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EQUITORSION CONFORM MAPPINGS OF GENERALIZED RIEMANNIAN SPACES

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Abstract. We define an equitorsion conform mapping of two generalized Riemannian spaces and obtain some invariant geometric objects of this mapping, generalizing the tensor of conform curvature.

0. Introduction

A generalized Riemannian space GR_N in the sense of Eisenhart's definition [5] is a differentiable N-dimensional manifold, equipped with nonsymmetric basic tensor g_{ij}.

The use of non-symmetric basic tensor and non-symmetric connection became especially actual after appearance of the works of A. Einstein [1]–[4] related to creation of the Unified Field Theory (UFT). Remark that at UFT the symmetric part $g_{\underline{ij}}$ of the basic tensor g_{ij} is related to the gravitation, and antisymmetric one $g_{\underset{\vee}{ij}}$ to the electromagnetism. M Prvanović [14] and S. Minčić [8] gave geometric interpretations of the torsion and curvature tensors of non-symmetric affine connection.

Consider two N-dimensional generalized Riemannian spaces GR_N and $G\overline{R}_N$. Generalized Cristoffel's symbols of the first kind of the space GR_N and $G\overline{R}_N$ are given by

$$\Gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}) \quad \text{and} \quad \overline{\Gamma}_{i.jk} = \frac{1}{2}(\overline{g}_{ji,k} - \overline{g}_{jk,i} + \overline{g}_{ik,j}), \tag{0.1}$$

where, for example, $g_{ij,k} = \partial g_{ij}/\partial x^k$. Connection coefficients of these spaces are generalized Cristoffel's symbols of the second kind $\Gamma^i_{jk} = g^{\underline{ip}}\Gamma_{p.jk}$ and $\overline{\Gamma}^i_{jk} =$

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$\overline{g}^{\underline{ip}}\overline{\Gamma}_{p.jk}$, respectively, where $(g^{\underline{ij}}) = (g_{\underline{ij}})^{-1}$ and $\underline{ij}$ denote symmetrisation with division by indices i and j. Generally it is $\Gamma^i_{jk} \neq \Gamma^i_{kj}$. We suppose that $g = det(g_{ij}) \neq 0$, $\overline{g} = \det(\overline{g}_{ij}) \neq 0$, $\underline{g} = \det(g_{\underline{ij}}) \neq 0$, $\underline{\overline{g}} = \det(\overline{g}_{\underline{ij}}) \neq 0$.

One says that a reciprocal one-valued mapping $f : GR_N \to G\overline{R}_N$ is conform if for the basic tensors g_{ij} and $\overline{g}_{ij}$ of these spaces the condition

$$\overline{g}_{ij} = e^{2\psi}\, g_{ij} \tag{0.2}$$

is satisfied, where ψ is an arbitrary function of x's, and the spaces are considered in the common by this mapping system of local coordinates x^i. In this case for the Cristoffel's symbols of the first kind of the spaces GR_N and $G\overline{R}_N$ the relation

$$\overline{\Gamma}_{i.jk} = e^{2\psi}(\Gamma_{i.jk} + g_{ji}\psi_{,k} - g_{jk}\psi_{,i} + g_{ik}\psi_{,j}) \tag{0.3}$$

holds true, and for the Cristoffel's symbols of the second kind

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + g^{\underline{ip}}(g_{jp}\psi_{,k} - g_{jk}\psi_{,p} + g_{pk}\psi_{,j}) \tag{0.4}$$

holds. Let us denote $\psi_k = \psi_{,k} = \partial\psi/\partial x^k$ and $\psi^i = g^{\underline{ip}}\psi_p$. Now from (0.4) we have

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + g^{\underline{ip}}\,(g_{\underline{jp}}\,\psi_k - g_{\underline{jk}}\,\psi_p + g_{\underline{pk}}\,\psi_j) + g^{\underline{ip}}\,(g_{\underset{\vee}{jp}}\psi_k - g_{\underset{\vee}{jk}}\psi_p + g_{\underset{\vee}{pk}}\psi_j),$$

i.e.

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\,\psi_k + \delta^i_k\,\psi_j - \psi^i g_{\underline{jk}} + \xi^i_{jk}, \tag{0.5}$$

where

$$\xi^i_{jk} = g^{\underline{ip}}\,(g_{\underset{\vee}{jp}}\,\psi_k - g_{\underset{\vee}{jk}}\,\psi_p + g_{\underset{\vee}{pk}}\,\psi_j) = -\xi^i_{kj} \tag{0.5$'$}$$

and $\underset{\vee}{ij}$ denotes an antisymmetrisation with division. In the corresponding points $M(x)$ and $\overline{M}(x)$ of conform mapping we can put

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + P^i_{jk} \quad (i, j, k = 1, \ldots, N), \tag{0.6}$$

where P^i_{jk} is the deformation tensor of the connection Γ of GR_N according to the conform mapping $f : GR_N \to G\overline{R}_N$.

Notice that in GR_N we have

$$\Gamma^p_{\underset{\vee}{ip}} = 0, \tag{0.7}$$

(eq. (2.10) in [13]).

In a generalized Riemannian space one can define four kinds of covariant derivatives [10, 11]. For example, for a tensor a^i_j in GR_N we have

$$a^i_{j\underset{1}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{jm}a^i_p,$$
$$a^i_{j\underset{2}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{mj}a^i_p,$$
$$a^i_{j\underset{3}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{mj}a^i_p,$$
$$a^i_{j\underset{4}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{jm}a^i_p.$$

Denote by $\underset{\theta}{|}, \underset{\overline{\theta}}{|}$ a covariant derivative of the kind θ in GR_N and $G\overline{R}_N$ respectively. We have [7]

$$g_{\underline{ij}\underset{\theta}{|}ma} = \overline{g}_{\underline{ij}\underset{\overline{\theta}}{|}ma} \equiv 0.$$

In the case of the space GR_N we have five independent curvature tensors [9] (in [9] $\underset{5}{R}$ is denoted by $\underset{2}{\tilde{R}}$):

$$\begin{aligned}
\underset{1}{R}{}^i{}_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{pm}, \\
\underset{2}{R}{}^i{}_{jmn} &= \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{mp}, \\
\underset{3}{R}{}^i{}_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{nm}(\Gamma^i_{pj} - \Gamma^i_{jp}), \\
\underset{4}{R}{}^i{}_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{mn}(\Gamma^i_{pj} - \Gamma^i_{jp}), \\
\underset{5}{R}{}^i{}_{jmn} &= \frac{1}{2}(\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{pn} + \Gamma^p_{mj}\Gamma^i_{np} \\
&\quad - \Gamma^p_{jn}\Gamma^i_{mp} - \Gamma^p_{nj}\Gamma^i_{pm}).
\end{aligned}$$

We use the conform mapping $f\colon GR_N \to G\overline{R}_N$ to obtain tensors $\underset{\theta}{\overline{R}}{}^i_{jmn}$ ($\theta = 1, \dots, 5$), where for example

$$\underset{1}{\overline{R}}{}^i_{jmn} = \overline{\Gamma}^i_{jm,n} - \overline{\Gamma}^i_{jn,m} + \overline{\Gamma}^p_{jm}\overline{\Gamma}^i_{pn} - \overline{\Gamma}^p_{jn}\overline{\Gamma}^i_{pm}. \tag{0.8}$$

In the case of conform mapping $f\colon R_N \to \overline{R}_N$ of Riemannian spaces R_N and $\overline{R}_N$ [6, 15] we have an invariant geometric object

$$C^i{}_{jmn} = R^i{}_{jmn} + \delta^i_m P_{jn} - \delta^i_n P_{jm} + P^i_m g_{jn} - P^i_n g_{mj} \tag{0.9}$$

where

$$P_{jm} \equiv \frac{1}{N-2}(R_{jm} - \frac{1}{2(N-1)} R g_{jm}),$$

and $R^i{}_{jmn}$ is Riemann-Cristoffel's curvature tensor of the space R_N, R_{jm} Ricci's tensor and R a scalar curvature.

The object $C^i{}_{jmn}$ is called a conform curvature tensor [6, 15]. Having a conform mapping of two generalized Riemannian spaces, we cannot find a generalization of the tensor of conform curvature as an invariant of conform mapping in general case. For that reason we define a special conform mapping.

A mapping $f\colon GR_N \to G\overline{R}_N$ is an equitorsion conform mapping if the torsion tensors of the spaces GR_N and $G\overline{R}_N$ are equal. Then from (0.5) and (0.6) we have

$$\xi^i_{jk} = 0. \tag{0.10}$$

In [12] we have investigated equitorsion geodesic mappings of generalized Riemannian spaces.

1. Equitorsion conform curvature tensor of the first kind

Using (0.6), we get a relation between the first kind curvature tensors of the spaces GR_N and $G\overline{R}_N$ [12, 16]

$$\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i{}_{jmn} + P^i_{\underset{1}{jm|n}} - P^i_{\underset{1}{jn|m}} + P^p_{jm}P^i_{pn} - P^p_{jn}P^i_{pm} + 2\Gamma^p_{\underset{\vee}{mn}}P^i_{jp}.$$

Substituting P with respect to $(0.5, 6, 10)$, and using $(0.7')$, we obtain

$$\begin{aligned}\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i{}_{jmn} &+ \delta^i_j\,(\psi_{\underset{1}{m|n}} - \psi_{\underset{1}{n|m}}) + \delta^i_m\,(\psi_{\underset{1}{j|n}} - \psi_j\psi_n)\\ &- \delta^i_n\,(\psi_{\underset{1}{j|m}} - \psi_j\psi_m) - (\psi^i{}_{\underset{1}{|n}} - \psi_n\psi^i)g_{\underline{jm}} + (\psi^i{}_{\underset{1}{|m}} - \psi_m\psi^i)g_{\underline{jn}}\\ &- \delta^i_n\,\psi^p\psi_p g_{\underline{jm}} + \delta^i_m\,\psi^p\psi_p g_{\underline{jn}} + 2\delta^i_j\,\Gamma^p_{\underset{\vee}{mn}}\psi_p + 2\Gamma^i_{\underset{\vee}{mn}}\psi_j - 2\Gamma_{j.\underset{\vee}{mn}}\psi^i.\end{aligned} \tag{1.1}$$

Denoting

$$\underset{1}{\psi}_{ij} = \psi_{\underset{1}{i|j}} - \psi_i\psi_j, \quad \underset{1}{\psi}{}^i_j = g^{\underline{ip}}\,\underset{1}{\psi}_{pj} \tag{1.2a}$$

$$\Delta_1\psi = g^{\underline{pq}}\,\psi_p\psi_q = \psi_p\psi^p \tag{1.2b}$$

and using the relation

$$\underset{1}{\psi}_{mn} - \underset{1}{\psi}_{nm} = -2\Gamma^p_{\underset{\vee}{mn}}\psi_p \tag{1.3}$$

in (1.1), we get

$$\begin{aligned}\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i{}_{jmn} &+ \delta^i_m\,\underset{1}{\psi}_{jn} - \delta^i_n\,\underset{1}{\psi}_{jm} + \underset{1}{\psi}{}^i_m g_{\underline{jn}} - \underset{1}{\psi}{}^i_n g_{\underline{jm}}\\ &+ (\delta^i_m\,g_{\underline{jn}} - \delta^i_n\,g_{\underline{jm}})\Delta_1\psi + 2\Gamma^i_{\underset{\vee}{mn}}\psi_j - 2\Gamma_{j.\underset{\vee}{mn}}\psi^i.\end{aligned} \tag{1.4}$$

Further, let us denote

$$\underset{1}{\Delta}_2\psi = g^{\underline{pq}}\,\psi_{\underset{1}{p|q}} \tag{1.5}$$

Then we have

$$\underset{1}{\psi}{}^p_p = \underset{1}{\psi}_{pq}g^{\underline{pq}} = (\psi_{\underset{1}{p|q}} - \psi_p\psi_q)g^{\underline{pq}} = \underset{1}{\Delta}_2\psi - \Delta_1\psi.$$

Contracting by indices i and n in (1.4) we get

$$\underset{1}{\overline{R}}_{jm} = \underset{1}{R}_{jm} - (N-2)\underset{1}{\psi}_{jm} - [\underset{1}{\Delta}_2\psi + (N-2)\Delta_1\psi]g_{\underline{jm}} - 2\Gamma_{j.\underset{\vee}{mp}}\psi^p. \tag{1.6}$$

From (0.2) we get

$$\overline{g}^{\underline{ij}} = e^{-2\psi}\,g^{\underline{ij}}. \tag{1.7}$$

In (1.6) multiplying by $g^{\underline{jm}}$ and contracting by j and then by m we get

$$e^{2\psi}\underset{1}{\overline{R}} = \underset{1}{R} - 2(N-1)\underset{1}{\Delta}_2\psi - (N-1)(N-2)\Delta_1\psi, \tag{1.8}$$

where $\underset{1}{\overline{R}} = \overline{g}^{\underline{pq}} \underset{1}{\overline{R}}_{pq}$, and $\underset{1}{R} = g^{\underline{pq}} \underset{1}{R}_{pq}$ are scalar curvature of the first kind of the spaces $G\overline{R}_N$ and GR_N respectively. From (1.8) we have

$$\underset{1}{\Delta}_2 \psi = \frac{1}{2(N-1)} (\underset{1}{R} - e^{2\psi} \underset{1}{\overline{R}}) - \frac{N-2}{2} \Delta_1 \psi. \tag{1.9}$$

Substituting (1.9) in (1.6) we get

$$\begin{aligned}(N-2)\underset{1}{\psi}_{jm} = \underset{1}{R}_{jm} - \underset{1}{\overline{R}}_{jm} - \frac{1}{2(N-1)} (\underset{1}{R} - e^{2\psi} \underset{1}{\overline{R}}) g_{\underline{jm}} \\ - \frac{N-2}{2} \Delta_1 \psi g_{\underline{jm}} - 2\Gamma_{\underset{\vee}{j.mp}} \psi^p.\end{aligned} \tag{1.10}$$

Let us denote in the space GR_N

$$\underset{1}{P}_{jm} \equiv \frac{1}{N-2} (\underset{1}{R}_{jm} - \frac{1}{2(N-1)} \underset{1}{R} g_{\underline{jm}}) \tag{1.10$'$}$$

and analogously $\underset{1}{\overline{P}}_{jm}$ in the space $G\overline{R}_N$. In this case for $\underset{1}{\psi}_{jm}$ we obtain

$$\underset{1}{\psi}_{jm} = \underset{1}{P}_{jm} - \underset{1}{\overline{P}}_{jm} - \frac{1}{2} \Delta_1 \psi g_{\underline{jm}} - \frac{2}{N-2} \Gamma_{\underset{\vee}{j.mp}} \psi^p. \tag{1.11}$$

Substituting (1.11) in (1.4), we get

$$\begin{aligned}\underset{1}{\overline{R}}^i_{jmn} = \underset{1}{R}^i{}_{jmn} + \delta^i_m (\underset{1}{P}_{jn} - \underset{1}{\overline{P}}_{jn}) - \delta^i_n (\underset{1}{P}_{jm} - \underset{1}{\overline{P}}_{jm}) \\ + \underset{1}{P}^i{}_m g_{\underline{jn}} - \underset{1}{\overline{P}}^i{}_m \overline{g}_{\underline{jn}} - \underset{1}{P}^i{}_n g_{\underline{jm}} + \underset{1}{\overline{P}}^i{}_n \overline{g}_{\underline{jm}} \\ + \frac{2}{N-2} (\delta^i_n \Gamma_{\underset{\vee}{j.mp}} - \delta^i_m \Gamma_{\underset{\vee}{j.np}} + \Gamma^i_{\underset{\vee}{np}} g_{\underline{jm}} - \Gamma^i_{\underset{\vee}{mp}} g_{\underline{jn}}) \psi^p \\ + 2\Gamma^i_{\underset{\vee}{mn}} \psi_j - 2\Gamma_{\underset{\vee}{j.mn}} \psi^i.\end{aligned} \tag{1.12}$$

We can see that it follows from (0.2)

$$\psi_i = \frac{1}{2N} (\frac{\partial}{\partial x^i} \ln \overline{g} - \frac{\partial}{\partial x^i} \ln g) \tag{1.13}$$

where $g = \det(g_{ij})$, $\overline{g} = \det(\overline{g}_{ij})$. From (0.10) and (1.13) we obtain

$$\Gamma_{\underset{\vee}{j.nm}} \psi^i = \frac{1}{2N} \overline{\Gamma}_{\underset{\vee}{j.nm}} \overline{g}^{\underline{ip}} \frac{\partial}{\partial x^p} \ln \overline{g} - \frac{1}{2N} \Gamma_{\underset{\vee}{j.nm}} g^{\underline{ip}} \frac{\partial}{\partial x^p} \ln g \tag{1.14}$$

and

$$\Gamma^i_{\underset{\vee}{qn}} g_{\underline{mj}} \psi^q = \frac{1}{2N} \overline{\Gamma}^i_{\underset{\vee}{qn}} \overline{g}_{\underline{mj}} \overline{g}^{\underline{pq}} \frac{\partial}{\partial x^p} \ln \overline{g} - \frac{1}{2N} \Gamma^i_{\underset{\vee}{qn}} g_{\underline{mj}} g^{\underline{pq}} \frac{\partial}{\partial x^p} \ln g. \tag{1.15}$$

Taking into account (1.13, 14,15), we can write the relation (1.12) in the form

$$\underset{1}{\overline{C}}^i{}_{jmn} = \underset{1}{C}^i{}_{jmn}, \tag{1.16}$$

where

$$\begin{aligned}\underset{1}{C}{}^i{}_{jmn} &= \underset{1}{R}{}^i{}_{jmn} + \delta^i_m \underset{1}{P}{}_{jn} - \delta^i_n \underset{1}{P}{}_{jm} + \underset{1}{P}{}^i{}_m g_{\underline{jn}} - \underset{1}{P}{}^i{}_n g_{\underline{jm}} \\ &+ \frac{1}{N(N-2)}(\delta^i_m \Gamma_{\underset{\vee}{j.np}} - \delta^i_n \Gamma_{\underset{\vee}{j.mp}} + \Gamma^i_{\underset{\vee}{mp}} g_{\underline{jn}} - \Gamma^i_{\underset{\vee}{np}} g_{\underline{jm}}) g^{\underline{pq}} \frac{\partial}{\partial x^q} \ln g \\ &+ \frac{1}{N}(\Gamma_{\underset{\vee}{j.mn}} g^{\underline{ip}} - \Gamma^i_{\underset{\vee}{mn}} \delta^p_j) \frac{\partial}{\partial x^p} \ln g\end{aligned} \tag{1.17}$$

and analogously for $\underset{1}{\overline{C}}{}^i{}_{jmn}$. From (1.16) we see that the tensor $\underset{1}{C}{}^i{}_{jmn}$ is an invariant of equitorsion conform mapping, and one can call it the equitorsion conform curvature tensor of the first kind. So, we have

THEOREM 1. *Let generalized Riemannian spaces GR_N and $G\overline{R}_N$ be defined by virtue of their nonsymmetric basic tensors g_{ij} and $\overline{g}_{ij}$ respectively. The equitorsion conform curvature tensor of the first kind* $\underset{1}{C}{}^i{}_{jmn}$ (1.17) *is an invariant of the equitorsion conform mapping $f\colon GR_N \to G\overline{R}_N$, defined by* (0.2), (0.5), (0.10), *i.e.* (1.16) *is in force, where the tensor* $\underset{1}{P}$ *is given by* (1.10′).

2. Equitorsion conform curvature tensor of the second kind

For the second kind curvature tensors of the spaces GR_N and $G\overline{R}_N$ we get the relation [12, 16]

$$\underset{2}{\overline{R}}{}^i_{jmn} = \underset{2}{R}{}^i{}_{jmn} + P^i_{mj\underset{2}{|}n} - P^i_{nj\underset{2}{|}m} + P^p_{mj} P^i_{np} - P^p_{nj} P^i_{mp} + 2\Gamma^p_{\underset{\vee}{nm}} P^i_{pj},$$

i.e., using (0.5,6,10) one obtains

$$\begin{aligned}\underset{2}{\overline{R}}{}^i_{jmn} &= \underset{2}{R}{}^i{}_{jmn} + \delta^i_m \underset{2}{\psi}{}_{jn} - \delta^i_n \underset{2}{\psi}{}_{jm} + \underset{2}{\psi}{}^i_m g_{\underline{nj}} - \underset{2}{\psi}{}^i_n g_{\underline{mj}} \\ &+ (\delta^i_m g_{\underline{mj}} - \delta^i_n g_{\underline{mj}}) \Delta_1 \psi + 2\Gamma^i_{\underset{\vee}{nm}} \psi_j - 2\Gamma^p_{\underset{\vee}{nm}} \psi^i g_{\underline{pj}},\end{aligned} \tag{2.1}$$

where

$$\underset{2}{\psi}{}_{ij} = \psi_{i\underset{2}{|}j} - \psi_i \psi_j, \quad \underset{2}{\psi}{}^i_j = g^{\underline{ip}} \underset{2}{\psi}{}_{ij}, \quad \Delta_1 \psi = g^{\underline{pq}} \psi_p \psi_q \tag{2.2}$$

Now, analogously to previous case, we get the invariant object of the equitorsion conform mapping $f : GR_N \to G\overline{R}_N$

$$\begin{aligned}\underset{2}{C}{}^i{}_{jmn} &= \underset{2}{R}{}^i{}_{jmn} + \delta^i_m \underset{2}{P}{}_{jn} - \delta^i_n \underset{2}{P}{}_{jm} + \underset{2}{P}{}^i{}_m g_{\underline{nj}} - \underset{2}{P}{}^i{}_n g_{\underline{mj}} \\ &+ \frac{1}{N(N-2)}(\delta^i_m \Gamma_{\underset{\vee}{j.pn}} - \delta^i_n \Gamma_{\underset{\vee}{j.pm}} + \Gamma^i_{\underset{\vee}{pm}} g_{\underline{nj}} - \Gamma^i_{\underset{\vee}{pn}} g_{\underline{mj}}) g^{\underline{pq}} \frac{\partial}{\partial x^q} \ln g \\ &+ \frac{1}{N}(\Gamma_{\underset{\vee}{j.nm}} g^{\underline{ip}} - \Gamma^i_{\underset{\vee}{nm}} \delta^p_j) \frac{\partial}{\partial x^p} \ln g\end{aligned} \tag{2.3}$$

where

$$\underset{2}{P}_{jm} \equiv \frac{1}{N-2}(\underset{2}{R}_{jm} - \frac{1}{2(N-1)}\underset{2}{R}g_{\underline{mj}}), \tag{2.4}$$

$\underset{2}{R}_{jm}$ is Ricci's curvature tensor of the second kind and $\underset{2}{R}$ is a scalar curvature tensor of the second kind. The object $\underset{2}{C}{}^i{}_{jmn}$ is a tensor and we call it equitorsion conform curvature tensor of the second kind. Accordingly, we have

Theorem 2. *Starting from the curvature tensor $\underset{2}{R}{}^i{}_{jmn}$, under conditions as in Theorem 1, one obtains an invariant tensor $\underset{2}{C}{}^i{}_{jmn}$ (2.3) of the equitorsion conform mapping of generalized Riemannian spaces, where $\underset{2}{P}$ is given according to* (2.4).

3. Equitorsion conform curvature tensor of the third kind

In the case of the third kind curvature tensors of the spaces GR_N and $G\overline{R}_N$ we get the relation [12, 16]

$$\begin{aligned}\underset{3}{\overline{R}}{}^i_{jmn} = \underset{3}{R}{}^i{}_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^p_{jm}P^i_{np} - P^p_{nj}P^i_{pm}\\ + 2P^p_{nm}\Gamma^i_{\underset{\vee}{pj}} + 2P^p_{nm}P^i_{\underset{\vee}{pj}}\end{aligned}$$

i.e., because of (0.5,6,10), (1.2a,b) and (2.2),

$$\begin{aligned}\underset{3}{\overline{R}}{}^i_{jmn} &= \underset{3}{R}{}^i{}_{jmn} + \delta^i_m \underset{2}{\psi}_{jn} - \delta^i_n \underset{2}{\psi}_{jm} - \underset{2}{\psi}{}^i_n g_{\underline{jm}} + \underset{1}{\psi}{}^i_m g_{\underline{nj}}\\ &+ (\delta^i_m g_{\underline{nj}} - \delta^i_n g_{\underline{jm}})\Delta_1\psi + 2\psi_m\Gamma^i_{\underset{\vee}{nj}} + 2\psi_n\Gamma^i_{\underset{\vee}{mj}} - 2\psi^p g_{\underline{nm}}\Gamma^i_{\underset{\vee}{pj}}\end{aligned} \tag{3.1}$$

Also, the following is satisfied

$$\underset{2}{\psi}_{mn} = \underset{1}{\psi}_{mn} + 2\Gamma^p_{\underset{\vee}{mn}}\psi_p, \quad \underset{2}{\psi}{}^i_n = \underset{1}{\psi}{}^i_n + 2g^{\underline{ip}}\Gamma^q_{\underset{\vee}{pn}}\psi_q. \tag{3.2}$$

From (3.1), (3.2) and (0.10) we get

$$\begin{aligned}\underset{3}{\overline{R}}{}^i_{jmn} &= \underset{3}{R}{}^i{}_{jmn} + \delta^i_m \underset{1}{\psi}_{jn} - \delta^i_n \underset{1}{\psi}_{jm} + \underset{1}{\psi}{}^i_m g_{\underline{nj}} - \underset{1}{\psi}{}^i_n g_{\underline{jm}}\\ &+ (\delta^i_m g_{\underline{nj}} - \delta^i_n g_{\underline{jm}})\Delta_1\psi + 2\psi_m\Gamma^i_{\underset{\vee}{nj}} + 2\psi_n\Gamma^i_{\underset{\vee}{mj}} - 2\psi^p g_{\underline{nm}}\Gamma^i_{\underset{\vee}{pj}}\\ &+ 2\delta^i_m\Gamma^p_{\underset{\vee}{jn}}\psi_p - 2g^{\underline{ip}}\Gamma^q_{\underset{\vee}{pn}}\psi_q g_{\underline{jm}}.\end{aligned} \tag{3.3}$$

Contracting (3.3) with respect to i and n, and using (1.5), we get

$$\underset{3}{\overline{R}}_{jm} = \underset{3}{R}_{jm} - (N-2)\underset{1}{\psi}_{jm} - [\underset{1}{\Delta}_2\psi + (N-2)\Delta_1\psi]g_{\underline{jm}} - \psi^p\Gamma_{\underset{\vee}{m.pj}}. \tag{3.4}$$

Multiplying (3.4) by $\overline{g}^{\underline{jm}} = e^{-2\psi}g^{\underline{jm}}$ and contracting we get

$$\underset{1}{\Delta}_2\psi = \frac{1}{2(N-1)}(\underset{3}{R} - e^{2\psi}\underset{3}{\overline{R}}) - \frac{N-2}{2}\Delta_1\psi. \tag{3.5}$$

Substituting (3.5) in (3.4) and denoting

$$\underset{3}{P}_{jm} = \frac{1}{N-2}(\underset{3}{R}_{jm} - \frac{1}{2(N-1)}\underset{3}{R} g_{\underline{jm}}) \tag{3.6}$$

in GR_N and analogously in $G\overline{R}_N$, in this case for $\underset{1}{\psi}_{jm}$ we obtain

$$\underset{1}{\psi}_{jm} = \underset{3}{P}_{jm} - \underset{3}{\overline{P}}_{jm} - \frac{1}{2}\Delta_1\psi g_{\underline{jm}} - \frac{2}{N-2}\Gamma_{\underset{\vee}{m.pj}}\psi^p. \tag{3.7}$$

Substituting (3.7) in (3.3) and using (1.14,15) we get

$$\underset{3}{\overline{C}}{}^i{}_{jmn} = \underset{3}{C}{}^i{}_{jmn} \tag{3.8}$$

where

$$\begin{aligned}\underset{3}{C}{}^i{}_{jmn} &= \underset{3}{R}{}^i{}_{jmn} + \delta^i_m \underset{3}{P}_{jn} - \delta^i_n \underset{3}{P}_{jm} + \underset{3}{P}{}^i{}_m g_{\underline{nj}} - \underset{3}{P}{}^i{}_n g_{\underline{jm}}\\ &+ \frac{1}{N(N-2)}(\delta^i_m \Gamma_{\underset{\vee}{n.pj}} - \delta^i_n \Gamma_{\underset{\vee}{m.pj}})g^{\underline{pq}}\frac{\partial}{\partial x^q}\ln g\\ &+ \frac{1}{N}(g^{\underline{ip}}\Gamma^q_{\underset{\vee}{pn}} g_{\underline{jm}} - \delta^q_m \Gamma^i_{\underset{\vee}{nj}} - \delta^q_n \Gamma^i_{\underset{\vee}{mj}}\\ &+ \Gamma^i_{\underset{\vee}{pj}} g_{\underline{nm}} g^{\underline{pq}} - \delta^i_m \Gamma^q_{\underset{\vee}{jn}})\frac{\partial}{\partial x^q}\ln g\end{aligned} \tag{3.9}$$

And analogously for $\underset{3}{\overline{C}}{}^i{}_{jmn}$ of the space $G\overline{R}_N$. From (3.8) we can see that the tensor $\underset{3}{C}{}^i{}_{jmn}$ is an invariant of equitorsion conform mapping, and one can call it the equitorsion conform curvature tensor of the third kind. Now we have

THEOREM 3. *From the curvature tensor* $\underset{3}{R}{}^i{}_{jmn}$, *under the conditions as in Theorem 1, we obtain an invariant tensor* $\underset{3}{C}{}^i{}_{jmn}$ (3.9) *of the equitorsion conform mapping* $f\colon GR_N \to G\overline{R}_N$, *where* $\underset{3}{P}$ *is given according to* (3.6).

4. Equitorsion conform curvature tensor of the fourth kind

For curvature tensors of the fourth kind we get [12, 16]

$$\begin{aligned}\underset{4}{\overline{R}}{}^i_{jmn} &= \underset{4}{R}{}^i{}_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^p_{jm}P^i_{np} - P^p_{nj}P^i_{pm}\\ &+ 2P^p_{mn}\Gamma^i_{\underset{\vee}{pj}} + 2P^p_{mn}P^i_{\underset{\vee}{pj}}\end{aligned}$$

i.e.

$$\begin{aligned}\underset{4}{\overline{R}}{}^i_{jmn} &= \underset{4}{R}{}^i{}_{jmn} + \delta^i_m \underset{1}{\psi}_{jn} - \delta^i_n \underset{1}{\psi}_{jm} + \underset{1}{\psi}{}^i_m g_{\underline{nj}} - \underset{1}{\psi}{}^i_n g_{\underline{jm}}\\ &+ (\delta^i_m g_{\underline{nj}} - \delta^i_n g_{\underline{jm}})\Delta_1\psi + 2\psi_n\Gamma^i_{\underset{\vee}{mj}} + 2\psi_m\Gamma^i_{\underset{\vee}{nj}} - 2\psi^p g_{\underline{mn}}\Gamma^i_{\underset{\vee}{pj}}\\ &+ 2\delta^i_m \Gamma^p_{\underset{\vee}{jn}}\psi_p - 2g^{\underline{ip}}\Gamma^q_{\underset{\vee}{pn}}\psi_q g_{\underline{jm}}.\end{aligned}$$

In this case, analogously to previous case, we get an invariant object of the equitorsion conform mapping in the form

$$\begin{aligned}
\underset{4}{C}{}^i{}_{jmn} = \underset{4}{R}{}^i{}_{jmn} + \delta^i_m \underset{4}{P}{}_{jn} - \delta^i_n \underset{4}{P}{}_{jm} + \underset{4}{P}{}^i{}_m g_{\underline{nj}} - \underset{4}{P}{}^i{}_n g_{\underline{jm}} \\
+ \frac{1}{N(N-2)}(\delta^i_m \Gamma_{\underset{\vee}{n.pj}} - \delta^i_n \Gamma_{\underset{\vee}{m.pj}}) g^{\underline{pq}} \frac{\partial}{\partial x^q} \ln g \\
+ \frac{1}{N}(g^{\underline{ip}} \Gamma^q_{\underset{\vee}{pn}} g_{\underline{jm}} - \delta^q_m \Gamma^i_{\underset{\vee}{nj}} - \delta^q_n \Gamma^i_{\underset{\vee}{mj}} \\
+ \Gamma^i_{\underset{\vee}{pj}} g_{\underline{nm}} g^{\underline{pq}} - \delta^i_m \Gamma^q_{\underset{\vee}{jn}}) \frac{\partial}{\partial x^q} \ln g,
\end{aligned} \tag{4.1}$$

$$\underset{4}{P}{}_{jm} = \frac{1}{N-2}(\underset{4}{R}{}_{jm} - \frac{1}{2(N-1)} \underset{4}{R} g_{\underline{jm}}), \tag{4.2}$$

where $\underset{4}{R}{}_{jm}$ is Ricci's curvature tensor of the fourth kind and $\underset{4}{R}$ a scalar curvature of the fourth kind. The object $\underset{4}{C}{}^i{}_{jmn}$ is a tensor and we call it equitorsion conform curvature tensor of the fourth kind of the equitorsion conform mapping. So, the next theorem is valid.

THEOREM 4. *From the curvature tensor* $\underset{4}{R}{}^i{}_{jmn}$*, under the conditions as in Theorem 1, one obtains an invariant tensor* $\underset{4}{C}{}^i{}_{jmn}$ (4.1) *of the equitorsion conform mapping of generalized Riemannian spaces, where* $\underset{4}{P}$ *is given with respect to* (4.2).

5. Equitorsion conform curvature tensor of the fifth kind

For the curvature tensors of the fifth kind of the spaces GR_N and $G\overline{R}_N$ we find the relation [12, 16]

$$\begin{aligned}
\underset{5}{\overline{R}}{}^i_{jmn} = \underset{5}{R}{}^i{}_{jmn} + \frac{1}{2}(P^i_{jm\underset{3}{|}n} - P^i_{jn\underset{4}{|}m} + P^i_{mj\underset{4}{|}n} - P^i_{nj\underset{3}{|}m} + \\
+ P^p_{jm} P^i_{pn} - P^p_{jn} P^i_{mp} + P^p_{mj} P^i_{np} - P^p_{nj} P^i_{pm})
\end{aligned}$$

i.e.

$$\begin{aligned}
\underset{5}{\overline{R}}{}^i_{jmn} = \underset{5}{R}{}^i{}_{jmn} + \frac{1}{2}[\delta^i_m (\psi_{j\underset{3}{|}n} + \psi_{j\underset{4}{|}n} - 2\psi_j\psi_n) - \delta^i_n (\psi_{j\underset{3}{|}m} + \psi_{j\underset{4}{|}m} - 2\psi_j\psi_m) \\
+ (\psi^i{}_{\underset{3}{|}m} + \psi^i{}_{\underset{4}{|}m} - 2\psi_m\psi^i) g_{\underline{jn}} - (\psi^i{}_{\underset{3}{|}n} + \psi^i{}_{\underset{4}{|}n} - 2\psi_n\psi^i) g_{\underline{mj}} \\
+ 2(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}})\psi_p\psi^p].
\end{aligned} \tag{5.1}$$

Let us denote

$$\underset{34}{\psi}{}_{jn} = \frac{1}{2}(\psi_{j\underset{3}{|}n} + \psi_{j\underset{4}{|}n} - 2\psi_j\psi_n), \quad \underset{34}{\psi}{}^i_j = g^{\underline{ip}} \underset{34}{\psi}{}_{pj}, \quad \Delta_1\psi = g^{\underline{pq}} \psi_p\psi_q. \tag{5.2}$$

Then

$$\begin{aligned}
\underset{5}{\overline{R}}{}^i_{jmn} = \underset{5}{R}{}^i{}_{jmn} + \delta^i_m \underset{34}{\psi}{}_{jn} - \delta^i_n \underset{34}{\psi}{}_{jm} + \underset{34}{\psi}{}^i_m g_{\underline{jn}} - \underset{34}{\psi}{}^i_n g_{\underline{mj}} \\
+ (\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}})\Delta_1\psi.
\end{aligned} \tag{5.3}$$

Contracting by indices i, n and denoting

$$\underset{5}{\overline{R}}{}^{p}_{jmp} = \underset{5}{\overline{R}}_{jm}, \quad \underset{5}{R}{}^{p}_{jmp} = \underset{5}{R}_{jm}, \quad \underset{34}{\Delta}\psi = \frac{1}{2} g^{\underline{pq}}(\psi_{p\underset{3}{|}q} + \psi_{p\underset{4}{|}q}), \tag{5.4}$$

we obtain

$$\underset{5}{\overline{R}}_{jm} = \underset{5}{R}_{jm} - (N-2)\underset{34}{\psi}{}^{j}_{m} - [\underset{34}{\Delta}\psi + (N-2)\Delta_1\psi] g_{\underline{jm}}, \tag{5.5}$$

wherefrom, multiplying by $\overline{g}^{\underline{jm}} = e^{-2\psi} g^{\underline{jm}}$ and contracting by j and then by m one obtains

$$\underset{34}{\Delta}\psi = \frac{1}{2(N-1)}(\underset{5}{R} - e^{2\psi}\underset{5}{\overline{R}}) - \frac{N-2}{2}\Delta_1\psi. \tag{5.6}$$

From (5.5) and (5.6) we get

$$\underset{34}{\psi}_{jm} = \underset{5}{P}_{jm} - \underset{5}{\overline{P}}_{jm} - \frac{1}{2}\Delta_1\psi g_{\underline{jm}} \tag{5.7}$$

where we denoted

$$\underset{5}{P}_{jm} = \frac{1}{N-2}(\underset{5}{R}_{jm} - \frac{1}{2(N-1)}\underset{5}{R} g_{\underline{jm}}) \tag{5.8}$$

in GR_N and analogously $\underset{5}{\overline{P}}_{jm}$ in $G\overline{R}_N$.

Analogously to previous cases eliminating $\underset{34}{\psi}_{jm}$ from (5.3) we can write

$$\underset{5}{\overline{C}}{}^{i}{}_{jmn} = \underset{5}{C}{}^{i}{}_{jmn}, \tag{5.9}$$

where we denoted

$$\underset{5}{C}{}^{i}{}_{jmn} = \underset{5}{R}{}^{i}{}_{jmn} + \delta^i_m \underset{5}{P}_{jn} - \delta^i_n \underset{5}{P}_{jm} + \underset{5}{P}{}^{i}_{m} g_{\underline{nj}} - \underset{5}{P}{}^{i}_{n} g_{\underline{jm}}. \tag{5.10}$$

The object $\underset{5}{C}{}^{i}{}_{jmn}$ is an invariant of the equitorsion conform mapping. We call it equitorsion conform curvature tensor of the fifth kind. So, we have

Theorem 5. *Starting from the curvature tensor* $\underset{5}{R}{}^{i}{}_{jmn}$*, under the conditions as in the Theorem 1, we obtain an invariant tensor* $\underset{5}{C}{}^{i}{}_{jmn}$ (5.10) *of the equitorsion mapping* $f\colon GR_N \to G\overline{R}_N$*, where* $\underset{5}{P}$ *is given according to* (5.8).

If $GR_N(G\overline{R}_N)$ reduces to $R_N(\overline{R}_N)$, then the objects $\underset{\theta}{C}{}^{i}{}_{jmn}$ $(\theta = 1,\dots,5)$ reduce to the conform curvature tensor (0.9).

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