МАТЕМАТИЧКИ ВЕСНИК
61, 3 (2009), 219–226
September 2009

UDK 517.521
оригинални научни рад
research paper

ON L^1-CONVERGENCE OF CERTAIN GENERALIZED MODIFIED TRIGONOMETRIC SUMS

Karanvir Singh and Kulwinder Kaur

Abstract. In this paper we define new modified generalized sine sums $K_{nr}(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}(\triangle^r a_{k-1} - \triangle^r a_{k+1})\tilde{S}_k^{r-1}(x)$ and study their L^1-convergence under a newly defined class $\mathbf{K}^\alpha$. Our results generalize the corresponding results of Kaur, Bhatia and Ram [6] and Kaur [7].

1. Introduction

Let

$$K(x) = \sum_{k=1}^{\infty} a_k \cos kx \tag{1.1}$$

and

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\triangle a_{j-1} - \triangle a_{j+1})\sin kx. \tag{1.2}$$

Using modified cosine sums

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n}\triangle a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}(\triangle a_j)\cos kx$$

of Garett and Stanojević [4], Kaur and Bhatia [5] proved the following theorem under the class of generalized semi-convex coefficients.

THEOREM 1. *If $\{a_n\}$ is a generalized semi-convex null sequence, then $g_n(x)$ converges to $K(x)$ in the L^1-metric if and only if $\triangle a_n \log n = o(1)$, as $n \to \infty$.*

The above mentioned result motivated the authors [6] to define a new modified sums (1.2) and to study these sums under a different class $\mathbf{K}$ of coefficients in the following way.

AMS Subject Classification: 42A20, 42A32.
Keywords and phrases: L^1-convergence; conjugate Cesàro means; generalized sine sums.

THEOREM 2. *Let k be a positive real number. If*

$$a_k = o(1), \qquad k \to \infty \tag{1.3}$$

and

$$\sum_{k=1}^{\infty} k|\triangle^2 a_{k-1} - \triangle^2 a_{k+1}| < \infty, \tag{1.4}$$

then $K_n(x)$ converges to $K(x)$ in the L^1-norm.

Any sequence satisfying (1.3) and (1.4) is said to belong to the class $\mathbf{K}$ [6].

In particular in [6] the following corollary to Theorem 2 is proved.

THEOREM 3. *If $\{a_n\}$ belongs to the class $\mathbf{K}$, then the necessary and sufficient condition for L^1-convergence of the cosine series (1.1) is* $\lim_{n\to\infty} a_n \log n = 0$.

DEFINITION. Let α be a positive real number. If (1.3) holds and

$$\sum_{k=1}^{\infty} k^{\alpha}|\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1}| < \infty, \quad (a_0 = 0),$$

then we say that $\{a_n\}$ belongs to the class $\mathbf{K}^{\alpha}$.

For $\alpha = 1$, the class $\mathbf{K}^{\alpha}$ reduces to the class $\mathbf{K}$.

Applying Abel's transformation to $K_n(x)$, we can easily see that

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}(\triangle a_{k-1} - \triangle a_{k+1})\tilde{S}_k^0(x),$$

where $\tilde{S}_k^0(x) = \tilde{D}_k(x) = \sin x + \sin 2x + \sin 3x + \cdots + \sin nx$. So in [7] the author studied the L^1-convergence of $K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}(\triangle a_{k-1} - \triangle a_{k+1})\tilde{S}_k^0(x)$ to $K(x)$ by proving the following result.

THEOREM 4. *Let the sequence $\{a_k\}$ belong to class $\mathbf{K}^{\alpha}$. Then $K_n(x)$ converges to $K(x)$ in the L^1-norm.*

If we take $\alpha = 1$, then this theorem reduces to Theorem 2.

It is natural to seek ways to prove the L^1-convergence of the generalized sums of the form

$$K_{nr}(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}(\triangle^r a_{k-1} - \triangle^r a_{k+1})\tilde{S}_k^{r-1}(x), \tag{1.5}$$

where r is any real number greater than or equal to 1. It is obvious that for $r = 1$, $K_{nr}(x)$ reduces to $K_n(x)$.

The purpose of this paper is to prove the L^1-convergence of $K_{nr}(x)$ to $K(x)$ under the class $\mathbf{K}^{\alpha}$.

2. Notation and Formulae

We use the following notations [10].

Given a sequence $S_0, S_1, S_2, \ldots$, we define the sequence $S_0^\alpha, S_1^\alpha, S_2^\alpha, \ldots$ for every $\alpha = 0, 1, 2, \ldots$ by the conditions $S_n^0 = S_n$, $S_n^\alpha = S_0^\alpha + S_1^{\alpha-1} + S_2^{\alpha-1} + \cdots + S_n^{\alpha-1}$ ($\alpha = 1, 2, \ldots;\ n = 0, 1, 2, \ldots$). Similarly we define the sequence of numbers A_0^α, $A_1^\alpha, A_2^\alpha, \ldots$ for $\alpha = 0, 1, 2, \ldots$ by the conditions $A_n^0 = 1$, $A_n^\alpha = A_0^{\alpha-1} + A_1^{\alpha-1} + A_2^{\alpha-1} + \cdots + A_n^{\alpha-1}$ ($\alpha = 1, 2, \ldots;\ n = 0, 1, 2, \ldots$). Let $\sum a_n$ be a given infinite series. The conjugate Cesàro sums of order α of $\sum a_n$ for any real number α are defined by

$$\tilde{S}_n^\alpha(a_p) = \tilde{S}_n^\alpha = \sum_{p=0}^{n} A_{n-p}^\alpha a_p = \sum_{p=0}^{n} A_{n-p}^{\alpha-1} \tilde{S}_p,$$

where $\tilde{S}_n = \tilde{S}_n^0 = \tilde{D}_n$, and A_p^α denotes the binomial coefficients. The conjugate Cesàro means $\tilde{T}_n^\alpha$ of order r of $\sum a_n$ will be defined by

$$\tilde{T}_n^\alpha = \frac{\tilde{S}_n^\alpha}{A_n^\alpha}.$$

The following formulae will also be needed:

$$\tilde{S}_n^\alpha(\tilde{S}_p^r) = \tilde{S}_n^{\alpha+r+1},$$

$$\tilde{S}_n^{\alpha+1} - \tilde{S}_{n-1}^{\alpha+1} = \tilde{S}_n^\alpha, \qquad \sum_{p=0}^{n} A_{n-p}^\alpha A_p^\beta = A_n^{\alpha+\beta+1}.$$

The differences of order α of the sequence $\{a_n\}$ for any positive integer α are defined by the equations $\triangle^\alpha a_n = \triangle(\triangle^{\alpha-1} a_n)$, $n = 0, 1, 2, \ldots$ $\triangle^1 a_n = a_n - a_{n+1}$. Since $A_m^{-\alpha-1} = 0$ for $m \geq \alpha + 1$, we have

$$\triangle^\alpha a_n = \sum_{m=0}^{\alpha} A_m^{-\alpha-1} a_{n+m} = \sum_{m=0}^{\infty} A_m^{-\alpha-1} a_{n+m}. \tag{2.1}$$

If the series (2.1) is convergent for some α which is not a positive integer, then we denote the differences

$$\triangle^\alpha a_n = \sum_{m=0}^{\infty} A_m^{-\alpha-1} a_{n+m} \qquad n = 0, 1, 2, 3, \ldots .$$

The broken differences $\triangle_n^\alpha a_p$ are defined by

$$\triangle_n^\alpha a_p = \sum_{m=0}^{n-p} A_m^{-\alpha-1} a_{p+m}.$$

By repeated partial summation of order r, we have

$$\sum_{p=0}^{n} a_p b_p = \sum_{p=0}^{n} \tilde{S}_p^{\alpha-1}(a_p) \triangle_n^\alpha b_p.$$

3. Lemmas

We need the following lemmas for the proof of our result.

LEMMA 3.1 [3] *If* $r \geq 0$, $p \geq 0$,

(i) $\epsilon_n = o(n)^{-p}$, *and*

(ii) $\sum_{n=0}^{\infty} A_n^{r+p} |\triangle^{r+1} \epsilon_n| < \infty$,

then

(iii) $\sum_{n=0}^{\infty} A_n^{\lambda+p} |\triangle^{\lambda+1} \epsilon_n| < \infty$, *for* $-1 \leq \lambda \leq r$ *and*

(iv) $A_n^{\lambda+p} \triangle^{\lambda} \epsilon_n$ *is of bounded variation for* $0 \leq \lambda \leq r$ *and tends to zero as* $n \to \infty$.

LEMMA 3.2 [1] *Let* r *be a non-negative real number. If the sequence* $\{\epsilon_n\}$ *satisfies the conditions*

(i) $\epsilon_n = O(1)$, *and*

(ii) $\sum\limits_{n=1}^{\infty} n^r |\triangle^{r+1} \epsilon_n| < \infty$,

then $\triangle^{\beta} \epsilon_n = \sum\limits_{m=0}^{\infty} A_m^{r-\beta} \triangle^{r+1} \epsilon_{n+m}$, *for* $\beta > 0$.

LEMMA 3.3 [2] *If* $0 < \delta < 1$ *and* $0 \leq n < m$, *then*

$$\left| \sum_{i=0}^{m} A_{n-i}^{\delta-1} S_i \right| \leq \max_{0 \leq p \leq m} |S_p^{\delta}|.$$

LEMMA 3.4 [10] *Let* $\tilde{S}_n(x)$ *and* $\tilde{T}_n^{\alpha}$ *be the* n^{th} *partial sum and Cesàro mean of order* $\alpha > 0$, *respectively, of the series* $\sin x + \sin 2x + \sin 3x + \cdots + \sin nx + \cdots$. *Then*

(i) $\int_0^{\pi} |\tilde{S}_n(x)| \, dx \sim \log n$,

(ii) $\int_0^{\pi} |\tilde{T}_n^{\alpha}| \, dx$ *remains bounded for all* n.

4. Main result

The main result of this paper is the following theorem.

THEOREM 4.1. *Let* α *be a positive real number. If a sequence* $\{a_k\}$ *belongs to the class* $\mathbf{K}^{\alpha}$, *then for* $\alpha \leq r \leq \alpha + 1$

(i) $K_{nr}(x)$ *converges to* $K(x)$ *pointwise for* $0 < \delta \leq x \leq \pi$, *and*

(ii) $K_{nr}(x) \to K(x)$ *in the* L^1*-norm.*

If we take $\alpha = 1$ and $r = 1$, then Theorem 2 is obtained as a particular case and also Theorem 4 can be deduced as a special case of Theorem 4.1 if we take $r = 1$ in (1.5) .

Proof. We have

$$K(x) = \frac{1}{2\sin x}\sum_{k=1}^{\infty}(\triangle^r a_{k-1} - \triangle^r a_{k+1})\tilde{S}_k^{r-1}(x)$$

and

$$K_{nr}(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}(\triangle^r a_{k-1} - \triangle^r a_{k+1})\tilde{S}_k^{r-1}(x).$$

Case 1. Let $r = \alpha + 1$. Then

$$K_{nr}(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}(\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1})\tilde{S}_k^{\alpha}(x).$$

So $K_{nr}(x)$ converges to $K(x)$ pointwise for $0 < \delta \le x \le \pi$.

Now

$$\begin{aligned}
&\int_0^{\pi} |K(x) - K_{nr}(x)|\, dx \\
&= \int_0^{\pi}\left|\frac{1}{2\sin x}\sum_{k=n+1}^{\infty}(\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1})\tilde{S}_k^{\alpha}(x)\right| dx \\
&\le C\sum_{k=n+1}^{\infty}|(\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1})|\textstyle\int_0^{\pi}|\tilde{S}_k^{\alpha}(x)|\, dx \\
&= C\sum_{k=n+1}^{\infty}A_k^{\alpha}|(\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1})|\textstyle\int_0^{\pi}|\tilde{T}_k^{\alpha}(x)|\, dx \\
&\le C_1\sum_{k=n+1}^{\infty}A_k^{\alpha}|(\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1})| = o(1), \text{by hypothesis of Theorem 4.1.}
\end{aligned}$$

Therefore, $K_{nr}(x)$ converges to $K(x)$ as $n \to \infty$ in the L^1-norm.

Case 2. Let $\alpha < r < \alpha + 1$. Take $r = \alpha + 1 - \delta$ and $0 < \delta < 1$. Then

$$\begin{aligned}
K_{nr}(x) &= \frac{1}{2\sin x}\sum_{k=1}^{n}(\triangle^r a_{k-1} - \triangle^r a_{k+1})\tilde{S}_k^{r-1}(x) \\
&= \frac{1}{2\sin x}\sum_{k=1}^{n}(\triangle^{\alpha+1-\delta} a_{k-1} - \triangle^{\alpha+1-\delta} a_{k+1})\tilde{S}_k^{\alpha-\delta}(x).
\end{aligned}$$

Applying Abel's transformation of order $-\delta$ and using Lemma 3.2, we have

$$\begin{aligned}
&\frac{1}{2\sin x}\sum_{k=1}^{n}(\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1})\tilde{S}_k^{\alpha}(x) \\
&\quad = \frac{1}{2\sin x}\left[\sum_{k=1}^{n}\tilde{S}_k^{\alpha-\delta}(x)\sum_{m=1}^{n-k}A_m^{\delta-1}(\triangle^{\alpha+1} a_{m+k-1} - \triangle^{\alpha+1} a_{m+k+1})\right] \\
&\quad = \frac{1}{2\sin x}\left[\sum_{k=1}^{n}\tilde{S}_k^{\alpha-\delta}(x)\{(\triangle^{\alpha-\delta+1} a_{k-1} - \triangle^{\alpha-\delta+1} a_{k+1}) \right.\\
&\qquad\qquad \left. - \sum_{m=n-k+1}^{\infty}A_m^{\delta-1}(\triangle^{\delta+1} a_{m+k-1} - \triangle^{\delta+1} a_{m+k+1})\}\right]
\end{aligned}$$

$$= \frac{1}{2\sin x}\Bigg[\sum_{k=1}^{n} \tilde{S}_k^{\alpha-\delta}(x)(\triangle^{\alpha-\delta+1}a_{k-1} - \triangle^{\alpha-\delta+1}a_{k+1}) - R_n(x)\Bigg],$$

where

$$\begin{aligned} R_n(x) &= \sum_{k=1}^{n} \tilde{S}_k^{\alpha-\delta}(x)\{A_{n-k+1}^{\delta-1}(\triangle^{\delta+1}a_n - \triangle^{\delta+1}a_{n+2}) \\ &\quad + A_{n-k+2}^{\delta-1}(\triangle^{\delta+1}a_{n+1} - \triangle^{\delta+1}a_{n+3}) + \cdots\} \\ &= \sum_{k=1}^{n} \tilde{S}_k^{r-\delta}(x)\{A_{n-k+1}^{\delta-1}(\triangle^{\delta+1}a_{n+2} - \triangle^{\delta+1}a_n) \\ &\quad + \sum_{k=1}^{n} \tilde{S}_k^{r-\delta}(x)A_{n-k+2}^{\delta-1}(\triangle^{\delta+1}a_{n+3} - \triangle^{\delta+1}a_{n+1}) + \cdots\}. \end{aligned}$$

This implies that

$$K_{nr}(x) = \frac{1}{2\sin x}\Bigg[\sum_{k=1}^{n} \tilde{S}_k^{\alpha}(x)(\triangle^{\alpha+1}a_{k-1} - \triangle^{\alpha+1}a_{k+1}) + R_n(x)\Bigg].$$

Hence

$$\begin{aligned} &\int_0^\pi |K(x) - K_{nr}(x)|\,dx \\ &\quad \le C\int_0^\pi \Bigg|\frac{1}{2\sin x}\{\sum_{k=n+1}^{\infty} \tilde{S}_k^{\alpha}(x)(\triangle^{\alpha+1}a_{k-1} - \triangle^{\alpha+1}a_{k+1}) - R_n(x)\}\Bigg|\,dx \\ &\quad \le C\sum_{k=n+1}^{\infty} |(\triangle^{\alpha+1}a_{k-1} - \triangle^{\alpha+1}a_{k+1})| \textstyle\int_0^\pi |\tilde{S}_k^{\alpha}(x)|\,dx + \int_0^\pi |R_n(x)|\,dx \\ &\quad = C\sum_{k=n+1}^{\infty} A_k^{\alpha}|(\triangle^{\alpha+1}a_{k-1} - \triangle^{\alpha+1}a_{k+1})|(x) \textstyle\int_0^\pi |\tilde{T}_k^{\alpha}(x)|\,dx + \int_0^\pi |R_n(x)|\,dx \\ &\quad \le \sum_{k=n+1}^{\infty} A_k^{\alpha}|(\triangle^{\alpha+1}a_{k-1} - \triangle^{\alpha+1}a_{k+1})| + \textstyle\int_0^\pi |R_n(x)|\,dx \\ &\quad = o(1) + \int_0^\pi |R_n(x)|\,dx, \text{by hypotheses of Theorem 4.1.} \end{aligned} \tag{4.1}$$

Now

$$\begin{aligned} &\int_0^\pi |R_n(x)|\,dx \\ &\quad = \int_0^\pi \Bigg|\Bigg(\sum_{k=1}^{n} \tilde{S}_k^{\alpha-\delta}(x)A_{n-k+1}^{\delta-1}\Bigg)(\triangle^{\delta+1}a_n - \triangle^{\delta+1}a_{n+2}) \\ &\qquad + \Bigg(\sum_{k=1}^{n} \tilde{S}_k^{\alpha-\delta}(x)A_{n-k+2}^{\delta-1}\Bigg)(\triangle^{\delta+1}a_{n+1} - \triangle^{\delta+1}a_{n+3}) + \cdots\Bigg|\,dx \\ &\quad \le \int_0^\pi |(\triangle^{\delta+1}a_n - \triangle^{\delta+1}a_{n+2})|\Bigg|\sum_{k=1}^{n} \tilde{S}_k^{\alpha-\delta}(x)A_{n-k+1}^{\delta-1}\Bigg|\,dx \\ &\qquad + \int_0^\pi |(\triangle^{\delta+1}a_{n+1} - \triangle^{\delta+1}a_{n+3})|\Bigg|\sum_{k=1}^{n} \tilde{S}_k^{\alpha-\delta}(x)A_{n-k+2}^{\delta-1}\Bigg|\,dx + \cdots \end{aligned}$$

$$\begin{aligned}
&\le \int_0^\pi |(\triangle^{\delta+1}a_n - \triangle^{\delta+1}a_{n+2})| \max_{0\le p\le n+1} |\tilde{S}_p^\delta(x)|\,dx \\
&\qquad + \int_0^\pi |(\triangle^{\delta+1}a_{n+1} - \triangle^{\delta+1}a_{n+3})| \max_{0\le p\le n+2} |\tilde{S}_p^\delta(x)|\,dx + \cdots, \text{by Lemma 3.3} \\
&= |(\triangle^{\delta+1}a_n - \triangle^{\delta+1}a_{n+2})| A_{n+1}^\delta \int_0^\pi \max_{0\le p\le n+1} |\tilde{T}_p^\delta(x)|\,dx \\
&\qquad + |(\triangle^{\delta+1}a_{n+1} - \triangle^{\delta+1}a_{n+3})| A_{n+2}^\delta \int_0^\pi \max_{0\le p\le n+2} |\tilde{T}_p^\delta(x)| + \cdots \\
&= CA_{n+1}^\delta |(\triangle^{\delta+1}a_n - \triangle^{\delta+1}a_{n+2})| + CA_{n+2}^\delta |(\triangle^{\delta+1}a_{n+1} - \triangle^{\delta+1}a_{n+3})| + \cdots \\
&= o(1), \text{by Lemmas 3.1 and 3.4.}
\end{aligned}$$

Thus

$$\int_0^\pi |R_n(x)|\,dx = o(1), \quad n\to\infty.$$

Hence by (4.1)

$$\lim_{n\to\infty} \int_0^\pi |K(x) - K_{nr}(x)|\,dx = o(1).$$

Case 3. Let $\alpha = r$. In this case

$$K_{nr}(x) = \frac{1}{2\sin x} \sum_{k=1}^{n} (\triangle^\alpha a_{k-1} - \triangle^\alpha a_{k+1}) \tilde{S}_k^{\alpha-1}(x).$$

Applying Abel's transformation, we have

$$K_{nr}(x) = \frac{1}{2\sin x}\Big[\sum_{k=1}^{n} (\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1}) \tilde{S}_k^{\alpha}(x) + (\triangle^\alpha a_n - \triangle^\alpha a_{n+2})\tilde{S}_n^\alpha(x)\Big].$$

Since $\tilde{S}_k^\alpha(x)$ are bounded for $0 < \delta \le x \le \pi$,

$$K_{nr}(x) \to K(x) = \frac{1}{2\sin x} \sum_{k=1}^{\infty} (\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1}) \tilde{S}_k^{\alpha}(x)$$

pointwise for $0 < \delta \le x \le \pi$. So

$$\begin{aligned}
&\int_0^\pi |K(x) - K_{nr}(x)|\,dx \\
&\quad = \int_0^\pi \left| \frac{1}{2\sin x} \sum_{k=n+1}^{\infty} (\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1}) \tilde{S}_k^{\alpha}(x) - (\triangle^\alpha a_n - \triangle^\alpha a_{n+2})\tilde{S}_n^\alpha(x) \right| dx \\
&\quad \le C \sum_{k=n+1}^{\infty} \int_0^\pi |(\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1})|\, |\tilde{S}_k^{\alpha}(x)|\,dx \\
&\qquad + \int_0^\pi |(\triangle^\alpha a_n - \triangle^\alpha a_{n+2})|\, |\tilde{S}_n^\alpha(x)|\,dx \\
&\quad = \sum_{k=n+1}^{\infty} A_k^\alpha \textstyle\int_0^\pi |(\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1})||\tilde{T}_k^{\alpha}(x)|\,dx
\end{aligned}$$

$$+ A_n^\alpha |(\triangle^\alpha a_n - \triangle^\alpha a_{n+2})| \int_0^\pi |\tilde{T}_n^\alpha(x)|\, dx$$
$$\leq C \sum_{k=n+1}^{\infty} A_k^\alpha |(\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1})| + C_1 |(\triangle^\alpha a_n - \triangle^\alpha a_{n+2})|$$
$$= o(1) + o(1) = o(1), \text{by the hypotheses of Theorem 4.1 and Lemma 3.1.}$$

Thus the proof is complete. ■

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(received 16.06.2008)

Department of Applied Sciences, GZS College of Engineering and Technology, Bathinda-151001, Punjab, India

E-mail: `karanvir786@indiatimes.com, mathkaur@yahoo.co.in`