounded linear operator and $\|W_{\psi,\phi}\| = \|\psi\|$. If $\psi \equiv 1$, then $W_{\psi,\phi}$ is equal to the composition operator C_ϕ on $A(\mathbb{D})$.

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Definition 1.2. [3, p. 75] An inner function is a function M $\in H^\infty$ for which $|M^*| = 1$ almost everywhere with respect to the Lebesgue measure on $\mathbb{T}$, where $M^*(e^{i\theta}) = \lim_{r\to 1-} M(re^{i\theta})$ and $\mathbb{T}$ is the unit circle.

Remark 1.1. It is not difficult to show that every non-constant inner function ϕ in $A(\mathbb{D})$ is finite Blaschke product [3, p. 196].

Definition 1.3. A bounded linear operator T from a Banach space X to a Banach space Y is compact if given any bounded $\{x_n\}$ in X, there exists a subsequence $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ converges in Y.

In 1979, Herbert Kamowitz characterized the compactness of weighted composition operator on $A(\mathbb{D})$ for those ϕ which are non-constant. We state his result as follows.

Theorem 1.1. [4] *Let ψ, $\phi \in A(\mathbb{D})$ and $\|\phi\| \leq 1$ and suppose ϕ is a non-constant function. Then $W_{\psi,\phi}$ is a compact operator on $A(\mathbb{D})$ if and only if $|\phi(z)| < 1$ whenever $\psi(z) \neq 0$.*

Shapiro [7] has obtained similar condition on ϕ for C_ϕ to be compact on a class of Banach spaces X consisting of functions which are analytic on $\mathbb{D}$ and have continuous extension on the boundary. He showed that under some natural hypothesis on X if ϕ induces a compact composition operator on X, then $\phi(\mathbb{D})$ must be relatively compact subset of $\mathbb{D}$.

The essential norm of a composition operator on the Hardy space H^2, the Hilbert space consisting of all analytic functions f on $\mathbb{D}$ such that $\lim_{r\to 1-} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta < \infty$, was given by J. H. Shapiro in terms of Nevanlinna counting function [6]. He also proved that if ϕ is inner then $\|C_\phi\|_e = \|C_\phi\|$ (see [9]). In 1997, Cima and Matheson [2] calculated the essential norm of composition operators on H^2 in terms of Aleksandrov-Clark measures of the inducing holomorphic map. They also gave an application which relates essential norm of C_ϕ to the angular derivative of ϕ. In 2002, L. Zheng [10] showed that the essential norm of composition operator acting on H^∞, the space of bounded analytic functions on $\mathbb{D}$, is either 0 or 1. In 2007, Kriete and Moorhouse obtained a result ([5, Theorem 3.1]) which is stated in terms of Aleksandrov-Clark measures, and which gives an upper and lower bound for the essential norm of a weighted composition operator on H^2, under the assumption that multiplicative symbol belongs to H^∞.

2. Extremal non-compactness

We start this section with the following theorem which gives a condition on ψ and ϕ under which $W_{\psi,\phi}$ becomes extremally non-compact.

Theorem 2.1. *Let $\psi \in A(\mathbb{D})$ and $\phi \in A(\mathbb{D})$ be a non-constant function with $\|\phi\| \leq 1$. If there exists a point $z_0 \in \overline{\mathbb{D}}$ such that $|\phi(z_0)| = 1$ and $\|\psi\| = |\psi(z_0)|$, then $\|W_{\psi,\phi}\|_e = \|W_{\psi,\phi}\| = \|\psi\|$.*

Proof. Since $\|W_{\psi,\phi}\| = \|\psi\|$ and $\|W_{\psi,\phi}\|_e \leq \|W_{\psi,\phi}\|$, we get $\|W_{\psi,\phi}\|_e \leq \|\psi\|$. Now we show that under the given condition on ψ and ϕ $\|W_{\psi,\phi}\|_e \geq \|\psi\|$.

Let $\{r_n\}$ be a sequence of non-negative real numbers converging to 1 and

$$\psi_n(z) = \frac{z - r_n}{1 - r_n z}.$$

Then $\|\psi_n\| = 1$, ψ_n fixes 1 and -1 for all $n \in \mathbb{N}$ and $\psi_n(z) \to -1$ for all $z \in \mathbb{D}$.

Suppose that K is a compact operator on $A(\mathbb{D})$. We want to show that $\|W_{\psi,\phi} - K\| \geq \|\psi\|$. Since K is compact and $\|\psi_n\| = 1$, there is a subsequence $\{\psi_{n_j}\}_{j=1}^{\infty}$ of $\{\psi_n\}$ and $f \in A(\mathbb{D})$ such that $\lim_{j\to\infty} \|K\psi_{n_j} - f\| = 0$. To show $\|W_{\psi,\phi} - K\| \geq \|\psi\|$, it is enough to prove that $\limsup_{j\to\infty} \|(W_{\psi,\phi} - K)\psi_{n_j}\| \geq \|\psi\|$. But $\|(W_{\psi,\phi} - K)\psi_{n_j}\| \geq \|W_{\psi,\phi}\psi_{n_j} - f\| - \|K\psi_{n_j} - f\|$, hence

$$\limsup_{j\to\infty} \|(W_{\psi,\phi} - K)\psi_{n_j}\| \geq \limsup_{j\to\infty} \|W_{\psi,\phi}\psi_{n_j} - f\|.$$

It suffices to prove that $\limsup_{j\to\infty} \|W_{\psi,\phi}\psi_{n_j} - f\| \geq \|\psi\|$.

The fact that $\psi_n(z) \to -1$ as $n \to \infty$ for all $z \in \mathbb{D}$ implies that $(\psi_{n_j} \circ \phi)(z) \to -1$ as $j \to \infty$ for all $z \in \mathbb{D}$. This, in turn, gives $\lim_{j\to\infty} |\psi(z)(\psi_{n_j} \circ \phi)(z) - f(z)| = |-\psi(z) - f(z)|$ for all $z \in \mathbb{D}$. Without loss of generality, we can assume that $\phi(z_0) = 1$. Now $\lim_{j\to\infty} |W_{\psi,\phi}\psi_{n_j}(z_0) - f(z_0)| = \lim_{j\to\infty} |\psi(z_0)\psi_{n_j} \circ \phi(z_0) - f(z_0)| = |\psi(z_0) - f(z_0)|$.

If $|\psi(z_0) - f(z_0)| \geq \|\psi\|$, then $\|\psi\psi_{n_j} \circ \phi - f\| \geq |\psi(z_0)\psi_{n_j} \circ \phi(z_0) - f(z_0)| \to |\psi(z_0) - f(z_0)| \geq \|\psi\|$. So we get $\limsup_{j\to\infty} \|\psi\psi_{n_j} \circ \phi - f\| \geq \|\psi\|$. In case $|\psi(z_0) - f(z_0)| < \|\psi\|$, a simple computation gives $|-\psi(z_0) - f(z_0)| > \|\psi\|$. Further, let $\{z_m\}$ be a sequence in $\mathbb{D}$ such that $z_m \to z_0$. Then $f(z_m) \to f(z_0)$. Since ψ_n is continuous and $\psi_n(1) = 1$, it follows that

$$\begin{aligned}\limsup_{j\to\infty} |W_{\psi,\phi}\psi_{n_j}(z_m) - f(z_m)| &= \limsup_{j\to\infty} |\psi(z_m)\psi_{n_j} \circ \phi(z_m) - f(z_m)| \\ &= |-\psi(z_m) - f(z_m)| \quad \text{for each } z_m \in \mathbb{D}.\end{aligned}$$

Hence

$$\begin{aligned}\limsup_{j\to\infty} \|W_{\psi,\phi}\psi_{n_j} - f\| &\geq \lim_{m\to\infty} \lim_{j\to\infty} |\psi(z_m)\psi_{n_j} \circ \phi(z_m) - f(z_m)| \\ &= |-\psi(z_0) - f(z_0)| > \|\psi\|.\end{aligned}$$

This implies $\|W_{\psi,\phi} - K\| > \|\psi\|$. As K is arbitrary, it follows that $\|W_{\psi,\phi}\|_e \geq \|\psi\|$. Thus $\|W_{\psi,\phi}\|_e = \|\psi\|$.

If $\phi(z_0) \neq 1$, let $\varsigma_n(z) = \phi(z_0)\psi_n(\phi(z_0)^{-1}z)$. The same proof holds with ψ_n replaced by ς_n and the boundary points replaced by $\phi(z_0)$ and $-\phi(z_0)$ respectively. ∎

Corollary 2.1. *If ϕ is a non-constant inner function in $A(\mathbb{D})$, then $\|W_{\psi,\phi}\|_e = \|W_{\psi,\phi}\| = \|\psi\|$.*

Proof. Since $\psi \in A(\mathbb{D})$ and ϕ is a non-constant inner function in $A(\mathbb{D})$, it follows that there exists a point $z_0 \in \overline{\mathbb{D}}$ such that $\|\psi\| = |\psi(z_0)|$ and $|\phi(z_0)| = 1$. Hence by Theorem 2.1 $\|W_{\psi,\phi}\|_e = \|W_{\psi,\phi}\| = \|\psi\|$. ∎

Corollary 2.2. *If C_ϕ is a composition operator on $A(\mathbb{D})$, then its essential norm is either 0 or 1.*

Proof. If C_ϕ is compact then $\|C_\phi\|_e = 0$. If C_ϕ is non-compact on $A(\mathbb{D})$, then we know that there exists a point $z_0 \in \overline{\mathbb{D}}$ such that $|z_0| = 1$ and $|\phi(z_0)| = 1$. Now applying the Theorem 2.1 with the constant $\psi \equiv 1$ we get $\|C_\phi\|_e = 1$. ∎

Remark 2.1. The essential spectral radius of C_ϕ on $A(\mathbb{D})$ is either 0 or 1.

Proof. If $(C_\phi)^n = C_{\phi_n}$ is compact for some $n \geq 1$, then essential spectral radius $\rho_e(C_\phi) = 0$. Suppose $(C_\phi)^n$ is non-compact for each $n \geq 1$. Then by Corollary 2.2 $\|(C_\phi)^n\|_e = 1$ for all $n \geq 1$. Now by spectral radius formula we get $\rho_e(C_\phi) = 1$. ∎

The following Proposition demonstrates the existence of a non-compact operator of the form $W_{\psi,\phi}$ which is not extremally non-compact.

Proposition 2.1. *Let $\psi \in A(\mathbb{D})$ and $\phi \in A(\mathbb{D})$ be a non-constant function with $\|\phi\| \leq 1$. Suppose that there is a unique point $z_0 \in \overline{\mathbb{D}}$ with $|\phi(z_0)| = 1$. If $0 < |\psi(z_0)| < \|\psi\|$, then the operator $W_{\psi,\phi}$ is neither compact nor extremally non-compact.*

Proof. Theorem 1.1 dictates that the operator $W_{\psi,\phi}$ is not compact. Define the function $\psi_1(z) = \psi(z) - \psi(z_0)$. Observe that $\psi_1(z_0) = 0$. So the operator $W_{\psi_1,\phi}$ is compact. Furthermore

$$\|W_{\psi,\phi}\|_e \leq \|W_{\psi,\phi} - W_{\psi_1,\phi}\| = \|\psi - \psi_1\| = |\psi(z_0)| < \|\psi\| = \|W_{\psi,\phi}\|.$$

Therefore $W_{\psi,\phi}$ is not extremally non-compact. ∎

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