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# ON QUASI ALMOST LACUNARY STRONG CONVERGENCE DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULI

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**Abstract.** The idea of difference sequence sets, $X(\triangle) = \{ x = (x_k) : \triangle x \in X \}$, where $X = l_\infty$, $c$ or $c_0$ was introduced by Kizmaz [3], and then this subject has been studied and generalized by various mathematicians. In this article we define quasi almost $\triangle^m$-Lacunary strongly P-convergent sequences defined by sequence of moduli and give inclusion relations on these sequence spaces.

## 1. Preliminaries

The difference sequence space $X(\triangle)$ was introduced by Kizmaz [3] as follows

$$X(\triangle) = \{ x = (x_k) \in \omega : (\triangle x_k) \in X \} \quad \text{for } X = l_\infty,\ c \text{ or } c_0,$$

where $\triangle x_k = (x_k - x_{k+1})$ for all $k \in \mathbf{N}$.

The difference sequence spaces were generalized by Et and Colak [1] as follows

$$X(\triangle^m) = \{ x = (x_k) \in \omega : \triangle^m x = (\triangle^m x_k) \in X \} \quad \text{for } X = l_\infty,\ c \text{ or } c_0,$$

where $\triangle^m x_k = (\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1})$.

A sequence of positive integers $\theta = (k_r)$ is called "lacunary" if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted by $I_r = (k_{r-1}, k_r)$ and $q_r = \frac{k_r}{k_{r-1}}$. The space of lacunary strongly convergent sequence $L_\theta$ was defined by Freedman et al. [2] as:

$$L_\theta = \{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0 \text{ for some } L \}.$$

The double lacunary sequence was defined by E. Savas and R. F. Patterson [11] as follows: The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exists two increasing sequences of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \quad \text{as } r \to \infty$$

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and

$$l_0 = 0, \bar{h_s} = l_s - l_{s-1} \to \infty \quad \text{as} \quad s \to \infty.$$

Let $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h_s}$ and $\theta_{r,s}$ is determined by

$$Ir, s = \{ (k,l) : k_{r-1} < k \leq k_r \ \text{ and } \ l_{s-1} < l \leq l_s \},$$

$q_r = \frac{k_r}{k_{r-1}}$, $\bar{q_s} = \frac{l_s}{l_{s-1}}$ and $q_{r,s} = q_r \bar{q_s}$.

DEFINITION 1.1. A function $f \colon [0,\infty) \to [0,\infty)$ is called modular if

1. $f(t) = 0$ if and only if $t = 0$,
2. $f(t+u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
3. $f$ is increasing, and
4. $f$ is continuous from the right at 0.

Let $X$ be a sequence space. Then the sequence space $X(f)$ is defined as

$$X(f) = \{ x = (x_k) : (f(|x_k|)) \in X \}$$

for a modulus $f$ ([6],[8],[10]). Kolk [4], [5] gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$ i.e.

$$X(F) = \{ x = (x_k) : (f_k(|x_k|)) \in X \}.$$

A double sequence $x = (x_{k,l})$ has Pringsheim limit $L$ (denoted by $P\text{-}\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$ [9]. We shall denote it briefly as "P-convergent".

Recently Moricz and Rhoades [7] defined almost P-convergent sequences as follows: A double sequence $x = (x_{k,l})$ of real numbers is called almost $P$-convergent to a limit $L$ if

$$P\text{-}\lim_{p,q\to\infty} \sup_{m,n\geq 0} \frac{1}{pq} \sum_{k=m}^{m+p-1} \sum_{l=n}^{n+q-1} |x_{k,l} - L| = 0.$$

In this paper we introduce the following definition.

A double sequence $x = (x_{k,l})$ of elements of the real vector space $w$ (the space of bounded sequences) in a real normed space $X$ is said to be quasi almost P-convergent to a limit $L$ if

$$\left\| P\text{-}\lim_{p,q\to\infty} \sup_{m,n\geq 0} \frac{1}{pq} \sum_{k=m}^{m+p-1} \sum_{l=n}^{n+q-1} (x_{k,l} - L) \right\|_X = 0.$$

Let us denote the above set of sequences as $\bar{t}^2$.

For a sequence $F = (f_k)$ of moduli, we define the following sequence spaces:

$$[L_{\theta_{r,s}}, \triangle^m, F, P] = \{ x = (x_{k,l}) : P\text{-}\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} [f_k(\|\triangle^m x_{k+m,l+n} - L\|)]^{p_{k,l}} = 0,$$
$$\text{uniformly in } m \text{ and } n \text{ for some } L \}.$$

$$[L_{\theta_{r,s}}, \triangle^m, F, P]_0 = \{ x = (x_{k,l}) : P\text{-}\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} [f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}} = 0,$$
$$\text{uniformly in } m \text{ and } n \text{ for some } l \}.$$

We shall denote $[L_{\theta_{r,s}}, \triangle^m, F, P]$ and $[L_{\theta_{r,s}}, \triangle^m, F, P]_0$ as $[L_{\theta_{r,s}}, \triangle^m, F]$ and $[L_{\theta_{r,s}}, \triangle^m, F]_0$, respectively when $p_{k,l} = 1$ for all $k$ and $l$. If $x$ is in $[L_{\theta_{r,s}}, \triangle^m, F]$, we shall say that $x$ is quasi almost lacunary strongly P-convergent with respect to the sequence of moduli $F = (f_k)$. Also note that if $F(x) = x$, $p_{k,l} = 1$ for all $k$ and $l$ then $[L_{\theta_{r,s}}, \triangle^m, F, P] = [L_{\theta_{r,s}}, \triangle^m]$ and $[L_{\theta_{r,s}}, \triangle^m, F, P]_0 = [L^0_{\theta_{r,s}}, \triangle^m]$ which are defined as follows:

$$[L_{\theta_{r,s}}, \triangle^m] = \{ x = (x_{k,l}) : P\text{-}\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \|\triangle^m x_{k+m,l+n} - L\| = 0,$$
$$\text{uniformly in } m \text{ and } n \text{ for some } L \}.$$

and

$$[L^0_{\theta_{r,s}}, \triangle^m] = \{ x = (x_{k,l}) : P\text{-}\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \|\triangle^m x_{k+m,l+n}\| = 0,$$
$$\text{uniformly in } m \text{ and } n \}.$$

Again note that if $p_{k,l} = 1$ for all $k$ and $l$ then $[L_{\theta_{r,s}}, \triangle^m, F, P] = [L_{\theta_{r,s}}, \triangle^m, F]$ and $[L_{\theta_{r,s}}, \triangle^m, F, P]_0 = [L_{\theta_{r,s}}, \triangle^m, F]_0$.

We define

$$[L_{\theta_{r,s}}, \triangle^m, F] = \{ x = (x_{k,l}) : P\text{-}\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} [f_k(\|\triangle^m x_{k+m,l+n} - L\|)] = 0,$$
$$\text{uniformly in } m \text{ and } n \text{ for some } L \},$$

and

$$[L_{\theta_{r,s}}, \triangle^m, F]_0 = \{ x = (x_{k,l}) : P\text{-}\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} [f_k(\|\triangle^m x_{k+m,l+n}\|)] = 0,$$
$$\text{uniformly in } m \text{ and } n \}.$$

Now we extend quasi almost convergent double sequences to a sequence of moduli as follows: Let $F = (f_k)$ be a sequence of moduli and $P = (p_{k,l})$ be any factorable sequence of strictly positive real numbers, we define the following sequence space:

$$[\bar{t}^2, \triangle^m, F, P] = \{x = (x_{k,l}) : P\text{-}\lim_{pq} \frac{1}{p,q} \sum_{k,l=1,1}^{p,q} [f_k(\|\triangle^m x_{k+m,l+n} - L\|)]^{p_{k,l}} = 0,$$
$$\text{uniformly in } m \text{ and } n \text{ for some } L \}.$$

If we take $F(x) = x$, $p_{k,l} = 1$ for all $k$ and $l$, then $[\bar{t}^2, \triangle^m, F, P] = [\bar{t}^2, \triangle^m]$.

## 2. Main results

THEOREM 1. *Let $\theta_{r,s} = \{k_r, l_s\}$ be a double lacunary sequence with $\liminf_r q_r > 1$, and $\liminf_s \bar{q}_s > 1$. Then for any sequence of moduli $F = (f_k)$, $[\bar{t}^2, \triangle^m, F, P] \subset [L_{\theta_{r,s}}, \triangle^m, F, P]$.*

*Proof.* We need to show that $[\bar{t}^2, \triangle^m, F, P]_0 \subset [L_{\theta_{r,s}}, \triangle^m, F, P]_0$. The general inclusion follows by linearity. Suppose $\liminf_r q_r > 1$, and $\liminf_s \bar{q}_s > 1$; then there exists $\delta > 0$ such that $q_r > 1 + \delta$. This implies $\frac{h_r}{k_r} \le \frac{\delta}{\delta+1}$ and $\frac{h_s}{l_s} \le \frac{\delta}{\delta+1}$. Then for $x \in [\bar{t}^2, \triangle^m, F, P]_0$, we can write for each $m$ and $n$

$$
\begin{aligned}
A_{r,s} &= \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} [f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}} = \frac{1}{h_{r,s}} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} [f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}} \\
&= \frac{1}{h_{r,s}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} [f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}} \\
&= \frac{1}{h_{r,s}} \sum_{k=k_{r-1}+1}^{k_r} \sum_{l=1}^{l_{s-1}} [f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}} \\
&= \frac{1}{h_{r,s}} \sum_{l=l_{s-1}+1}^{l_s} \sum_{k=1}^{k_{r-1}} [f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}} \\
&= \frac{k_r k_s}{h_{r,s}} \left( \frac{1}{k_r l_s} \sum_{k=l}^{k_r} \sum_{l=1}^{l_s} [f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}} \right) \\
&= \frac{k_{r-1} l_{s-1}}{h_{r,s}} \left( \frac{1}{k_{r-1} l_{s-1}} \sum_{k=l}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} [f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}} \right) \\
&= \frac{1}{h_r} \sum_{k=k_{r-1}+1}^{k_r} \frac{l_{s-1}}{h_s} \frac{1}{l_{s-1}} \sum_{l=l}^{l_{s-1}} \sum_{l=1}^{l_{s-1}} [f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}} \\
&= \frac{1}{h_s} \sum_{l=l_{s-1}+1}^{l_s} \frac{k_{r-1}}{h_r} \frac{1}{k_{r-1}} \sum_{k=l}^{k_{r-1}} \sum_{k=1}^{k_{r-1}} [f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}}.
\end{aligned}
$$

Since $x \in [\bar{t}^2, \triangle^m, F, P]$ the last two terms tends to zero uniformly in $m, n$ in the Pringsheim sense, thus for each $m$ and $n$

$$
\begin{aligned}
A_{r,s} &= \frac{k_r k_s}{h_{r,s}} \left( \frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} [f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}} \right) \\
&= \frac{k_{r-1} l_{s-1}}{h_{r,s}} \left( \frac{1}{k_{r-1} l_{s-1}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} [f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}} \right) + o(1).
\end{aligned}
$$

Since $h_{rs} = k_r l_s - k_{r-1} l_{s-1}$ we are granted for each $m$ and $n$ the following:

$$\frac{k_r l_s}{h_{rs}} \le \frac{1+\delta}{\delta} \quad \text{and} \quad \frac{k_{r-1} l_{s-1}}{h_{rs}} \le \frac{1}{\delta}.$$

The terms

$$\frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} [f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}}$$

and

$$\frac{1}{k_{r-1}l_{s-1}}\sum_{k=1}^{k_{r-1}}\sum_{l=1}^{l_{s-1}}[f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}}$$

are both Pringsheim null sequences for all $m$ and $n$. Thus $A_{rs}$ is Pringsheim. ∎

Theorem 2. *Let* $\theta_{r,s}=\{k,l\}$ *be a double lacunary sequence with* $\limsup_r q_r<\infty$, *and* $\limsup_s \bar{q}_s<\infty$. *Then for any sequence of moduli* $F=(f_k)$, $[L_{\theta_{r,s}},\triangle^m,F,P]\subset[\bar{t}^2,\triangle^m,F,P]$.

*Proof.* Since $\limsup_r q_r<\infty$, and $\limsup_s \bar{q}_s<\infty$ there exists $G>0$ such that $q_r<G$ and $\bar{q}_s<G$ for all $r$ and $s$. Let $x\in[L_{\theta_{r,s}},\triangle^m,F,P]$ and $\epsilon>0$. Also there exist $r_0>0$ and $s_0>0$ such that for every $i\geq r_0$ and $j\geq s_0$ and $m$ and $n$,

$$D'_{i,j}=\frac{1}{h_{ij}}\sum_{(k,l)\in I_{i,j}}[f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}}<\epsilon.$$

Let $F'=\max\{D'_{i,j}:1\leq i\leq r_0 \textit{ and } 1\leq j\leq s_0\}$ and $p$ and $q$ be such that $k_{r-1}<p\leq k_r$ and $l_{s-1}<q\leq l_s$. Thus we obtain the following:

$$\begin{aligned}
&\frac{1}{pq}\sum_{k,l=1,1}^{p,q}[f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}}\\
&\leq\frac{1}{k_{r-1}l_{s-1}}\sum_{k,l=1,1}^{k_r l_s}[f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}}\\
&\leq\frac{1}{k_{r-1}l_{s-1}}\sum_{t,u=1,1}^{r,s}\left(\sum_{k,l\in I_{t,u}}[f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}}\right)\\
&=\frac{1}{k_{r-1}l_{s-1}}\sum_{t,u=1,1}^{r_0,s_0}h_{t,u}D'_{t,u}+\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<t\leq r)\cup(s_0<u\leq s)}h_{t,u}D'_{t,u}\\
&\leq\frac{F'}{k_{r-1}l_{s-1}}\sum_{t,u=1,1}^{r_0,s_0}h_{t,u}+\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<t\leq r)\cup(s_0<u\leq s)}h_{t,u}D'_{t,u}\\
&\leq\frac{F'k_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}}+(\sup_{t\geq r_0\cup u\geq s_0}D'_{t,u})\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<t\leq r)\cup(s_0<u\leq s)}h_{t,u}\\
&\leq\frac{F'k_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}}+\frac{1}{k_{r-1}l_{s-1}}\epsilon\sum_{(r_0<t\leq r)\cup(s_0<u\leq s)}h_{t,u}\\
&\leq\frac{F'k_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}}+\epsilon H^2.
\end{aligned}$$

Since $k_r$ and $l_s$ both approach infinity as both $p$ and $q$ approach infinity, it follows that

$$\frac{1}{pq}\sum_{k,l=1,1}^{p,q}[f_k(\|\triangle^m x_{k+m,l+n}\|)]^{p_{k,l}}\to 0,\text{uniformly in } m \text{ and } n.$$

Therefore $x\in[\bar{t}^2,\triangle^m,F,P]$. ∎

As a consequence we obtain

THEOREM 3. *Let* $\theta_{r,s} = \{k, l\}$ *be a double lacunary sequence with* $\liminf_{rs} q_{rs} \leq \limsup_{rs} q_{rs} < \infty$. *Then for any sequence of moduli* $F = (f_k)$, $[L_{\theta_{r,s}}, \triangle^m, F, P] = [\bar{t}^2, \triangle^m, F, P]$.

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