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OPEN COVERS AND FUNCTION SPACES

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Abstract. We investigate some closure properties of the space $C(X)$ of the continuous real-valued functions on a Tychonoff space X endowed with the compact-open topology and the pointwise convergence topology.

1. Introduction

In this paper we use the standard topological notation and terminology as in [3]. All spaces are assumed to be infinite Tychonoff. Let X be a topological space. Then:

$\mathcal{O}$ denotes the collection of all open covers of X;

Ω denotes the collection of all open ω-covers of X. A cover $\mathcal{U}$ of a set Y is called an *ω-cover* if Y is not a member of $\mathcal{U}$ and every finite subset of Y is contained in a member of $\mathcal{U}$ [4];

$\mathcal{K}$ denotes the collection of all open k-covers of X. A cover $\mathcal{U}$ of a space X is called a *k-cover* if X is not a member of $\mathcal{U}$ and every compact subset of X is contained in a member of $\mathcal{U}$ [2].

A space X is called a *k-Lindelöf space* if for each open k-cover $\mathcal{U}$ of X there is a $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V}$ is countable and $\mathcal{V} \in \mathcal{K}$. Each k-Lindelöf space is Lindelöf, so normal, too.

For a space X and a point $x \in X$, the symbol Ω_x denotes the set $\{A \subseteq X \setminus \{x\} : x \in \overline{A}\}$. Let $\mathcal{A}$ and $\mathcal{B}$ be two arbitrary sets (usually collections of open covers of a topological space X). The symbol $S_1(\mathcal{A}, \mathcal{B})$ [9] denotes the selection principle:

For each sequence $(A_n : n \in \mathbf{N})$ of elements of $\mathcal{A}$ there exists a sequence $(b_n : n \in \mathbf{N})$ such that, for each n, $b_n \in A_n$ and $\{b_n : n \in \mathbf{N}\}$ is an element of $\mathcal{B}$.

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The symbol $S_{fin}(\mathcal{A},\mathcal{B})$ [9] denotes the selection principle:

For each sequence $(A_n : n \in \mathbf{N})$ of elements of $\mathcal{A}$ there exists a sequence $(B_n : n \in \mathbf{N})$ such that, for each n, B_n is a finite subset of A_n and $\cup_{n\in\mathbf{N}} B_n$ is an element of $\mathcal{B}$.

Then the property $S_1(\mathcal{O},\mathcal{O})$ is called the *Rothberger* property [5,17], and the property $S_{fin}(\mathcal{O},\mathcal{O})$ is known as the *Menger* property [5,17].

For a space X by $C(X)$ we denote the set of all continuous real-valued functions defined on X. The symbol $\mathbf{0}$ denotes the constant to zero function defined on X. Then $C_p(X)$ is the set $C(X)$ endowed with the topology of pointwise convergence (the "pc-topology" for short). Typical basic open sets of $C(X)$ are of the form:

$$W(x_1,\ldots,x_k;U_1,\ldots,U_k) = \{f \in C_p(X) : f(x_i) \in U_i, i = 1,\ldots,k\}$$

where $x_1,\ldots,x_k$ are points of X and $U_1,\ldots,U_k$ are open sets of $\mathbf{R}$. For a subset S of X and a positive real number ε we let

$$O(S,\varepsilon) = \{g \in C(X) : |g(x)| < \varepsilon, \text{ for all } x \in S\}.$$

The standard local base at the point $\mathbf{0}$ consists of the sets $O(F,\varepsilon)$, where F is a finite subset of X and ε is a positive real number.

By $C_k(X)$ we denote the set $C(X)$ endowed with the compact-open topology (the "co-topology" for short). Typical basic open sets of $C_k(X)$ are of the form:

$$W(K_1,\ldots,K_n;U_1,\ldots,U_n) = \{f \in C(X) : f(K_i) \subset U_i, i = 1,\ldots,n\}$$

where $K_1,\ldots,K_n$ are compact subsets of X and $U_1,\ldots,U_n$ are open sets of $\mathbf{R}$. The standard local base at the point $\mathbf{0} \in C_k(X)$ consists of the sets $O(K,\varepsilon)$, where K is a compact subset of X and ε is a positive real number.

Since $C_p(X)$ and $C_k(X)$ are homogenous spaces we may always consider the point $\mathbf{0}$ when studying local properties of these spaces. Since we are considering two topologies of $C(X)$ we shall use the symbol $(\Omega_{\mathbf{0}})^p$ to denote $\Omega_{\mathbf{0}}$ in the space $C_p(X)$ and the symbol $(\Omega_{\mathbf{0}})^k$ to denote $\Omega_{\mathbf{0}}$ in the space $C_k(X)$. Many results in the literature show that for a Tychonoff space X closure properties of function spaces $C_p(X)$ and $C_k(X)$ can be characterized by covering properties of X [1,9].

By standard techniques of switching between k-covers of X and subsets of $C_k(X)$ one can easily see that the following holds.

LEMMA 1.1.$C_k(X)$ *has countable tightness iff* X *is k-Lindelöf.*

In the first section of the paper we investigate the selection principles $S_1(\mathcal{K},\Omega)$ and $S_{fin}(\mathcal{K},\Omega)$ and their relations with function spaces. In the remaining two sections we study a bitopological variant of the Pytkeev property and of the Reznichenko property in function spaces.

2. The selection principles $S_1(\mathcal{K},\Omega)$ and $S_{fin}(\mathcal{K},\Omega)$

Note that the following relations between classes of covers defined above hold:

$$\mathcal{K} \subseteq \Omega \subseteq \mathcal{O},$$

hence we have:

$$S_1(\mathcal{K},\mathcal{K}) \subseteq S_1(\mathcal{K},\Omega);$$
$$S_{fin}(\mathcal{K},\mathcal{K}) \subseteq S_{fin}(\mathcal{K},\Omega).$$

The next two lemmas follow from the definition of k-covers.

LEMMA 2.1. *If a k-cover $\mathcal{U}$ of a space X is the union of finitely many subfamilies, then at least one of them is also a k-cover of X.*

LEMMA 2.2. *If $\mathcal{U}$ is a k-cover of a space X, then each compact subset of X is contained in infinitely many elements of $\mathcal{U}$. For any finite set S the family $\mathcal{U} \setminus S$ is also a k-cover.*

In [2] the following was proved:

LEMMA 2.3. *If $\mathcal{U}$ is a k-cover of a space X^n, then there is a k-cover $\mathcal{V}$ of X such that $\{V^n : V \in \mathcal{V}\}$ refines $\mathcal{U}$.*

THEOREM 2.1. *A space X satisfies $S_1(\mathcal{K},\Omega)$ if and only if each finite power of X satisfies $S_1(\mathcal{K},\Omega)$.*

Proof. Let X belong to the class $S_1(\mathcal{K},\Omega)$ and let $(\mathcal{W}_n : n \in \mathbf{N})$ be a sequence of k-covers of X^m, for a fixed natural number m. By Lemma 2.3, for each natural number n there exists a k-cover $\mathcal{U}_n$ of X such that $\{U^m : U \in \mathcal{U}_n\}$ refines $\mathcal{W}_n$. Apply the fact that $X \in S_1(\mathcal{K},\Omega)$ to the sequence $(\mathcal{U}_n : n \in \mathbf{N})$. There is a sequence $(U_n : n \in \mathbf{N})$ such that, for each n, $U_n \in \mathcal{U}_n$ and $\{\mathcal{U}_n : n \in \mathbf{N}\}$ is a k-cover of X. For each n, let W_n be an element of $\mathcal{W}_n$ with $U^m \subset W_n$. Then the sequence $(W_n : n \in \mathbf{N})$ shows that X^m is in the class $S_1(\mathcal{K},\Omega)$. Let F be a finite subset of X^m. The union $\bigcup_{i \le m} p_i(F) = B$ of the projections onto X is a finite subset of X and thus there is an $n \in \mathbf{N}$ such that $B \subset U_n$. Hence $F \subseteq B^m \subseteq U_n^m \subseteq W_n$. ∎

In a similar way one can prove that:

THEOREM 2.2. *A space X satisfies $S_{fin}(\mathcal{K},\Omega)$ if and only if each finite power of X satisfies $S_{fin}(\mathcal{K},\Omega)$.*

Our intent in this section is to describe how these two principles affect the bitopological space $(C_k(X), C_p(X))$. Let us mention here that the principles $S_1(\Omega,\mathcal{K})$ and $S_{fin}(\Omega,\mathcal{K})$ have already been considered in [8] where Ramsey theoretical characterizations were established.

A space X *has countable fan tightness* [1] if for each $x \in X$ and each sequence $(A_n : n \in \mathbf{N})$ of elements of Ω_x there exists a sequence $(B_n : n \in \mathbf{N})$ of finite sets such that, for each n, $B_n \subseteq A_n$ and $x \in \overline{\cup_{n\in\mathbf{N}} B_n}$, i.e. if $S_{fin}(\Omega_x, \Omega_x)$ holds for each $x \in X$.

A space X *has countable strong fan tightness* [16] if for each $x \in X$ $S_1(\Omega_x, \Omega_x)$ holds.

The following theorem was proved in [6].

THEOREM 2.3. *For a space X the following are equivalent:*

(1) *$C_k(X)$ has countable strong fan tightness;*

(2) *X has the property $S_1(\mathcal{K}, \mathcal{K})$.*

THEOREM 2.4. *For a space X the following are equivalent:*

(1) *$C(X)$ satisfies $S_1((\Omega_{\mathbf{0}})^k, (\Omega_{\mathbf{0}})^p))$;*

(2) *X has the property $S_1(\mathcal{K}, \Omega)$.*

Proof. (1) $\Rightarrow$ (2) : Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of k-covers of X. For each pair of a compact subset K of X and an open subset $U \supseteq K$ of X let $f_{K,U}$ be any continuous function from X to $[0,1]$ such that $f_{K,U}(K) \subseteq \{0\}$ and $f_{K,U}(X \setminus U) \subseteq \{1\}$. For each n let $A_n = \{f_{K,U} : K \text{ compact in } X,\ K \subseteq U \in \mathcal{U}_n\}$. Then for each compact subset K of X there is a $f_{K,U} \in A_n$ so, as it can easily be verified, $\mathbf{0}$ is in the closure of each A_n, with respect to the compact-open topology. Since $C(X)$ satisfies $S_1((\Omega_{\mathbf{0}})^k, (\Omega_{\mathbf{0}})^p)$ there is a sequence $(f_{K_n,U_n} : n \in \mathbf{N})$ such that for each n, K_n is compact, $U_n \in \mathcal{U}_n$ and $\mathbf{0}$ belongs to the closure of $\{f_{K_n,U_n} : n \in \mathbf{N}\}$ with respect to the pointwise convergence topology. We claim that $\{U_n : n \in \mathbf{N}\} \in \Omega$. Let F be a finite subset of X. From the fact that $\mathbf{0}$ belongs to the closure of $\{f_{K_n,U_n} : n \in \mathbf{N}\}$ with respect to the pointwise convergence topology it follows that there is an $i \in \mathbf{N}$ such that $W = O(F, 1)$ contains the function f_{K_i,U_i}. Then $F \subseteq U_i$. Otherwise for some $x \in F$ one has $x \notin U_i$ so that $f_{K_i,U_i}(x) = 1$, contradicting $f_{K_i,U_i} \in W$.

(2) $\Rightarrow$ (1) : Let $(A_m : m \in \mathbf{N})$ be a sequence of subsets of $C(X) \setminus \{\mathbf{0}\}$ the closures of which all contain $\mathbf{0}$, with respect to the compact-open topology. If X is compact then the compact-open topology coincides with the topology of uniform convergence, so $C_k(X)$ is metrizable, thus first countable, which means that we can find a sequence $(a_n : n \in \mathbf{N})$, $a_n \in A_n$, converging uniformly to $\mathbf{0}$ so there is nothing to be proved. Let X be a noncompact space.

For a bijection $i : \mathbf{N}^2 \to \mathbf{N}$ put $A_{n,m} := A_{i(n,m)}$.

For each $n, m \in \mathbf{N}$ and every compact set $K \subseteq X$ the neighborhood $W = O(K, \frac{1}{n})$ of $\mathbf{0}$ intersects $A_{m,n}$, so there exists a continuous function $f_{K,m,n} \in A_{m,n}$ such that $|f_{K,m,n}(x)| < \frac{1}{n}$ for each $x \in K$. Since $f_{K,m,n}$ is a continuous function there is an open $U_{K,m,n}$ such that $f_{K,m,n}(U_{K,m,n}) \subseteq (-\frac{1}{n}, \frac{1}{n})$. Let $\mathcal{U}_{m,n} = \{U_{K,m,n} : K \text{ is a compact subset of } X\}$.

As for any compact subset K we have $K \neq X$, it can easily be achieved that none of the sets $U_{K,m,n}$ above coincides with X. So for each m and n, $\mathcal{U}_{m,n}$ is a k-cover of X. To each sequence $(\mathcal{U}_{m,n} : m \in \mathbf{N})$ apply the fact that X is an $S_1(\mathcal{K}, \Omega)$-space to obtain sequences $(U_{K_{m,n},m,n} : m \in \mathbf{N})$, with each $K_{m,n}$ compact, such that $\{U_{K_{m,n},m,n} : m \in \mathbf{N}\} \in \Omega$ for every $n \in \mathbf{N}$. Let us show that $\mathbf{0}$ belongs to the closure of $\{f_{K_{m,n},m,n} : m, n \in \mathbf{N}\}$ with respect to the pointwise convergence topology.

Let $W = O(F, \varepsilon)$ be a neighborhood of $\mathbf{0}$ in $C_p(X)$ and let n be a positive integer such that $\frac{1}{n} < \varepsilon$. Since F is a finite subset of X and $\{U_{K_{m,n},m,n} : m \in \mathbf{N}\} \in \Omega$ there is a $m \in \mathbf{N}$ such that $F \subseteq U_{K_{m,n},m,n}$. We have

$$f_{K_{m,n},m,n}(F) \subseteq f_{K_{m,n},m,n}(U_{K_{m,n},m,n}) \subseteq (-\frac{1}{n}, \frac{1}{n}) \subset (-\varepsilon, \varepsilon)$$

i.e. $f_{K_{m,n},m,n} \in W$. Since $f_{K_{m,n},m,n} \in A_{m,n}$ this ends the proof of the theorem. ∎

In a similar way one can show that

THEOREM 2.5. *For a space X the following are equivalent:*

(1) *$C(X)$ satisfies $S_{fin}((\Omega_{\mathbf{0}})^k, (\Omega_{\mathbf{0}})^p))$;*

(2) *X has the property $S_{fin}(\mathcal{K}, \Omega)$.*

3. The Pytkeev-type properties

For a space X and $x \in X$, a family $\mathcal{F}$ of subsets of X is called a *π-network at x* if every neighborhood of x contains an element of $\mathcal{F}$.

A space X is called a *Pytkeev space* [14] if $x \in \overline{A} \setminus A$ and $A \subseteq X$ imply the existence of a countable π-network at x consisting of infinite subsets of A.

An open ω-cover $\mathcal{U}$ is said to be *ω-shrinkable* [14] if there is a function C such that for each $U \in \mathcal{U}$ the set $C(U)$ is closed, $C(U) \subseteq U$ and $\{C(U) : U \in \mathcal{U}\}$ is an ω-cover of X.

In [14] the following theorem was proved:

THEOREM 3.1. *The following are equivalent:*

(1) *$C_p(X)$ is a Pytkeev space;*

(2) *If $\mathcal{U}$ is an ω-shrinkable open nontrivial ω-cover of X, there is a sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of countably infinite subfamilies of $\mathcal{U}$ such that $\{\cap\mathcal{U}_n : n \in \mathbf{N}\}$ is an ω-cover of X.*

Nontrivial means not containing the whole space as one of its elements.

When dealing with the space $C_k(X)$ we will need the "compact" version of "shrinkability". An open cover $\mathcal{U}$ is said to be *k-shrinkable* if there is a function C such that for each $U \in \mathcal{U}$ the set $C(U)$ is closed, $C(U) \subseteq U$ and $\{C(U) : U \in \mathcal{U}\}$ is a k-cover of X; the collection of all k-shrinkable open covers of X is denoted by $\mathcal{K}_{shr}$. A family $\mathcal{U}$ of subsets of a space X is said to be a *3-k-shrinkable cover* of X if there is a function g such that for each $U \in \mathcal{U}$ $g(U) = (V_U, Z_U)$, where $V_U \subseteq Z_U \subseteq U$, V_U is a cozero set, Z_U is a zero set and $\{V_U : U \in \mathcal{U}\}$ k-covers X. $\mathcal{U}$ is called an open 3-k-shrinkable cover provided that its elements are open subsets of X. The collection of all nontrivial open 3-k-shrinkable covers of X will be denoted by 3-$\mathcal{K}_{shr}$. Obviously one has 3-$\mathcal{K}_{shr} \subseteq \mathcal{K}_{shr} \subseteq \mathcal{K}$.

In the sequel we will need the following two lemmas which are proved exactly as Lemma 1.2 and Lemma 1.4, respectively, in [12], mostly by replacing the word *finite* by the word *compact*.

Lemma 3.1. *Every open k-cover can be refined by an open 3-k-shrinkable cover.*

Lemma 3.2. *Let X be a k-Lindelöf space. If $\varepsilon > 0$ and $\mathbf{0} \in \overline{A} \subseteq C_k(X)$ then there is a $B \subseteq A$ and a function $s : B \to (0, \varepsilon)$ such that at least one of the families $\{|f|^{\leftarrow}[0, s(f)) : f \in B\}$ or $\{|f|^{\leftarrow}[0, s(f)] : f \in B\}$ is a 3-k- shrinkable open cover of X.*

What could be called "k-shr-Lindelöf" does not differ from k-Lindelöf. More exactly:

Lemma 3.3. *A space X is k-Lindelöf iff every open k-shrinkable cover contains a countable k-shrinkable subcover.*

Proof. If every open k-shrinkable cover contains a countable k-shrinkable subcover then Lemma 3.1 implies that X is k-Lindelöf.

Now let X be k-Lindelöf and fix an open k-shrinkable cover $\mathcal{U}$ and a function C confirming that. For each $U \in \mathcal{U}$ choose a $f_U \in C(X)$ such that $f_U[C(U)] \subseteq \{0\}$, $f_U[X \setminus U] \subseteq \{1\}$ and put $T(U) := |f_U|^{\leftarrow}[0, 1/2] \subseteq U$. Clearly $\mathbf{0}$ is in the closure of $A = \{f_U : U \in \mathcal{U}\}$ with respect to the co-topology. By Lemma 1.1, $C_k(X)$ has countable tightness so there is a countable $B \subseteq A$ such that $\mathbf{0} \in \overline{B}$ with respect to the co-topology. By the construction of A there is a countable $\mathcal{V} \subseteq \mathcal{U}$ with $B = \{f_U : U \in \mathcal{V}\}$. Then T witnesses that $\mathcal{V}$ is k-shrinkable.

Indeed, let $K \subseteq X$ be compact. There is an $h \in B \cap O(K, 1/2)$ and a $U \in \mathcal{V}$ with $h = f_U$. Then $|f_U|[K] = |h|[K] \subseteq [0, 1/2)$ so $K \subseteq |f_U|^{\leftarrow}[0, 1/2] = T(U)$. ∎

Using the techniques as in [12] one can show

Proposition 3.1. *The following are equivalent:*

(1) *$C_k(X)$ is a Pytkeev space;*

(2) *If $\mathcal{U}$ is a nontrivial 3-k-shrinkable cover of X, there is a sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of countably infinite subfamilies of $\mathcal{U}$ such that $\{\bigcap \mathcal{U}_n : n \in \mathbf{N}\}$ is a k-cover of X.*

For two sets $\mathcal{A}$ and $\mathcal{B}$ the formula $Pyt(\mathcal{A}, \mathcal{B})$ (see [12]) abbreviates the statement:

> for each sequence $(A_n : n \in \mathbf{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbf{N})$, where each B_n is a countably infinite subset of A_n, such that for each $f \in \prod(B_n : n \in \mathbf{N})$ the set $\{f(n) : n \in \mathbf{N}\}$ is an element of $\mathcal{B}$.

For some topological properties it is possible to consider their "selective" versions (see e.g. [12,13]). In this paper we are interested in the selective bitopological versions of the Pytkeev and the Reznichenko properties.

Let τ_1 and τ_2 be two topologies on the same set X with $\tau_2 \subseteq \tau_1$. X has the selectively (τ_1, τ_2)-*Pytkeev property* at $x \in X$ if for each sequence $(A_n : n \in \mathbf{N})$ of subsets of X and $x \in \cap_{n\in\mathbf{N}}(\overline{A_n} \setminus A_n)$, with respect to the topology τ_1, there exists a sequence $(B_n : n \in \mathbf{N})$ such that each B_n is an infinite and countable subset of A_n and $\{B_n : n \in \mathbf{N}\}$ is a π-network at x, with respect to the topology τ_2. If this holds for each point of X then we say that X has the selectively (τ_1, τ_2)-*Pytkeev property*. This property has been already considered in the context of hyperspaces in [7]. In further text if $X = C(Y)$ for a space Y, τ_1 is the corresponding compact-open topology and τ_2 is the corresponding topology of pointwise convergence, then the letters k and p will denote τ_1 and τ_2, respectively, in the above notation. Obviously, $Pyt((\Omega_x)^{\tau_1}, (\Omega_x)^{\tau_2})$ is another way of saying that X has the selectively (τ_1, τ_2)-*Pytkeev property* at $x \in X$.

We now characterize this bitopological property considered on the set $C(X)$.

THEOREM 3.2. *For a k-Lindelöf space X the following are equivalent:*

(1) $C(X)$ *satisfies the selectively* (k, p)*-Pytkeev property;*

(2) $Pyt(\mathcal{K}_{shr}, \Omega)$ *holds.*

Proof. (1) $\Rightarrow$ (2). Let us first remark that X is normal.

Let $\mathcal{U}$ be a nontrivial k-shrinkable open cover of X and C a function such that for each $U \in \mathcal{U}$, $C(U) \subseteq U$, $C(U)$ is closed and such that $\{C(V) : V \in \mathcal{U}\} \in \mathcal{K}$. List injectively the finite subsets of X as $(F_\alpha : \alpha < |X|)$. Choose a $U_0 \in \mathcal{U}$ with $F_0 \subseteq C(U_0)$ and $f_0 \in C(X)$ with $f_0[C(U_0)] \subseteq \{0\}$, $f_0[X \setminus U_0] \subseteq \{1\}$. If (U_β, f_β) have been defined for all $\beta < \alpha$, so that $f_\beta \in C(X)$, $C(U_\beta) \subseteq f_\beta^{\leftarrow}\{0\}$, $X \setminus U_\beta \subseteq f_\beta^{\leftarrow}\{1\}$ and for all $\beta_1 < \beta_2 < \alpha$ $f_{\beta 1} \neq f_{\beta_2}$, $U_{\beta_1} \neq U_{\beta_2}$, proceed the recursive definition as follows: if $\{f_\beta^{\leftarrow}\{0\} : \beta < \alpha\}$ k-covers X then end defining; if $\{f_\beta^{\leftarrow}\{0\} : \beta < \alpha\}$ does not k-cover X take a finite $T_\alpha \subseteq X$ with $T_\alpha \subseteq f_\beta^{\leftarrow}\{0\}$ for no $\beta < \alpha$, a $U_\alpha \in \mathcal{U}$ with $T_\alpha \cup F_\alpha \subseteq C(U_\alpha)$ and $f_\alpha \in C(X)$ such that $C(U_\alpha) \subseteq f_\alpha^{\leftarrow}\{0\}$, $X \setminus U_\alpha \subseteq f_\alpha^{\leftarrow}\{1\}$; it is clear that $f_\alpha \neq f_\beta$ for all $\beta < \alpha$ and also, for each $\beta < \alpha$ we must have that $U_\alpha \neq U_\beta$ because otherwise $T_\alpha \subseteq C(U_\alpha) = C(U_\beta) \subseteq f_\beta^{\leftarrow}\{0\}$ for a $\beta < \alpha$, which is impossible. Having finished this recursive definition there is a $\beta_0 < \alpha$ such that $\{f_\beta^{\leftarrow}\{0\} : \beta < \beta_0\}$ k-covers X and with $U_{\beta_1} \neq U_{\beta_2}$ and $f_{\beta 1} \neq f_{\beta_2}$ for each $\beta_1 < \beta_2 < \beta_0$. Therefore, the function $g_\mathcal{U}$ such that for every $\beta < \beta_0$ $(g_\mathcal{U}(U_\beta) = f_\beta)$ and $dom(g_\mathcal{U}) \subseteq \mathcal{U}$ is correctly defined. For $g_\mathcal{U}$ the following hold: $dom(g_\mathcal{U}) \subseteq \mathcal{U}$, $ran(g_\mathcal{U}) \subseteq C(X)$, $\{g_\mathcal{U}(U)^{\leftarrow}\{0\} : U \in \underline{dom(g_\mathcal{U})}\}$ k-covers X and $X \setminus U \subseteq g_\mathcal{U}(U)^{\leftarrow}\{1\}$ for all $U \in dom(g_\mathcal{U})$. Clearly $\mathbf{0} \in \overline{ran(g_\mathcal{U})}$ with respect to the co-topology and as $X \notin \mathcal{U}$, $ran(g_\mathcal{U})$ does not contain the function $\mathbf{0}$.

Now let $C(X)$ satisfy the selectively (k, p)-Pytkeev property and let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of elements of $\mathcal{K}_{shr}$. For each n associate to $\mathcal{U}_n$ a function $g_{\mathcal{U}_n}$ as described above. $\mathbf{0} \in \overline{ran(g_{\mathcal{U}_n})} \setminus ran(g_{\mathcal{U}_n})$ with respect to the co-topology for each n, so by the selectively (k, p)-Pytkeev property, there is a sequence $(B_n : n \in \mathbf{N})$ with each B_n an infinite subset of $ran(g_{\mathcal{U}_n})$ such that $\{B_n : n \in \mathbf{N}\}$ is a π-network at $\mathbf{0}$ with respect to the pointwise-convergence topology. For each n there is an infinite $\mathcal{V}_n \subseteq dom(g_{\mathcal{U}_n}) \subseteq \mathcal{U}_n$ with $B_n = \{g_{\mathcal{U}_n}(U) : U \in \mathcal{V}_n\}$. We show that $\{\cap\mathcal{V}_n : n \in \mathbf{N}\} \in \Omega$.

Let F be a finite subset of X. As $\{B_n : n \in \mathbf{N}\}$ is a π-network at $\mathbf{0}$ with respect to the pointwise-convergence topology there is $n_0 \in \mathbf{N}$ such that $B_{n_0} \subseteq O(F,1)$. This actually means that $g_{\mathcal{U}_{n_0}}(U)[F] \subseteq (-1,1)$ for all $U \in \mathcal{V}_{n_0}$. But $g_{\mathcal{U}_{n_0}}(U)[X \setminus U] \subseteq \{1\}$ for each $U \in \mathcal{V}_{n_0}$. Therefore $F \subseteq U$ for every $U \in \mathcal{V}_{n_0}$, i.e. $F \subseteq \cap\mathcal{V}_{n_0}$.

(2) $\Rightarrow$ (1). Let $(A_n : n \in \mathbf{N})$ be a sequence of subsets of $C(X)$ with $\mathbf{0} \in \overline{A_n} \setminus A_n$, with respect to the co-topology, for each n.

If $A \subseteq C_k(X)$ and $\mathbf{0} \in \overline{A} \setminus A$, call A *small* if for every $\varepsilon > 0$ there is an $f \in A$ such that $|f|[X] \subseteq [0,\varepsilon)$. For a *small* A, given any $\delta > 0$ one can find an *injective* sequence $(f_n(A,\delta) : n \in \mathbf{N})$ of elements of A such that $|f_n(A,\delta)|[X] \subseteq [0,\delta)$ for all $n \in \mathbf{N}$.

If $A \subseteq C_k(X)$, $\mathbf{0} \in \overline{A} \setminus A$ and A is not *small*, there is a positive real number $\delta(A) > 0$ such that for each $f \in A$ we have $|f|^{\leftarrow}[0,\delta(A)) \neq X$.

Put $S := \{n \in \mathbf{N} : A_n \text{ is } \mathit{small}\}$.

Case 1. S is infinite. For each $n \in S$ let $B_n := \{f_m(A_n, 1/n) : m \in \mathbf{N}\}$. If $n \notin S$ choose arbitrarily an infinite $B_n \subseteq A_n$. Obviously for each $n \in \mathbf{N}$ B_n is an infinite subset of A_n. We show that $\{B_n : n \in \mathbf{N}\}$ is a π-network at $\mathbf{0}$ with respect to the pc-topology.

Let F be a finite subset of X and $\varepsilon > 0$. Take $n_0 \in \mathbf{N}$ with $1/n_0 < \varepsilon$ and $n \in S$ with $n \geq n_0$. Then $B_n \subseteq O(F,\varepsilon)$: if $h \in B_n$ then $h = f_m(A_n, 1/n)$ for a $m \in \mathbf{N}$, so $h[X] \subseteq (-1/n, 1/n) \subseteq (-\varepsilon,\varepsilon)$, i.e. $h \in O(F,\varepsilon)$.

Case 2. S is finite. Let $m_0 := \max S$. Fix $n > m_0$. The set A_n is not *small* thus $X \neq |f|^{\leftarrow}[0,\delta(A_n))$ for each $f \in A_n$. Let $\delta_n := \min\{\delta(A_n), 1/n\}$. Then $\delta_n \leq 1/n$ and if $f \in A_n$ we have that $X \neq |f|^{\leftarrow}[0,\delta_n)$. By Lemma 3.2 there is a function U_n with $dom(U_n) \subseteq A_n$ such that $\mathcal{U}_n := \{U_n(f) : f \in dom(U_n)\}$ is an open 3-k-shrinkable cover of X and such that $U_n(f) \subseteq |f|^{\leftarrow}[0,\delta_n)$ for each $f \in dom(U_n)$. $X \notin \{|f|^{\leftarrow}[0,\delta_n) : f \in A_n\}$ so $\mathcal{U}_n$ is nontrivial.

Apply the principle $Pyt(\mathcal{K}_{shr},\Omega)$ to $(\mathcal{U}_n : n > m_0)$ to get a sequence $(\mathcal{V}_n : n > m_0)$ with $\mathcal{V}_n \subseteq \mathcal{U}_n$, $|\mathcal{V}_n| = \omega$ for each $n > m_0$ and such that $\{\cap\mathcal{V}_n : n > m_0\} \in \Omega$. Then for each $n > m_0$ there is an infinite $B_n \subseteq A_n$ with $\{U_n(f) : f \in B_n\} = \mathcal{V}_n$. For $n \leq m_0$ choose any infinite $B_n \subseteq A_n$. We show that $\{B_n : n \in \mathbf{N}\}$ is a π-network at $\mathbf{0}$ with respect to the pc-topology.

Let F be a finite subset of X and $\varepsilon > 0$. Take $n_0 \in \mathbf{N}$ with $1/n_0 < \varepsilon$ and a finite $F_0 \subseteq X$ such that there is no $n \in \mathbf{N}$, $m_0 < n < n_0$, with $F_0 \subseteq \cap\mathcal{V}_n$. As $\{\cap\mathcal{V}_n : n > m_0\} \in \Omega$ there is a $k > m_0$ with $F \cup F_0 \subseteq \cap\mathcal{V}_k$. By the construction of F_0 we have $k \geq n_0$. Also, $F \subseteq \cap\{U_k(f) : f \in B_k\}$, so for each $f \in B_k$ we have $F \subseteq U_k(f) \subseteq |f|^{\leftarrow}[0,\delta_k) \subseteq |f|^{\leftarrow}[0,1/k) \subseteq |f|^{\leftarrow}[0,1/n_0) \subseteq |f|^{\leftarrow}[0,\varepsilon)$, i.e. $f \in O(F,\varepsilon)$. In other words $B_k \subseteq O(F,\varepsilon)$. ∎

A space X is called a (τ_1,τ_2)-*Pytkeev space*, if whenever $x \in \overline{A} \setminus A$ with respect to the τ_1 topology, there is a countable π-network at x with respect to the τ_2 topology consisting of infinite subsets of A.

Theorem 3.3. For a k-Lindelöf space X the following are equivalent:

(1) *$C(X)$ is a (k,p)-Pytkeev space;*

(2) *If $\mathcal{U}$ is a k-shrinkable open nontrivial cover of X, there is a sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of subfamilies of $\mathcal{U}$ such that $|\mathcal{U}_n| = \omega$ for each n and $\{\cap\mathcal{U}_n\}_{n\in\mathbf{N}}$ is an ω-cover of X.*

Proof. Practically repeat the proof of the previous theorem. ∎

4. The Reznichenko-type properties

In 1996 Reznichenko (at a seminar at the Moscow State University) introduced the following property of a space X:

For each $x \in X$ and $A \subseteq X$ with $x \in \overline{A}\setminus A$, there is a countably infinite pairwise disjoint family $\mathcal{F}$ of finite subsets of A such that for every neighborhood V of x the family $\{F \in \mathcal{F} : F \cap V = \emptyset\}$ is finite.

This property is referred to as the *weakly Fréchet-Urysohn property* [14,15], or the *Reznichenko property* [7,10]. Let us remark that every Pytkeev space is a Reznichenko space (see [11]).

In [14] it was shown

THEOREM 4.1. *For a space X the following are equivalent:*

(1) *$C_p(X)$ is a Reznichenko space;*

(2) *If $\mathcal{U}$ is a nontrivial ω-shrinkable open cover of X, then there is a sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of pairwise disjoint finite subsets of $\mathcal{U}$ such that for each finite F the set $\{n \in \mathbf{N} : F \subseteq U$ for some $U \in \mathcal{U}_n\}$ is cofinite in $\mathbf{N}$.*

In a similar way one can prove

THEOREM 4.2. *For a space X the following are equivalent:*

(1) *$C_k(X)$ is a Reznichenko space;*

(2) *If $\mathcal{U}$ is a nontrivial k-shrinkable open cover of X, there is a sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of pairwise disjoint finite subsets of $\mathcal{U}$ such that for each compact set K of X the set $\{n \in \mathbf{N} : K \subseteq U$ for some $U \in \mathcal{U}_n\}$ is cofinite in $\mathbf{N}$.*

Let τ_1 and τ_2 be two topologies on the same set X with $\tau_2 \subseteq \tau_1$. A space X satisfies the *selectively (τ_1, τ_2)-Reznichenko property* if for each $x \in X$ and each sequence $(A_n : n \in \mathbf{N})$ of subsets of X with $x \in \cap_{n\in\mathbf{N}}(\overline{A_n} \setminus A_n)$ with respect to the topology τ_1, there exists a sequence $(B_n : n \in \mathbf{N})$ such that B_n is a finite subset of A_n for each n, B_n and B_m are disjoint for distinct m and n and for every neighborhood V of x, with respect to the topology τ_2, the set $\{n \in \mathbf{N} : B_n \cap V = \emptyset\}$ is finite. In further text if $X = C(Y)$ for a space Y, τ_1 is the corresponding compact-open topology and τ_2 is the corresponding topology of pointwise convergence, then the letters k and p will stand for τ_1 and τ_2, respectively, in the above notation.

In [7] this property has been considered in the context of hyperspaces.

We now borrow some terminology from [12]. Let $\mathcal{A}$ and $\mathcal{F}$ be two sets (here one may look at $\mathcal{F}$ as a "list of certain properties"). $HL_0(\mathcal{A},\mathcal{F})$ denotes the following statement:

for each sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ such that each $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, if $n \neq m$ then $\mathcal{V}_n \cap \mathcal{V}_m = \emptyset$, and for every $F \in \mathcal{F}$ there exists n_0 such that for every $n \geq n_0$ there is a $U \in \mathcal{V}_n$ such that $U \in F$.

If we do not require that the $\mathcal{V}_n$-s must be pairwise disjoint we obtain the principle denoted by $HL(\mathcal{A},\mathcal{F})$. The corresponding games $GameHL_0(\mathcal{A},\mathcal{F})$ and $GameHL(\mathcal{A},\mathcal{F})$ are defined as it is customary with selection principles.

The next two general results will considerably simplify further study in this section.

Proposition 4.1. [12] *If for every $X \in \mathcal{A}$ there exists $Y \subseteq X$ such that $Y \in \mathcal{A}$ and $card(Y) = \omega$ and if $HL(\mathcal{A},\mathcal{F})$ holds, then ONE has no winning strategy in the game $GameHL(\mathcal{A},\mathcal{F})$.*

Proposition 4.2. [12] *Let for each $X \in \mathcal{A}$ and each finite set Y, $X \setminus Y \in \mathcal{A}$ hold. Then: ONE has no winning strategy in the game $GameHL(\mathcal{A},\mathcal{F})$ iff he has no winning strategy in the game $GameHL_0(\mathcal{A},\mathcal{F})$.*

Note 4.1. Obviously, if ONE has no winning strategy in the game $GameHL(\mathcal{A},\mathcal{F})$ $(GameHL_0(\mathcal{A},\mathcal{F}))$, then $HL(\mathcal{A},\mathcal{F})$ $(HL_0(\mathcal{A},\mathcal{F}))$ holds. Also, $HL_0(\mathcal{A},\mathcal{F})$ implies $HL(\mathcal{A},\mathcal{F})$. Thus, if $\mathcal{A}$ satisfies both the condition of Proposition 4.1 and Proposition 4.2, then $HL(\mathcal{A},\mathcal{F})$ is equivalent to $HL_0(\mathcal{A},\mathcal{F})$. As a consequence of this we have that the following holds:

Let τ_1 and τ_2 be two topologies on the same set X with $\tau_2 \subseteq \tau_1$ such that (X,τ_1) has countable tightness. If $x \in X$, for $\mathcal{A} = (\Omega_x)^{\tau_1}$ and a suitable $\mathcal{F}$, we obtain that X has the selectively (τ_1,τ_2)-Reznichenko property at x iff for each sequence $(A_n : n \in \mathbf{N})$ of subsets of X and $x \in \cap_{n\in\mathbf{N}}(\overline{A_n} \setminus A_n)$, with respect to the topology τ_1, there exists a sequence $(B_n : n \in \mathbf{N})$ such that B_n is a finite subset of A_n for each n and for every neighborhood V of x, with respect to the topology τ_2, the family $\{n \in \mathbf{N} : B_n \cap V = \emptyset\}$ is finite, i.e. such that the sequence $(B_n : n \in \mathbf{N})$ converges to x with respect to the τ_2 topology. Note that the B_n-s do not have to be pairwise disjoint. ■

We state our next bitopological result.

Theorem 4.3. *Let X be k-Lindelöf. Then the following are equivalent:*

(1) *$C(X)$ satisfies the selectively (k,p)-Reznichenko property;*

(2) *If $(\mathcal{U}_n : n \in \mathbf{N})$ is a sequence of nontrivial k-shrinkable open covers of X, there is a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ for each n, $\mathcal{V}_n$ and $\mathcal{V}_m$ are disjoint for distinct m and n and for each finite $F \subseteq X$ the set $\{n \in \mathbf{N} : K \subseteq U$ for some $U \in \mathcal{V}_n\}$ is cofinite in $\mathbf{N}$.*

Proof. (1) $\Rightarrow$ (2): As X is k-Lindelöf, by Note 4.1 and Lemma 3.3 we only need to show that for each sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of nontrivial k-shrinkable open covers of X there is a sequence $(\mathcal{R}_n : n \in \mathbf{N})$, with each $\mathcal{R}_n$ a finite subset of $\mathcal{U}_n$, such that for each finite $F \subseteq X$ for all but finitely many $n \in \mathbf{N}$ the set $\{U \in \mathcal{R}_n : F \subseteq U\}$ is not empty.

Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of nontrivial k-shrinkable open covers of X. For each $n \in \mathbf{N}$ associate to $\mathcal{U}_n$ in the way we did in Theorem 3.2 a function g_n such that $dom(g_n) \subseteq \mathcal{U}_n$, $ran(g_n) \subseteq C(X)$, $\{g_n(U)^{\leftarrow}\{0\} : U \in dom(g_n)\}$ k-covers X and $X \setminus U \subseteq g_n(U)^{\leftarrow}\{1\}$ for all $U \in dom(g_n)$. Clearly $\mathbf{0} \in \overline{ran(g_n)}$ with respect to the co-topology and as $X \notin \mathcal{U}_n$, $ran(g_n)$ does not contain the function $\mathbf{0}$. Apply the selectively (k,p)-Reznichenko property of $C(X)$ to the sequence $(ran(g_n) : n \in \mathbf{N})$ to obtain a sequence $(R_n : n \in \mathbf{N})$, with each R_n a finite subset of $ran(g_n)$, converging to $\mathbf{0}$ with respect to the pc-topology. For each n there is a finite subset $\mathcal{R}_n$ of $dom(g_n) \subseteq \mathcal{U}_n$ with $\{g_n(U) : U \in \mathcal{R}_n\} = R_n$. $C_k(X)$ has countable tightness, so by Note 4.1 it suffices to show that for each finite $F \subseteq X$ for all but finitely many $n \in \mathbf{N}$ the set $\{U \in \mathcal{R}_n : F \subseteq U\}$ is not empty, so fix such an F. As $(R_n : n \in \mathbf{N})$ converges to $\mathbf{0}$ with respect to the pc-topology there is $n_0 \in \mathbf{N}$ such that for all $n > n_0$ the set $O(F,1) \cap R_n$ is not empty. Fix $n > n_0$. There is an $f \in O(F,1) \cap R_n$ and a $U \in \mathcal{R}_n$ with $g_n(U) = f$. Since $g_n(U)[F] = f[F] \subseteq (-1,1)$ and $X \setminus U \subseteq g_n(U)^{\leftarrow}\{1\}$, it follows that $F \subseteq U$. Thus $\{U \in \mathcal{R}_n : F \subseteq U\}$ is not empty.

(2) $\Rightarrow$ (1): As X is k-Lindelöf, by Note 4.1 and Lemma 1.1 we only need to show that for each sequence $(A_n : n \in \mathbf{N})$ of subsets of $C(X)$ with $x \in \cap_{n\in\mathbf{N}}(\overline{A_n} \setminus A_n)$ with respect to the co-topology, there exists a sequence $(B_n : n \in \mathbf{N})$ converging to $\mathbf{0}$ with respect to the pc-topology, such that for each n, B_n is a finite subset of A_n.

Let $(A_n : n \in \mathbf{N})$ be a sequence of subsets of $C(X)$ with $\mathbf{0} \in \overline{A_n} \setminus A_n$, with respect to the co-topology, for each n. By Lemma 3.2 for each $n \in \mathbf{N}$ there is a function U_n with $dom(U_n) \subseteq A_n$ such that $\mathcal{U}_n := \{U_n(f) : f \in dom(U_n)\}$ is an open 3-k-shrinkable cover of X and such that $U_n(f) \subseteq |f|^{\leftarrow}[0,1/n)$ for each $f \in dom(U_n)$. Set $S := \{n \in \mathbf{N} : X \notin \mathcal{U}_n\}$.

Case 1. S is finite. Then there would be a sequence $(f_n : n > \max S)$, with $f_n \in A_n$ for each $n > \max S$, uniformly converging to $\mathbf{0}$ so this would end the proof.

Case 2. S is infinite. For each $n \notin S$ pick an $f_n \in A_n$ with $f_n[X] \subseteq (-1/n, 1/n)$. As for each $n \in S$ the cover $\mathcal{U}_n$ is nontrivial, we can apply the condition (2) of this theorem to the sequence $(\mathcal{U}_n : n \in S)$ to get a sequence $(\mathcal{V}_n : n \in S)$, where for each $n \in S$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, such that for each finite $F \subseteq X$ for all but finitely many $n \in S$ the set $\{U \in \mathcal{V}_n : F \subseteq U\}$ is not empty. For each $n \in S$ there is a finite $C_n \subseteq dom(U_n) \subseteq A_n$ with $\mathcal{V}_n = \{U_n(f) : f \in C_n\}$. Put $B_n := C_n$ if $n \in S$ and $B_n := \{f_n\}$ if $n \notin S$. We show that $(B_n : n \in \mathbf{N})$ is as required.

Let F be a finite subset of X and $\varepsilon > 0$. By the construction of $(\mathcal{V}_n : n \in S)$ there is $n_0 \in S$ such that for each $n \in S$ with $n > n_0$ the set $\{U \in \mathcal{V}_n : F \subseteq U\}$ is not empty. Without loss of generality we may suppose that $1/n_0 < \varepsilon$. Fix $n > n_0$. If $n \notin S$ then $f_n[F] \subseteq f_n[X] \subseteq (-1/n, 1/n) \subseteq (-\varepsilon, \varepsilon)$, so $O(F,\varepsilon) \cap B_n$ is not empty. If $n \in S$ then there is a $U \in \mathcal{V}_n$ with $F \subseteq U$ and an $f \in C_n$ with $U = U_n(f)$. Since

$F \subseteq U = U_n(f) \subseteq |f|^{\leftarrow}[0,1/n) \subseteq f^{\leftarrow}(-\varepsilon,\varepsilon)$, thus again $O(F,\varepsilon)\cap B_n = O(F,\varepsilon)\cap C_n$ is not empty. ■

A space X has the (τ_1,τ_2)-*Reznichenko property* if $A \subseteq X$ and $x \in \overline{A}\setminus A$ with respect to τ_1 topology imply the existence of a countably infinite disjoint family $\mathcal{F}$ of subsets of A such that for every neighborhood V of x with respect to the τ_2 topology, the family $\{F \in \mathcal{F} : F \cap V = \emptyset\}$ is finite.

THEOREM 4.4. *For a k-Lindelöf space X the following are equivalent:*

(1) *$C(X)$ has the (k,p)-Reznichenko property;*

(2) *If $\mathcal{U}$ is a nontrivial k-shrinkable open k-cover of X, there is a sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of pairwise disjoint finite subsets of $\mathcal{U}$ such that for each finite set F the set $\{n \in \mathbf{N} : F \subseteq U$ for some $U \in \mathcal{U}_n\}$ is cofinite in $\mathbf{N}$.*

Proof. (1) $\Rightarrow$ (2): If $\mathcal{U} \in \mathcal{K}_{shr}$ then as in Theorem 3.2 construct a function g with $dom(g_{\mathcal{U}}) \subseteq \mathcal{U}$, $ran(g) \subseteq C(X)$, $X \setminus U \subseteq g(U)^{\leftarrow}\{1\}$ for all $U \in dom(g)$ and $\{g(U)^{\leftarrow}\{0\} : U \in dom(g)\}$ k-covers X. We have that $\mathbf{0} \in \overline{ran(g_{\mathcal{U}})} \setminus ran(g)$ with respect to the co-topology, so we can apply the condition (1) to $ran(g)$ to get a sequence $(R_n : n \in \mathbf{N})$ of finite pairwise disjoint subsets of $ran(g)$ converging to $\mathbf{0}$ with respect to the pc-topology. Pick an $h : ran(g) \to dom(g)$ with $h(g(U)) = U$ for each $U \in dom(g)$. Put $\mathcal{U}_n := \{h(a) : a \in R_n\}$. Then $(\mathcal{U}_n : n \in \mathbf{N})$ is a sequence of finite pairwise disjoint subsets of $\mathcal{U}$. It is not difficult to check using the methods of previous theorems that this sequence is as required.

(2) $\Rightarrow$ (1): Let the condition (2) be satisfied.

Claim. Let $\mathbf{0} \in \overline{A}\setminus A$ with respect to the co-topology and $\delta > 0$. Then there is a sequence $(A_n : n \in \mathbf{N})$ of pairwise disjoint finite subsets of A and a $B \subseteq A$, with $\mathbf{0} \in \overline{B}\setminus B$ with respect to the co-topology, such that $(\bigcup_{n\in\mathbf{N}} A_n) \cap B = \emptyset$ and the sequence $(A_n : n \in \mathbf{N})$ is δ-*converging to* $\mathbf{0}$, i.e. for each finite subset F of X for all but finitely many n the set $A_n \cap O(F,\delta)$ is not empty.

Proof of the claim. If for each $\varepsilon > 0$ there is an $f \in A$ with $f[X] \subseteq (-\varepsilon,\varepsilon)$, there exists an injective sequence $(f_n : n \in \mathbf{N})$ of elements of A uniformly converging to $\mathbf{0}$ so, in this case there is nothing to prove. Thus we may suppose that there is a $\delta_1 > 0$ with $f[X] \subseteq (-\delta_1,\delta_1)$ for no $f \in A$. Put $\delta_0 := \min\{\delta,\delta_1\}$.

By Lemma 4.1, let U be a function with $dom(U) \subseteq A$, $U(f) \subseteq |f|^{\leftarrow}[0,\delta_0)$ for every $f \in dom(U)$, such that $ran(U)$ is a 3-k-shrinkable open cover of X. By the assumption made above $ran(U)$ is nontrivial so by the condition (2) there is a sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of pairwise disjoint finite subsets of $ran(U)$ such that for each finite subset F of X for all but finitely many n the set $\{V \in \mathcal{U}_n : F \subseteq V\}$ is not empty. For each n there is a finite subset A_n of $dom(U) \subseteq A$ with $\mathcal{U}_n = \{U(f) : f \in A_n\}$. If $n \neq m$ and $g \in A_n \cap A_m$ then $U(g) \in \mathcal{U}_n \cap \mathcal{U}_m$, which is impossible. Thus $A_n \cap A_m = \emptyset$ for distinct n and m.

For each finite $F \subseteq X$ we have that for all but finitely many n the set $A_n \cap O(F,\delta)$ is not empty: there is an n_0 with $\{V \in \mathcal{U}_n : F \subseteq V\} \neq \emptyset$ for each $n > n_0$. Fix an $n > n_0$ and a $V \in \mathcal{U}_n$ with $F \subseteq V$. Then $V = U(f)$ for some $f \in A_n$. Hence $F \subseteq V = U(f) \subseteq |f|^{\leftarrow}[0,\delta_0) \subseteq |f|^{\leftarrow}[0,\delta)$, i.e. $f \in O(F,\delta) \cap A_n$.

If $\mathbf{0}$ belongs to the closure of the set $(A\setminus\bigcup_{n\in\mathbf{N}} A_n)\cup(\bigcup_{n\in\mathbf{N}} A_{2n})$ with respect to the co-topology then let $B := (A\setminus\bigcup_{n\in\mathbf{N}} A_n)\cup(\bigcup_{n\in\mathbf{N}} A_{2n})$ and let $C_n := A_{2n-1}$. If $\mathbf{0}$ belongs to the closure of the set $(A\setminus\bigcup_{n\in\mathbf{N}} A_n)\cup(\bigcup_{n\in\mathbf{N}} A_{2n-1})$ with respect to the co-topology then let $B := (A\setminus\bigcup_{n\in\mathbf{N}} A_n)\cup(\bigcup_{n\in\mathbf{N}} A_{2n-1})$ and let $C_n := A_{2n}$. Then it easy to see that the sequence $(C_n :\in \mathbf{N})$ and the set B are as required.

Now we prove the theorem. Let $\mathbf{0} \in \overline{A}\setminus A$ with respect to the co-topology. By the above *Claim* let $(H^1_n : n \in \mathbf{N})$ be a sequence of pairwise disjoint finite subsets of A which $1-$converges to $\mathbf{0}$ and $B_1 \subseteq A$ with $\mathbf{0} \in \overline{B_1}\setminus B_1$, $(\bigcup_{n\in\mathbf{N}} H^1_n)\cap B_1 = \emptyset$. If the sequences $(H^i_n : n \in \mathbf{N})$ and sets $B_i \subseteq A$ have been defined for $1 \le i \le k$ so that:

(*i*) $\bigcup\{H^i_n : 1 \le i \le k,\ n \in \mathbf{N}\}\cap B_k = \emptyset$;

(*ii*) $\mathbf{0} \in \overline{B_k}\setminus B_k$;

(*iii*) $(H^i_n : n \in \mathbf{N})$ $(1/i)-$converges to $\mathbf{0}$ for each $1 \le i \le k$,

then let by the above *Claim* $(H^{k+1}_n : n \in \mathbf{N})$ be a sequence of pairwise disjoint finite subsets of B_k which $1/(k+1)-$converges to $\mathbf{0}$ and $B_{k+1} \subseteq B_k$ with $\mathbf{0} \in \overline{B_{k+1}}\setminus B_{k+1}$, $(\bigcup_{n\in\mathbf{N}} H^{k+1}_n)\cap B_{k+1} = \emptyset$.

Having finished the construction we set $A_n := \bigcup\{H^i_n : 1 \le i \le n\}$. Obviously for distinct n and m the sets A_n and A_m are disjoint finite subsets of A. We show that $(A_n : n \in \mathbf{N})$ converges to $\mathbf{0}$ with respect to the pc-topology.

Let F be a finite subset of X and $\varepsilon > 0$. Fix an $m_0 > 1/\varepsilon$ and an $n_0 > m_0$ such that $H^{m_0}_n \cap O(F, 1/m_0) \ne \emptyset$ for every $n \ge n_0$. If $n \ge n_0$ then $H^{m_0}_n \subseteq \bigcup\{H^i_n : 1 \le i \le n\} = A_n$, so $\emptyset \ne A_n \cap O(F, 1/m_0) \subseteq A_n \cap O(F, \varepsilon)$. ■

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