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SPECTRAL MULTIPLICITY OF CERTAIN GAUSSIAN PROCESSES

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Abstract. In this paper we compare our conditions under which the spectral multiplicity of a Gaussian process given by an integral expression equals the one with the Cramer's regularity conditions [1].

1. Introduction

Let $x(t)$, $t \in (a,b) \subset \mathbf{R}$ be a second-order real-valued process with $Ex(t) = 0$ for each t. Let $H(x,t)$ be the linear closure generated by $x(s)$, $s \in (a,t]$ in the Hilbert space H of all random variables with finite variance ($Ex^2(t) < \infty$). We will suppose that $x(t)$, $t \in (a,b)$ is left-continuous and purely nondeterministic (i.e. $\bigcap_{t>a} H(x,t) = 0$).

Let such $x(t)$ be a Gaussian process given by an integral representation

$$x(t) = \int_a^t g(t,u)\,dz(u), \qquad u \leqslant t, \quad t \in [a,b], \tag{1}$$

where the kernel $g(t,u)$ and Gaussian process $z(u)$ satisfy the conditions 1 and 2:

1. The process $z(u)$ has orthogonal increments such that $Ez(u) = 0$ and $Ez^2u = F(u)$, where $F(u)$ is a non-decreasing function, left-continuous everywhere on (a,b).

2. The non-random function $g(t,u)$, $u \leqslant t$ is such that

$$E\,x^2(t) = \int_a^t g^2(t,u)\,dF(u) < \infty, \quad \text{for each } t \in (a,b).$$

If $H(x,t) = H(z,t)$, $t \in (a,b)$, the expansion (1) satisfying the conditions 1, 2 is the *canonical representation* for the process $x(t)$ and the *multiplicity* of $x(t)$ equals to one. In general ($H(x,t) \subset H(z,t)$), this representation may not be canonical and its multiplicity may be unknown. The main question here is to determine spectral

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multiplicity of $x(t)$. Before we consider this problem, let us note some very well known facts about equivalence of the Gaussian processes.

We say that Gaussian processes $x_1(t)$ and $x_2(t)$ are equivalent if the Gaussian measures P_{x_1} and P_{x_2} induced by $x_1(t)$ and $x_2(t)$ on the same space (X,β) (β is a Borel σ-field), are equivalent.

According to the fact that a Gaussian process is uniquely determined by the mean $Ex(t)$, $t\in T$, and the covariance function $B(s,t)=E(x(s)-Ex(s))(x(t)-Ex(t))$, $s,t\in T$, in order to find conditions for equivalence of two Gaussian processes, it is sufficient to consider two particular cases: a) the case of different means but the same covariance functions; and b) the case of the same means and different covariance functions (see [6]). Here, the case b) will be considered, because we have assumed that for our processes $Ex(t)=0$ for each t. In this case (see [6]), two Gaussian processes $x_1(t)$ and $x_2(t)$, given by (1), are equivalent if and only if there exists $y\in H(z_1)\otimes H(z_2)$, $y=\int_a^b\int_a^b h(u,v)\,dz_1(u)\,dz_2(v)$, such that

$$\int_a^b\int_a^b h^2(u,v)\,dF_1(u)\,dF_2(v)<\infty,$$

and the following equation is satisfied

$$B_1(s,t)-B_2(s,t)=\int_a^s\int_a^t h(u,v)g_1(s,u)g_2(t,v)\,dF_1(u)\,dF_2(v),\qquad s,t\in T, \quad (2)$$

where $B_i(s,t)$ are covariance functions of $x_i(t)=\int_a^t g_i(s,t)\,dz_i(u)$, $t\in T$.

For equivalent processes $x_1(t)$ and $x_2(t)$, the spectral multiplicity is the same (see [6]).

2. Main result

One of the problems here is to find out a criterion for processes given by (1) to have multiplicity equal to one. Here the main idea is to settle equivalence of two Gaussian processes one of which has already multiplicity one.

Lemma. *If two Gaussian processes $x_1(t)$ and $x_2(t)$ given by (1) are equivalent, then the discontinuities of the partial derivatives $B_1(s,t)$ and $B_2(s,t)$ at the diagonal $s=t$ must be the same,*

$$f_1(t)g_1^2(t,t)=f_2(t)g_2^2(t,t). \quad (3)$$

Proof. If the covariance functions $B_i(s,t)$ of these processes have continuous partial derivatives $\partial B_i(s,t)/\partial t$ and $\partial B_i(s,t)/\partial s$, then the equality holds for all s,t except for $s=t$. At $s=t$ there is a jump equal to

$$g_i^2(t,t)f_i(t),\qquad i=1,2,$$

(see [1], p. 18). For equivalent processes $x_1(t)$ and $x_2(t)$, the difference $B_1(s,t)-B_2(s,t)$, from (2), has everywhere continuous partial derivatives of the first order, so the discontinuities of partial derivatives of B_1 and B_2 must cancel out in the difference for each t. ∎

THEOREM. *Let $x(t)$, $t \in [a,b] = T$, be a process given by* (1), *where $z(u)$, $u \in [a,b]$, is a Gaussian process such that:*

A. the function $f(u) = \partial F(u)/\partial u = \partial E z^2(u)/\partial u$ is continuous and $f(u) \neq 0$, for all $t \in T$,

B. $g(t,t) \neq 0$, for all $t \in T$, and

C. $\dfrac{1}{f(t)} \left(\dfrac{g(t,u)}{g(t,t)}\right)'_t \in L^2(f(t)dt \times f(u)du)$, *i.e.*

$$\int_a^b \int_a^b \frac{1}{f(t)} \left(\left(\frac{g(t,u)}{g(t,t)} \right)'_t \right)^2 f(u)\, dt\, du < \infty,$$

then the process $x(t)$ has multiplicity one.

Proof. Let us introduce the process $y(t) = \int_a^t g(t,t)\, dz(u)$, $u \leqslant t$, $u,t \in T$, where $z(u)$ is the given Gaussian process. Now, the necessary condition for the equivalence of $x(t)$ and $y(t)$ is satisfied (see the previous Lemma and (3)).

The difference between their covariance functions is

$$B_1(s,t) - B_2(s,t) = \int_0^{s \wedge t} (g(s,u)g(t,u) - g(s,s)g(t,t)) f(u)\, du.$$

According to (2), to find out the necessary and sufficient condition for equivalence of $x(t)$ and $y(t)$, we have to solve the following integral equation, with $h(u,v)$ as the unknown function,

$$\int_a^{s \wedge t} \left(g(s,u) \frac{g(t,u)}{g(t,t)} - g(s,s) \right) f(u)\, du = \int_a^s \int_a^t h(u,v) g(s,u) f(u) f(v)\, du\, dv,$$

$s,t \in T$. If we suppose $\min\{s,t\} = s$, after some calculation we obtain for $u < s < t$,

$$h(u,t) = \frac{1}{f(t)} \left(\frac{g(t,u)}{g(t,t)} \right)'_t .$$

The same holds when we suppose $\min\{s,t\} = t$. Now, the necessary and sufficient condition for equivalence of processes $x(t)$ and $y(t)$ is

$$\int_a^b \int_a^b \frac{1}{f(t)} \left(\left(\frac{g(t,u)}{g(t,t)} \right)'_t \right)^2 f(u)\, dt\, du < \infty, \qquad u \leqslant t.$$

If this condition is satisfied, the spectral multiplicity of $x(t)$ and $y(t)$ will be the same and equal to one, because the process $y(t)$ has multiplicity one (see [2]). The proof is completed. ∎

3. Conclusion

The Cramer's regularity conditions R_1, R_2, R_3 (see [1], p. 13) and the property of canonical representation ensure the unit multiplicity for stochastic processes (Theorem 5.1 in [1]). We have shown (see [3]) that the characteristic of canonical representation was not a consequence of regularity conditions R_1, R_2, R_3 in the case of stochastic processes given by (1). So, the characteristic representation as an assumption cannot be omitted in Theorem 5.2 [1]. To ensure the unit multiplicity for Gaussian processes given by (1) it remains to use Theorem 5.1 from [1] and its conditions: canonical representation and regularity conditions R_1, R_2, R_3.

Note that there is no assumption of canonical representation (1) in our Theorem. Instead of Cramer's condition R_3 ($f(u)$ equals to zero at at most finite number of isolated points), we take the condition A: $f(u)$ is continuous and $f(u) \neq 0$, for all $t \in T$. As we have shown, A and the condition B and C imply that Gaussian processes $x(t)$ and $z(t)$ are equivalent and of the same spectral type. Moreover, it is clear that in our case of such $f(u)$, the conditions R_1 and R_2 imply the condition C. The converse does not hold.

So, according to the previous considerations, the next theorem holds.

THEOREM. *Let $x(t)$, $t \in [a,b] = T$, be a process given by* (1), *where $z(u)$, $u \in [a,b]$, is a Gaussian process. If $x(t)$ satisfies:*

R_1. the functions $g(t,u)$ and $\partial g(t,u)/\partial t$ are bounded and continuous for all $u,t \in T$,

R_2. $g(t,t) = 1$, for all $t \in T$, and

A. the function $f(u) = \partial F(u)/\partial u = \partial Ez^2(u)/\partial u$ is continuous and $f(u) \neq 0$, for all $t \in T$,

then the process $x(t)$ has multiplicity equal to one.

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