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INFINITESIMAL DEFORMATIONS OF CURVATURE TENSORS AT NON-SYMMETRIC AFFINE CONNECTION SPACE

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Abstract. At the present work we consider infinitesimal deformations of geometric objects, especially of curvature tensors, at a space L_N of non-symmetric affine connection.

0. Introduction

We consider a space L_N of a non-symmetric affine connection L^i_{jk} with the torsion tensor $T^i_{jk} = L^i_{jk} - L^i_{kj}$ at local coordinates x^i $(i = 1, \dots, N)$. Basic information on infinitesimal deformations one can find at [4–11].

Let x^i, $i = 1, \dots, N$ be local coordinates at L_N and let us define infinitesimal deformations of space L_N in the next manner.

Definition 0.1. A transformation $f\colon L_N \to L_N\colon x = (x^1, \dots, x^N) \equiv (x^i) \mapsto \bar{x} = (\bar{x}^1, \dots, \bar{x}^N) \equiv (\bar{x}^i)$, where

$$\bar{x} = x + z(x)\varepsilon, \tag{0.1}$$

or in local coordinates

$$\bar{x}^i = x^i + z^i(x^j)\varepsilon, \quad i, j = 1, \dots, N, \tag{0.1$'$}$$

where ε is an infinitesimal, is called *infinitesimal deformation of a space* L_N, determined by the vector $z = (z^i)$, which is called *the field of infinitesimal deformation* (0.1′).

Under deformation of a space according to (0.1′) some geometrical objects are deformed. The fact that geometric object is given at point x at local coordinates x^i, is denoted by $\mathcal{A}(i, x)$. Under deformation (0.1′) we get geometric object $\bar{\mathcal{A}}$,

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which can be considered at the same point and at the same coordinate system (see [6,7,10,11]).

REMARK 0.1. In this study of infinitesimal deformations according to (0.1′), quantities of an order higher then the first with respect to ε are neglected.

DEFINITION 0.2. The magnitude $\mathcal{DA}$, the difference between deformed object $\bar{\mathcal{A}}$ and initial object $\mathcal{A}$ at the same coordinate system and at the same point with respect to (0.1′), i.e.

$$\mathcal{DA} = \bar{\mathcal{A}}(i,x) - \mathcal{A}(i,x), \tag{0.2}$$

is called *Lie difference (Lie differential)*, and the magnitude

$$\mathcal{L}_z\mathcal{A} = \lim_{\varepsilon\to 0}\frac{\mathcal{DA}}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{\bar{\mathcal{A}}(i,x) - \mathcal{A}(i,x)}{\varepsilon} \tag{0.2′}$$

is *Lie derivative* of geometric object $\mathcal{A}(i,x)$ with respect to the vector field $z = (z^i(x^j))$.

Using the relation (0.2), for deformed object $\bar{\mathcal{A}}(i,x)$ we have

$$\bar{\mathcal{A}}(i,x) = \mathcal{A}(i,x) + \mathcal{DA}, \tag{0.2″}$$

and thus we can express $\bar{\mathcal{A}}$, finding previously $\mathcal{DA}$.

From (0.2′) we get

$$\frac{\mathcal{D}A}{\varepsilon} = \mathcal{L}_z\mathcal{A} + \varepsilon_1, \varepsilon_1 \to 0, \quad \text{if} \quad \varepsilon \to 0,$$

and $\mathcal{D}A = \varepsilon\mathcal{L}_z\mathcal{A} + \varepsilon\varepsilon_1$, i.e.

$$\mathcal{D}A = \varepsilon\mathcal{L}_z\mathcal{A}, \tag{0.3}$$

where $\varepsilon\varepsilon_1$, as an infinitesimal value of the higher order with respect to ε, is omitted.

As it is known (see [6,7]), for an arbitrary tensor $t^{i_1,\dots,i_u}_{j_1,\dots,j_v}$ we have

$$\begin{aligned}\mathcal{D}t^{i_1\dots i_u}_{j_1\dots j_v} &= [t^{i_1\dots i_u}_{j_1\dots j_v,p}z^p - \sum_{\alpha=1}^{u} z^{i_\alpha}_{,p}\binom{p}{i_\alpha}t^{i_1\dots i_u}_{j_1\dots j_v} + \sum_{\beta=1}^{v} z^p_{,j_\beta}\binom{j_\beta}{p}t^{i_1\dots i_u}_{j_1\dots j_v}]\varepsilon \\ &= \mathcal{L}_z t^{i_1\dots i_u}_{j_1\dots j_v}\varepsilon,\end{aligned} \tag{0.4}$$

where we denoted, for instance, $z^i_{,p} = \partial z^i/\partial x^p$, and

$$\binom{p}{i_\alpha}t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_{\alpha-1}p i_{\alpha+1}\dots i_u}_{j_1\dots j_v}, \qquad \binom{j_\beta}{p}t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_u}_{j_1\dots j_{\beta-1}p j_{\beta+1}\dots j_v}.$$

For the *connection coefficients* we have

$$\mathcal{D}L^i_{jk} = (L^i_{jk,p}z^p + z^i_{,jk} - z^i_{,p}L^p_{jk} + z^p_{,j}L^i_{pk} + z^p_{,k}L^i_{jp})\varepsilon = \mathcal{L}_z L^i_{jk}\varepsilon. \tag{0.5}$$

Because of non-symmetry of the connection, at L_N we can consider two types of covariant derivatives for a vector and four types for general tensor. So, denoting by $\underset{\theta}{|}(\theta = 1, \dots, 4)$ the derivative of the type θ, we have ([1]–[3]):

$$t^{i_1\dots i_u}_{j_1\dots j_v\underset{\substack{1\\2\\3\\4}}{|}m} = t^{i_1\dots i_u}_{j_1\dots j_v,m} + \sum_{\alpha=1}^{u} L^{i_\alpha}_{\substack{pm\\mp\\pm\\mp}} \binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} - \sum_{\beta=1}^{v} L^{p}_{\substack{j_\beta m\\mj_\beta\\mj_\beta\\j_\beta m}} \binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v}. \qquad (0.6a-d)$$

By virtue of Ricci-type identities twelve curvature tensors in L_N are obtained [1,2,3]. Among these tensors only five of them are independent:

$$\underset{1}{R}^i_{jmn} = L^i_{jm,n} - L^i_{jn,m} + L^p_{jm}L^i_{pn} - L^p_{jn}L^i_{pm}, \qquad (0.7)$$

$$\underset{2}{R}^i_{jmn} = L^i_{mj,n} - L^i_{nj,m} + L^p_{mj}L^i_{np} - L^p_{nj}L^i_{mp}, \qquad (0.8)$$

$$\underset{3}{R}^i_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{nm}T^i_{pj}, \qquad (0.9)$$

$$\underset{4}{R}^i_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{mn}T^i_{pj}, \qquad (0.10)$$

$$\underset{5}{R}^i_{jmn} = \frac{1}{2}(L^i_{jm,n} + L^i_{mj,n} - L^i_{jn,m} - L^i_{nj,m} + L^p_{jm}L^i_{pn} + L^p_{mj}L^i_{np} - L^p_{jn}L^i_{mp} - L^p_{nj}L^i_{pm}). \qquad (0.11)$$

If we define symmetric connection

$$\underset{0}{L}^i_{jk} = \frac{1}{2}(L^i_{jk} + L^i_{kj}), \qquad (0.12)$$

its curvature tensor

$$\underset{0}{R}^i_{jmn} = \underset{0}{L}^i_{jm,n} - \underset{0}{L}^i_{jn,m} + \underset{0}{L}^p_{jm}\underset{0}{L}^i_{pn} - \underset{0}{L}^p_{jn}\underset{0}{L}^i_{pm} \qquad (0.13)$$

is an ordinary Riemann-Christoffel's curvature tensor of symmetric connection. Denoting by semicolon (;) covariant derivative with respect to symmetric connection then according to [3], we have

$$\underset{1}{R}^i_{jmn} = \underset{o}{R}^i_{jmn} + \frac{1}{2}T^i_{jm;n} - \frac{1}{2}T^i_{jn;m} + \frac{1}{4}T^p_{jm}T^i_{pn} - \frac{1}{4}T^p_{jn}T^i_{pm}, \qquad (0.14)$$

$$\underset{2}{R}^i_{jmn} = \underset{0}{R}^i_{jmn} - \frac{1}{2}T^i_{jm;n} + \frac{1}{2}T^i_{jn;m} + \frac{1}{4}T^p_{jm}T^i_{pn} - \frac{1}{4}T^p_{jn}T^i_{pm}, \qquad (0.15)$$

$$\underset{3}{R}^i_{jmn} = \underset{0}{R}^i_{jmn} + \frac{1}{2}T^i_{jm;n} + \frac{1}{2}T^i_{jn;m} - \frac{1}{4}T^p_{jm}T^i_{pn} + \frac{1}{4}T^p_{jn}T^i_{pm} - \frac{1}{2}T^p_{mn}T^i_{pj}, \qquad (0.16)$$

$$\underset{4}{R}^i_{jmn} = \underset{0}{R}^i_{jmn} + \frac{1}{2}T^i_{jm;n} + \frac{1}{2}T^i_{jn;m} - \frac{1}{4}T^p_{jm}T^i_{pn} + \frac{1}{4}T^p_{jn}T^i_{pm} + \frac{1}{2}T^p_{mn}T^i_{pj} \qquad (0.17)$$

$$\underset{5}{R}^i_{jmn} = \underset{0}{R}^i_{jmn} + \frac{1}{4}T^p_{jm}T^i_{pn} + \frac{1}{4}T^p_{jn}T^i_{pm}. \qquad (0.18)$$

At (0.14–18) all the addends at the right side are tensors.

At the present work we will consider infinitesimal deformations of cited five curvature tensors.

1. Infinitesimal deformations of curvature tensor $\underset{1}{R}$

1.1. According to the (0.7) for deformed first curvature tensor at L_N, i.e. for $\underset{1}{\bar{R}}$, we have

$$\underset{1}{\bar{R}}{}^i_{jmn}(x) = \bar{L}^i_{jm,n} - \bar{L}^i_{jn,m} + \bar{L}^p_{jm}\bar{L}^i_{pn} - \bar{L}^p_{jn}\bar{L}^i_{pm}, \tag{1.1}$$

and with respect to $\bar{L}^i_{jm}(x) = L^i_{jm}(x) + \mathcal{D}L^i_{jm}$, we have

$$\begin{aligned}\underset{1}{\bar{R}}{}^i_{jmn} &= (L^i_{jm} + \mathcal{D}L^i_{jm})_{,n} - (L^i_{jn} + \mathcal{D}L^i_{jn})_{,m} \\ &\quad + (L^p_{jm} + \mathcal{D}L^p_{jm})(L^i_{pn} + \mathcal{D}L^i_{pn}) - (L^p_{jn} + \mathcal{D}L^p_{jn})(L^i_{pm} + \mathcal{D}L^i_{pm}).\end{aligned}$$

Developing this and omitting the members of the form $\mathcal{D}L^{\cdots}_{\cdots}\cdot\mathcal{D}L^{\cdots}_{\cdots}$, as they include $(\varepsilon)^2$, we have

$$\begin{aligned}\underset{1}{\bar{R}}{}^i_{jmn} &= L^i_{jm,n} + (\mathcal{D}L^i_{jm})_{,n} - L^i_{jn,m} - (\mathcal{D}L^i_{jn})_{,m} + L^p_{jm}L^i_{pn} \\ &\quad + L^p_{jm}\mathcal{D}L^i_{pn} + (\mathcal{D}L^p_{jm})L^i_{pn} - L^p_{jn}L^i_{pm} - L^p_{jn}\mathcal{D}L^i_{pm} - (\mathcal{D}L^p_{jn})L^i_{pm}.\end{aligned} \tag{1.2}$$

As $\mathcal{D}L^i_{jm}$ is a tensor, we can consider covariant derivative:

$$(\mathcal{D}L^i_{jm})_{\underset{1}{|}n} = (\mathcal{D}L^i_{jm})_{,n} + L^i_{pn}\mathcal{D}L^p_{jm} - L^p_{jn}\mathcal{D}L^i_{pm} - L^p_{mn}\mathcal{D}L^i_{jp}$$

where

$$(\mathcal{D}L^i_{jm})_{,n} + L^i_{pn}\mathcal{D}L^p_{jm} = (\mathcal{D}L^i_{jm})_{\underset{1}{|}n} + L^p_{jn}\mathcal{D}L^i_{pm} + L^p_{mn}\mathcal{D}L^i_{jp} \tag{1.3}$$

and in the same manner

$$(\mathcal{D}L^i_{jn})_{,m} + L^i_{pm}\mathcal{D}L^p_{jn} = (\mathcal{D}L^i_{jn})_{\underset{1}{|}m} + L^p_{jm}\mathcal{D}L^i_{pn} + L^p_{nm}\mathcal{D}L^i_{jp} \tag{1.3$'$}$$

If we have in mind (0.7), (1.3, 3′), the equation (1.2) becomes

$$\begin{aligned}\underset{1}{\bar{R}}{}^i_{jmn} &= \underset{1}{R}{}^i_{jmn} + (\mathcal{D}L^i_{jm})_{\underset{1}{|}n} + L^p_{jn}\mathcal{D}L^i_{pm} + L^p_{mn}\mathcal{D}L^i_{jp} \\ &\quad - (\mathcal{D}L^i_{jn})_{\underset{1}{|}m} - L^p_{im}\mathcal{D}L^i_{pn} - L^p_{nm}\mathcal{D}L^i_{jp} + L^p_{jm}\mathcal{D}L^i_{pn} - L^p_{jn}\mathcal{D}L^i_{pm},\end{aligned}$$

i.e.

$$\underset{1}{\bar{R}}{}^i_{jmn} = \underset{1}{R}{}^i_{jmn} + (\mathcal{D}L^i_{jm})_{\underset{1}{|}n} - (\mathcal{D}L^i_{jn})_{\underset{1}{|}m} + T^p_{mn}\mathcal{D}L^i_{jp}. \tag{1.4}$$

From here

$$\mathcal{D}\underset{1}{R}{}^i_{jmn} = \underset{1}{\bar{R}}{}^i_{jmn} - \underset{1}{R}{}^i_{jmn} = (\mathcal{D}L^i_{jm})_{\underset{1}{|}n} - (\mathcal{D}L^i_{jn})_{\underset{1}{|}m} + T^p_{mn}\mathcal{D}L^i_{jp}, \tag{1.4$'$}$$

and (dividing with ε):

$$\mathcal{L}_z\underset{1}{R}{}^i_{jmn} = (\mathcal{L}_zL^i_{jm})_{\underset{1}{|}n} - (\mathcal{L}_zL^i_{jn})_{\underset{1}{|}m} + T^p_{mn}\mathcal{L}_zL^i_{jp}. \tag{1.4$''$}$$

1.2. We can also start from (0.14), and then we have

$$\begin{aligned}\underset{1}{\bar R}{}^i_{jmn} &= \underset{0}{\bar R}{}^i_{jmn} + \frac12 \bar T^i_{jm;n} - \frac12 \bar T^i_{jn;m} + \frac14 \bar T^p_{jm}\bar T^i_{pn} - \frac14 \bar T^p_{jn}\bar T^i_{pm} \\ &= \underset{0}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac12 (T^i_{jm} + \mathcal{D}T^i_{jm})_{;n} - \frac12 (T^i_{jn} + \mathcal{D}T^i_{jn})_{;m} \\ &\quad + \frac14 (T^p_{jm} + \mathcal{D}T^p_{jm})(T^i_{pn} + \mathcal{D}T^i_{pn}) - \frac14 (T^p_{jn} + \mathcal{D}T^p_{jn})(T^i_{pm} + \mathcal{D}T^i_{pm}).\end{aligned}$$

Omitting the members containing $\mathcal{D}T \cdot \mathcal{D}T$, we get

$$\begin{aligned}\underset{1}{\bar R}{}^i_{jmn} &= \underset{0}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac12 T^i_{jm;n} + \frac12 (\mathcal{D}T^i_{jm})_{;n} - \frac12 T^i_{jn;m} - \frac12 (\mathcal{D}T^i_{jn})_{;m} \\ &\quad + \frac14 (T^p_{jm}T^i_{pn} + T^p_{jm}\mathcal{D}T^i_{pn} + \mathcal{D}T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm} - T^p_{jn}\mathcal{D}T^i_{pm} - \mathcal{D}T^p_{jn}T^i_{pm}).\end{aligned}$$

So, with respect to (0.14), one obtains

$$\underset{1}{\bar R}{}^i_{jmn} = \underset{1}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac12 (\mathcal{D}T^i_{jm})_{;n} - \frac12 (\mathcal{D}T^i_{jn})_{;m} + \frac14 \mathcal{D}(T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}), \quad (1.5)$$

wherefrom

$$\mathcal{D}\underset{1}{R}{}^i_{jmn} = \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac12 (\mathcal{D}T^i_{jm})_{;n} - \frac12 (\mathcal{D}T^i_{jn})_{;m} + \frac14 \mathcal{D}(T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}) \quad (1.5')$$

where ; denotes covariant derivative with respect to $\underset{0}{L}$ (0.12), and (dividing with ε):

$$\mathcal{L}_z\underset{1}{R}{}^i_{jmn} = \mathcal{L}_z\underset{0}{R}{}^i_{jmn} + \frac12 (\mathcal{L}_zT^i_{jm})_{;n} - \frac12 (\mathcal{L}_zT^i_{jn})_{;m} + \frac14 \mathcal{L}_z(T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}). \quad (1.5'')$$

2. Infinitesimal deformations of curvature tensor $\underset{2}{R}$

2.1. In the similar manner as for $\underset{1}{R}$, using (0.8), we get

$$\underset{2}{\bar R}{}^i_{jmn} = \underset{2}{R}{}^i_{jmn} + (\mathcal{D}L^i_{mj})_{\underset{2}{|}n} - (\mathcal{D}L^i_{nj})_{\underset{2}{|}m} + T^p_{nm}\mathcal{D}L^i_{pj}. \quad (2.1)$$

and corresponding expressions for $\mathcal{D}\underset{2}{R}{}^i_{jmn}$ and $\mathcal{L}_z\underset{2}{R}{}^i_{jmn}$.

2.1. Starting from (0.15), we obtain

$$\underset{2}{\bar R}{}^i_{jmn} = \underset{2}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} - \frac12 (\mathcal{D}T^i_{jm})_{;n} + \frac12 (\mathcal{D}T^i_{jn})_{;m} + \frac14 \mathcal{D}(T^p_{jm}T^i_{pn} - T^p_{jn}T^i_{pm}). \quad (2.2)$$

3. Infinitesimal deformations of curvature tensor $\underset{3}{R}$

3.1. Starting from (0.9), we obtain

$$\begin{aligned}\underset{3}{\bar R}{}^i_{jmn} &= L^i_{jm,n} + (\mathcal{D}L^i_{jm})_{,n} - L^i_{nj,m} - (\mathcal{D}L^i_{nj})_{,m} + L^p_{jm}L^i_{np} + L^p_{jm}\mathcal{D}L^i_{np} + \mathcal{D}L^p_{jm}L^i_{np} \\ &\quad - L^p_{nj}L^i_{pm} - L^p_{nj}\mathcal{D}L^i_{pm} - \mathcal{D}L^p_{nj}L^i_{pm} + L^p_{nm}T^i_{pj} + L^p_{nm}\mathcal{D}T^i_{pj} + \mathcal{D}L^p_{nm}T^i_{pj}. \quad (3.1)\end{aligned}$$

In relation to

$$(\mathcal{D}L^i_{jm})_{\underset{4}{|}n} = (\mathcal{D}L^i_{jm})_{,n} + L^i_{np}\mathcal{D}L^p_{jm} - L^p_{jn}\mathcal{D}L^i_{pm} - L^p_{mn}\mathcal{D}L^i_{jp},$$

$$(\mathcal{D}L^i_{nj})_{\underset{3}{|}m} = (\mathcal{D}L^i_{nj})_{,m} + L^i_{pm}\mathcal{D}L^p_{nj} - L^p_{mn}\mathcal{D}L^i_{pj} - L^p_{mj}\mathcal{D}L^i_{np},$$

one obtains respectively

$$(\mathcal{D}L^i_{jm})_{,n} + L^i_{np}\mathcal{D}L^p_{jm} = (\mathcal{D}L^i_{jm})_{\underset{4}{|}n} + L^p_{jn}\mathcal{D}L^i_{pm} + L^p_{mn}\mathcal{D}L^i_{jp},$$

$$(\mathcal{D}L^i_{nj})_{,m} + L^i_{pm}\mathcal{D}L^p_{nj} = (\mathcal{D}L^i_{nj})_{\underset{3}{|}m} + L^p_{mn}\mathcal{D}L^i_{pj} + L^p_{mj}\mathcal{D}L^i_{np}.$$

Using these equations and (0.9), the equation (3.1) becomes

$$\begin{aligned}\underset{3}{\bar{R}}{}^i_{jmn} = \underset{3}{R}{}^i_{jmn} &+ (\mathcal{D}L^i_{jm})_{\underset{4}{|}n} + L^p_{jn}\mathcal{D}L^i_{pm} + L^p_{mn}\mathcal{D}L^i_{jp} - (\mathcal{D}L^i_{nj})_{\underset{3}{|}m}\\ &- L^p_{mn}\mathcal{D}L^i_{pj} - L^p_{mj}\mathcal{D}L^i_{np} + L^p_{jm}\mathcal{D}L^i_{np} - L^p_{nj}\mathcal{D}L^i_{pm} + L^p_{nm}\mathcal{D}T^i_{pj} + T^i_{pj}\mathcal{D}L^p_{nm},\end{aligned}$$

that is

$$\begin{aligned}\underset{3}{\bar{R}}{}^i_{jmn} = \underset{3}{R}{}^i_{jmn} &+ (\mathcal{D}L^i_{jm})_{\underset{4}{|}n} - (\mathcal{D}L^i_{nj})_{\underset{3}{|}m}\\ &+ T^p_{jn}\mathcal{D}L^i_{pm} + T^p_{jm}\mathcal{D}L^i_{np} + L^p_{mn}\mathcal{D}T^i_{jp} + \mathcal{D}(T^i_{pj}L^p_{nm}). \qquad (3.2)\end{aligned}$$

3.2. According to (0.16), we get

$$\begin{aligned}\underset{3}{\bar{R}}{}^i_{jmn} = \underset{0}{R}{}^i_{jmn} &+ \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac{1}{2}T^i_{jm;n} + \frac{1}{2}(\mathcal{D}T^i_{jm})_{;n} + \frac{1}{2}T^i_{jn;m} + \frac{1}{2}(\mathcal{D}T^i_{jn})_{;m}\\ &- \frac{1}{4}T^p_{jm}T^i_{pn} - \frac{1}{4}T^p_{jm}\mathcal{D}T^i_{pn} - \frac{1}{4}\mathcal{D}T^p_{jm}T^i_{pn} + \frac{1}{4}T^p_{jn}T^i_{pm} + \frac{1}{4}T^p_{jn}\mathcal{D}T^i_{pm}\\ &+ \frac{1}{4}\mathcal{D}T^p_{jn}T^i_{pm} - \frac{1}{2}T^p_{mn}T^i_{pj} - \frac{1}{2}T^p_{mn}\mathcal{D}T^i_{pj} - \frac{1}{2}\mathcal{D}T^p_{mn}T^i_{pj}\end{aligned}$$

i.e.

$$\begin{aligned}\underset{3}{\bar{R}}{}^i_{jmn} = \underset{3}{R}{}^i_{jmn} &+ \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac{1}{2}(\mathcal{D}T^i_{jm})_{;n} + \frac{1}{2}(\mathcal{D}T^i_{jn})_{;m}\\ &+ \frac{1}{4}\mathcal{D}(T^i_{pm}T^p_{jn} - T^i_{pn}T^p_{jm} - 2T^i_{pj}T^p_{mn}). \qquad (3.3)\end{aligned}$$

4. Infinitesimal deformations of curvature tensor $\underset{4}{R}$

4.1. Starting from (0.10) and using the procedure which is similar to that used for $\underset{3}{R}$, we obtain

$$\begin{aligned}\underset{4}{\bar{R}}{}^i_{jmn} = \underset{4}{R}{}^i_{jmn} &+ (\mathcal{D}L^i_{jm})_{\underset{4}{|}n} - (\mathcal{D}L^i_{nj})_{\underset{3}{|}m}\\ &+ T^p_{jm}\mathcal{D}L^i_{np} + T^p_{jn}\mathcal{D}L^i_{pm} + L^p_{mn}\mathcal{D}T^i_{jp} + \mathcal{D}(T^i_{pj}L^p_{mn}) \qquad (4.1)\end{aligned}$$

4.2. Using (0.17), we obtain for $\underset{4}{R}$

$$\underset{4}{\bar{R}}{}^i_{jmn} = \underset{4}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac{1}{2}(\mathcal{D}T^i_{jm})_{;n} + \frac{1}{2}(\mathcal{D}T^i_{jn})_{;m}$$
$$+ \frac{1}{4}\mathcal{D}(T^i_{pm}T^p_{jn} - T^i_{pn}T^p_{jm} + 2T^i_{pj}T^p_{mn}). \quad (4.2)$$

5. Infinitesimal deformations of the tensor $\underset{5}{R}$

5.1 According to (0.11) we have

$$\begin{aligned}2\underset{5}{\bar{R}}{}^i_{jmn} &= L^i_{jm,n} + (\mathcal{D}L^i_{jm}),n + L^i_{mj,n} + (\mathcal{D}L^i_{mj}),n - L^i_{jn,m} - (\mathcal{D}L^i_{jn}),m \\ &- L^i_{nj,m} - (\mathcal{D}L^i_{nj}),m + L^p_{jm}L^i_{pn} + L^p_{jm}\mathcal{D}L^i_{pn} + (\mathcal{D}L^p_{jm})L^i_{pn} + L^p_{mj}L^i_{np} \\ &+ L^p_{mj}\mathcal{D}L^i_{np} + (\mathcal{D}L^p_{mj})L^i_{np} - L^p_{jn}L^i_{mp} - L^p_{jn}\mathcal{D}L^i_{mp} \\ &- (\mathcal{D}L^p_{jn})L^i_{mp} - L^p_{nj}L^i_{pm} - L^p_{nj}\mathcal{D}L^i_{pm} - (\mathcal{D}L^p_{nj})L^i_{pm}. \quad (5.1)\end{aligned}$$

By differentiating the Lie differences one obtains

$$\begin{aligned}(\mathcal{D}L^i_{jm})_{\underset{3}{|}n} &= (\mathcal{D}L^i_{jm})_{,n} + L^i_{pn}\mathcal{D}L^p_{jm} - L^p_{nj}\mathcal{D}L^i_{pm} - L^p_{nm}\mathcal{D}L^i_{jp},\\ (\mathcal{D}L^i_{mj})_{\underset{4}{|}n} &= (\mathcal{D}L^i_{mj})_{,n} + L^i_{np}\mathcal{D}L^p_{mj} - L^p_{mn}\mathcal{D}L^i_{pj} - L^p_{jn}\mathcal{D}L^i_{mp},\\ (\mathcal{D}L^i_{nj})_{\underset{3}{|}m} &= (\mathcal{D}L^i_{nj})_{,m} + L^i_{pm}\mathcal{D}L^p_{nj} - L^p_{mn}\mathcal{D}L^i_{pj} - L^p_{mj}\mathcal{D}L^i_{np},\\ (\mathcal{D}L^i_{jn})_{\underset{4}{|}} &= (\mathcal{D}L^i_{jn})_{,m} + L^i_{mp}\mathcal{D}L^p_{jn} - L^p_{jm}\mathcal{D}L^i_{pn} - L^p_{nm}\mathcal{D}L^i_{jp}.\end{aligned}$$

If we substitute at (5.1) three members from the right sides of previous four equations, we get

$$\underset{5}{\bar{R}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn} + \frac{1}{2}[(\mathcal{D}L^i_{jm})_{\underset{3}{|}n} - (\mathcal{D}L^i_{nj})_{\underset{3}{|}m} + (\mathcal{D}L^i_{mj})_{\underset{4}{|}n} - (\mathcal{D}L^i_{jn})_{\underset{4}{|}m}]. \quad (5.2)$$

5.2. If we start from (0.18), we have

$$\underset{5}{\bar{R}}{}^i_{jmn} = \underset{0}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac{1}{4}[T^p_{jm}T^i_{pn} + T^p_{jm}\mathcal{D}T^i_{pn}$$
$$+ (\mathcal{D}T^p_{jm})T^i_{pn} + T^p_{jn}T^i_{pm} + T^p_{jn}\mathcal{D}T^i_{pm} + (\mathcal{D}T^p_{jn})T^i_{pm}].$$

According to (0.18) the previous equation becomes:

$$\underset{5}{\bar{R}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac{1}{4}[T^p_{jm}\mathcal{D}T^i_{pn} + (\mathcal{D}T^p_{jm})T^i_{pn} + T^p_{jn}\mathcal{D}T^i_{pm} + (\mathcal{D}T^p_{jn})T^i_{pm}],$$

i.e.

$$\underset{5}{\bar{R}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn} + \mathcal{D}\underset{0}{R}{}^i_{jmn} + \frac{1}{4}\mathcal{D}(T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm}). \quad (5.3)$$

REFERENCES

[**1**] Minčić, S. M., *Ricci identities in the space of non-symmetric affine connexion*, Mat. Vesnik **10(25)**, 2 (1973), 161–172.

[**2**] Minčić, S. M., *New commutation formulas in the non-symetric affine connexion space*, Publ. Inst. Math. (Beograd)(N.S), **22(36)** (1977), 189–199.

[**3**] Minčić, S. M., *Independent curvature tensors and pseudotensors of spaces with non-symmetric affine connexion*, Coll. Math. Soc. János Bolyai, **31**. Dif. geom., Budapest (1979), 445–460.

[**4**] Stojanović, R., *Osnovi diferencijalne geometrije*, Građevinska knjiga, Beograd, 1963.

[**5**] Schouten, J. A., Ricci Calculus, Springer Verlag, Berlin-Götingen-Heildelberg, 1954.

[**6**] Yano, K., *Sur la theorie des deformations infinitesimales*, Journal of Fac. of Sci. Univ. of Tokyo, **6** (1974), 1–75.

[**7**] Yano, K., *The Theory of Lie Derivatives and Its Applications*, N. Holland Publ. Co., Amsterdam, 1957.

[**8**] Ivanova-Karatopraklieva, I., Sabitov, I. Kh., *Surface deformation*, J. Math. Sci., New York **70**, 2 (1994), 1685–1716.

[**9**] Ivanova-Karatopraklieva, I., Sabitov, I. Kh., *Bending of surfaces, II*, J. Math. Sci., New York **74**, 3 (1995), 997–1043.

[**10**] Minčić, S. M., Velimirović, L. S., Stanković M.S., *Infinitesimal deformations of a non-symmetric affine connection space*, Filomat **15** (2001).

[**11**] Velimirović, L. S., Minčić, S. M., Stanković, M. S., *Infinitesimal deformations and Lie derivative of a non-symmetric affine connection space*, (submitted).

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