

DYNAMICS OF MULTIPARAMETER OPERATOR SEMIGROUPS INDUCED BY SEMIFLOWS

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Abstract. In this paper, we investigate various dynamical properties of multiparameter strongly continuous semigroups of operators induced by semiflows. We also provide many illustrative examples and useful remarks concerning issues under our consideration.

1. Introduction and preliminaries

Unless stated otherwise, we will always assume henceforth that E is a separable Fréchet space over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and that the topology of E is induced by the fundamental system $(p_k)_{k \in \mathbb{N}}$ of increasing seminorms. A linear continuous mapping $T: E \rightarrow E$ is said to be hypercyclic if there exists an element $x \in E$ whose orbit $\text{Orb}(x, T) := \{T^k x : k \in \mathbb{N}_0\}$ is dense in E , while T is said to be topologically transitive if for any pair of open non-empty subsets U, V of E there exists $k \in \mathbb{N}$ such that $T^k(U) \cap V \neq \emptyset$. Using the Baire theorem, we can simply prove that T is hypercyclic if and only if T is topologically transitive; see the research monographs [1] by F. Bayart, E. Matheron, [8] by K.-G. Grosse-Erdmann, A. Peris and [19] by M. Kostić for basic source of information about linear topological dynamics.

The hypercyclic tuples of operators were introduced and seriously analyzed by L. Kérchy in [17] (2005) and N. Feldman in [6] (2008). Let us recall that a tuple $(T_1, \dots, T_n)$ of commuting linear continuous operators on E , where $n \in \mathbb{N} \setminus \{1\}$, is said to be hypercyclic if there exists $x \in E$ such that the orbit $\{T_1^{k_1} \dots T_n^{k_n} x : k_1 \in \mathbb{N}_0, \dots, k_n \in \mathbb{N}_0\}$ is dense in E . A tuple $(T_1, \dots, T_n)$ of commuting linear continuous operators on E is hypercyclic if and only if it is topologically transitive, which means that for any pair of open non-empty subsets U, V of E there exist integers $k_1 \in \mathbb{N}_0, \dots, k_n \in \mathbb{N}_0$ such that $T_1^{k_1} \dots T_n^{k_n}(U) \cap V \neq \emptyset$. In contrast with hypercyclic

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operators, hypercyclic tuples of operators exist on finite-dimensional spaces (cf. [6, 26] and references cited therein for more details in this direction).

A linear continuous mapping $T: E \rightarrow E$ is said to be topologically mixing if for any pair of open non-empty subsets U, V of E there exists $k_0 \in \mathbb{N}$ such that $T^k(U) \cap V \neq \emptyset$ for all $k_0 \in \mathbb{N}$ with $k \geq k_0$. The topologically mixing tuples of operators were considered in [9]; cf. also [10, 25, 27] for more details in this direction. Following [9, Definition 1.1], it is said that a tuple $(T_1, \dots, T_n)$ of commuting linear continuous operators on E is topologically mixing if for any pair of open non-empty subsets U, V of E there exists an integer $k_0 \in \mathbb{N}$ such that, for every integers $k_1 \geq k_0, \dots, k_n \geq k_0$ we have $T_1^{k_1} \dots T_n^{k_n}(U) \cap V \neq \emptyset$. In contrast to hypercyclic tuples of operators, topologically mixing tuples of operators do not exist on finite-dimensional spaces.

On the other hand, the hypercyclic and chaotic strongly continuous one-parameter semigroups of linear operators were systematically analyzed by W. Desch, W. Schappacher and G. F. Webb in [5] (1997); cf. also [7, 15, 16, 18, 19, 21, 24] and references quoted therein. The hypercyclic multiparameter strongly continuous semigroups of linear operators were introduced and analyzed by M. Janfada and A. N. Baghan in [14] (2019); cf. [20, pp. 363–365, pp. 593–595] and [14, Section 2, Section 3] for the notion and more details about multiparameter strongly continuous semigroups of operators. In [23], we have recently analyzed dynamical properties of strongly continuous semigroups of operators over topological monoids.

The main aim of this research article is to continue the study raised in [23]. We consider here various dynamical properties of multiparameter strongly continuous semigroups induced by semiflows and the associated abstract n -parameter Cauchy problem $(ACP)_M$; cf. the research articles [15, 16] by T. Kalmes, the research monograph [18] and the research article [21] for the basic results established in the one-dimensional framework. Because of a great similarity with the corresponding analyses of one-parameter strongly continuous semigroups induced by semiflows, the proofs of our structural results are omitted.

The organization of paper can be briefly summarized as follows. We first recall the notion of weighted function spaces used in our work and the notion of various dynamical properties of arbitrary operator families over general sets. In Subsection 1.1, we recall the basic definitions and results concerning the semigroups over topological monoids. After that, we focus our attention to the special class of monoids $M \subseteq \mathbb{R}^n$ having the form (2) clarified below; cf. Theorem 1.2 for the main result concerning the existence and uniqueness of strong solutions to the associated abstract n -parameter Cauchy problem $(ACP)_M$. Our main results are given in Section 2, where we analyze dynamics of multiparameter semigroups of operators induced by semiflows. In Definition 2.1, we recall the notion of a semiflow over topological monoid M ; see also Remark 2.2 and Example 2.3. After that, we consider the strong continuity of operator semigroups induced by semiflows (see Theorem 2.6 and Theorem 2.7). We formulate our main results in Subsection 2.1; cf. Theorem 2.8, Theorem 2.9, Theorem 2.11 and Example 2.12.

Notation and preliminaries

We assume henceforth that $d, n \in \mathbb{N}$ and Ω is an open, non-empty subset of $\mathbb{R}^d$. If $1 \leq p < +\infty$, then the Banach space $L_\rho^p(\Omega, \mathbb{K})$, where $\rho : \Omega \rightarrow (0, \infty)$ is a Lebesgue measurable function and $\rho : \Omega \rightarrow (0, \infty)$ is a locally integrable function, consists of those Lebesgue measurable functions $f : \Omega \rightarrow \mathbb{K}$ such that

$$\|f\|_p := \left(\int_\Omega |f(x)|^p \rho(x) dx \right)^{1/p} < +\infty.$$

Further on, the Banach space $C_{0,\rho}(\Omega, \mathbb{K})$ consists of those continuous functions $f : \Omega \rightarrow \mathbb{K}$ such that, for every $\epsilon > 0$, $\{x \in \Omega : |f(x)|\rho(x) \geq \epsilon\}$ is a compact subset of Ω ; here, $\rho : \Omega \rightarrow (0, \infty)$ is an upper semicontinuous function and the norm of an element $f \in C_{0,\rho}(\Omega, \mathbb{K})$ is given by $\|f\| := \sup_{x \in \Omega} |f(x)|\rho(x)$. We omit the term “ $\mathbb{K}$ ” from the notation if the field $\mathbb{K}$ is known.

Unless stated otherwise, we will always assume henceforth that M is an infinite set, $f : M \rightarrow [0, +\infty)$, $\sup_{t \in M} f(t) = +\infty$ and S is a closed subset of $\mathbb{K}$ such that $S \setminus \{0\} \neq \emptyset$. We will use the following notion [23].

DEFINITION 1.1. Suppose that $(R(t))_{t \in M} \subseteq L(E)$ is an operator family on E . Then we say that $(R(t))_{t \in M}$ is:

- (i) hypercyclic, if there exists an element $x \in E$ whose orbit $\text{Orb}(x, R) := \{R(t)x : t \in M\}$ is dense in E ,
- (ii) chaotic, if $(R(t))_{t \in M}$ is hypercyclic and the set of periodic points of $(R(t))_{t \in M}$, defined by $\{x \in E : R(t_0)x = x \text{ for some } t_0 \in M \setminus \{0\}\}$, is dense in E ,
- (iii) topologically transitive, if for every pair of open non-empty sets U, V of E , there exists $t \in M$ such that $R(t)U \cap V \neq \emptyset$,
- (iv) S-hypercyclic, if there exists an element $x \in E$ such that its S-projective orbit $\{cR(t)x : c \in S, t \in M\}$ is dense in E ; $\text{HC}_S(R)$ denotes the set of all elements $x \in E$ whose S-projective orbit is dense in E ,
- (v) S-topologically transitive, if for every pair of open non-empty sets U, V of E , there exist $c \in S$ and $t \in M$ such that $cR(t)U \cap V \neq \emptyset$,
- (vi) (f, S) -topologically mixing, if for every pair of open non-empty sets U, V of E , there exists $r > 0$ such that, for every $t \in M$ with $f(t) \geq r$, there exists $c_t \in S$ such that $c_t R(t)U \cap V \neq \emptyset$.

1.1 Semigroups over topological monoids

If $(M, +)$ is a monoid with the neutral element 0 and (M, τ) is a topological space, then we say that $(M, +, \tau)$ is:

- (i) a semitopological monoid, if the operations $x \mapsto x + a$, $x \in M$ and $x \mapsto a + x$, $x \in M$ are continuous for every fixed element $a \in M$;
- (ii) a topological monoid, if the operation $+: M^2 \rightarrow M$ is continuous;

(iii) a topological group, if $(M, +)$ is a group as well as the mappings $+: M^2 \rightarrow M$ and $-: M^2 \rightarrow M$ are continuous, where $-$ denotes the inverse operation of $+$.

It is clear that every topological monoid is a semitopological monoid as well as that every topological group is a topological monoid; observe, also, that we do not assume a priori the commutativity of operation $+$ in the above definition.

The notion of a semigroup over topological monoid naturally generalizes the notion of usually considered one-parameter strongly continuous semigroup of bounded linear operators. If $M = [0, +\infty)^n$, these semigroups are usually called multiparameter semigroups (cf. [2, 12]). So, let $(M, +)$ be a monoid with the neutral element 0. By a semigroup on E defined over a monoid M we mean any operator-valued function $T: M \rightarrow L(E)$ such that $T(0) = I$, the identity operator on E , and $T(t+s) = T(t)T(s)$ for all $t, s \in M$. If $(M, +, \tau)$ is a topological monoid, then we say that a semigroup $T: M \rightarrow L(E)$, denoted also by $(T(t))_{t \in M}$, is strongly continuous if, for every $x \in E$, the mapping $t \mapsto T(t)x$, $t \in M$ is strongly continuous at $t = 0$; cf. the research article [4] by R. Dahya and references cited therein for more details. If $(M, +, \tau)$ is a topological group, then we also say that $(T(t))_{t \in M}$ is a strongly continuous group of operators.

Suppose now that $M = [0, +\infty)^n$ and $(T(t))_{t \in M}$ is strongly continuous. If $(e_1, \dots, e_n)$ is the standard basis of $\mathbb{R}^n$, then we set $T_i(t_i) := T(t_i e_i)$, $t_i \geq 0$; clearly, $(T_i(t_i))_{t_i \geq 0}$ is a strongly continuous semigroup on E ($i \in \mathbb{N}_n \equiv \{1, 2, \dots, n\}$). Furthermore, if $T_i(\cdot)$ is generated by A_i ($i \in \mathbb{N}_n$), then

$$T(t_1, \dots, t_n) = T_1(t_1) \cdot \dots \cdot T_n(t_n), \quad (t_1, \dots, t_n) \in [0, +\infty)^n, \quad (1)$$

and the tuple $(A_1, A_2, \dots, A_n)$ is said to be the infinitesimal generator of $(T(t))_{t \in [0, +\infty)^n}$. We can simply prove the following result; cf. also [2, Proposition 1.1.8] and [13, Lemma 2.4]: Suppose that A_i is the infinitesimal generator of a strongly continuous semigroup $(T_i(t))_{t \geq 0}$ for $1 \leq i \leq n$. If $(T(t))_{t \in [0, +\infty)^n}$ is defined through (1), then $(A_1, A_2, \dots, A_n)$ is the infinitesimal generator of multiparameter strongly continuous semigroup $(T(t))_{t \in [0, +\infty)^n}$ if and only if $T_i(t_i)T_j(t_j) = T_j(t_j)T_i(t_i)$ for all $1 \leq i, j \leq n$, $t_i \geq 0$ and $t_j \geq 0$.

Let us consider now the monoid $M \subseteq \mathbb{R}^n$ given by

$$M := \left\{ t_1 a_1 + \dots + t_n a_n : t_i \in D_i \ (1 \leq i \leq n) \right\}, \quad (2)$$

where $(a_1, \dots, a_n)$ is a normed basis of $\mathbb{R}^n$ and D_i is either $[0, +\infty)$, $(-\infty, 0]$ or $\mathbb{R}$ for $1 \leq i \leq n$. If $(T(t))_{t \in M}$ is a strongly continuous semigroup, where M has the form (2), and $i \in \mathbb{N}_n$, then we define the *directional infinitesimal generator* A_i of $(T(t))_{t \in M}$ by

$$D(A_i) := \left\{ x \in E : \lim_{t_i \rightarrow 0} \frac{T(t_i a_i)x - x}{t_i} \text{ exists in } E \right\} \quad (3)$$

and

$$A_i x := \lim_{t_i \rightarrow 0} \frac{T(t_i a_i)x - x}{t_i}, \quad x \in D(A_i). \quad (4)$$

It is clear that for each $i \in \mathbb{N}_n$ the operator A_i is the infinitesimal generator of a strongly continuous semigroup $(T(t_i a_i))_{t_i \geq 0}$. The tuple $A := (A_1, \dots, A_n)$ is said to be the *directional infinitesimal generator* of $(T(t))_{t \in M}$.

Let us consider now the n -parameter abstract Cauchy problem

$$(ACP)_M : \begin{cases} u \in C(M : E) \cap C^1(M^\circ : E), \\ u_{t_i; a_i}(t) = A_i u(t), \quad t \in M^\circ, \quad 1 \leq i \leq n, \\ u(0) = x, \end{cases}$$

where $u_{t_i; a_i}(t) := \lim_{h_i \rightarrow 0} \frac{u(t + h_i a_i) - u(t)}{h_i}$, $t \in M$, provided that the above limit exists in E . Then we know the following.

THEOREM 1.2 ([23]). *Suppose that $(T(t))_{t \in M}$ is a strongly continuous semigroup, where M has the form (2), and for each $i \in \mathbb{N}_n$ the operator A_i is defined through (3)-(4). Then, for every $x \in \bigcap_{i \in \mathbb{N}_n} D(A_i)$, there exists a unique strong solution of problem $(ACP)_M$, which is given by $u(t) := T(t)x$, $t \in M$.*

If $(T(t))_{t \in M}$ is S-topologically transitive and exponentially equicontinuous, i.e., for each $k \in \mathbb{N}$ there exist an integer $m \in \mathbb{N}$, a real number $c > 0$ and real numbers $\omega_1, \dots, \omega_n \in \mathbb{R}$ such that $p_k(T(t)x) \leq ce^{\omega_1 t_1 + \dots + \omega_n t_n} p_m(x)$, $x \in E$, $t = (t_1, \dots, t_n) \in M$, then [23, Theorem 4.6] shows that the set $HC_S(T) \cap D_\infty(A_1) \cap \dots \cap D_\infty(A_n)$ is dense in E , where $D_\infty(A) \equiv \bigcap_{k \in \mathbb{N}} D(A^k)$. For the sake of brevity, we will not consider here the multiparameter strongly continuous semigroups whose directional infinitesimal generators are abstract first-order partial differential operators with non-vanishing zero-order terms (cf. [15, 16] for some results obtained in this direction).

2. Dynamics of multiparameter semigroups of operators induced by semiflows

We start by recalling the following notion.

DEFINITION 2.1. If $(M, +)$ is a topological monoid and (Ω, τ) is a topological space, then we say that a continuous mapping $\varphi : M \times \Omega \rightarrow \Omega$ is a semiflow over M if $\varphi(0, x) = x$, $x \in \Omega$, $\varphi(t + s, x) = \varphi(t, \varphi(s, x))$, $t, s \in M$, $x \in \Omega$ and the mapping $x \mapsto \varphi(t, x)$, $x \in \Omega$ is injective for all $t \in M$. It is also said that $\varphi : M \times \Omega \rightarrow \Omega$ is a (topological) semiflow with phase space Ω and phase monoid M .

By $\varphi(t, \cdot)^{-1}$ we denote the inverse mapping of $\varphi(t, \cdot)$, i.e., $y = \varphi(t, x)^{-1}$ if $x = \varphi(t, y)$, $t \in M$.

In connection with this notion, we have the following observation.

REMARK 2.2. Recently, P. J. Chocano has investigated the semiflows on finite topological spaces; [3, Definition 2.7] is a special case of Definition 2.1 with $M = [0, +\infty)$. But, the author has not required in this definition that the mapping $x \mapsto \varphi(t, x)$, $x \in \Omega$ is injective for all $t \geq 0$; the injectiveness of above mapping has not been required in a general definition of a (topological) semiflow with phase space Ω and phase monoid M , as well (see, e.g., [11]). The continuity of mapping $x \mapsto \varphi(t, x)$, $x \in \Omega$

($t \geq 0$) has been imposed by T. Kalmes [15, 16] in order to make the Brouwer theorem susceptible to applications.

We continue by providing some illustrative examples.

EXAMPLE 2.3. (i) If $(T(t))_{t \in M} \subseteq L(E)$ is a strongly continuous semigroup of operators over M and for each $t \in M$ the operator $T(t)$ is injective, then the mapping $\varphi : M \times E \rightarrow E$, defined by $\varphi(t, x) := T(t)x$, $t \in M$, $x \in E$, is a semiflow over M ; here, $\Omega = E$. Furthermore, if $(T(t))_{t \in M} \subseteq L(E)$ is a strongly continuous semigroup of operators over M , $\emptyset \neq \Omega \subseteq E$, for each $t \in M$ the operator $T(t)|_\Omega$ is injective, and $T(t)\Omega \subseteq \Omega$, $t \in M$, then the mapping $\varphi : M \times \Omega \rightarrow \Omega$, defined by $\varphi(t, x) := T(t)x$, $t \in M$, $x \in \Omega$, is a semiflow over M . For instance, we have the following:

1. If $(a_1, \dots, a_n)$ is a normed basis of $\mathbb{R}^n$, $D_i = [0, +\infty)$ for $1 \leq i \leq n$, and M has the form (2), then the expression

$$\varphi(t_1 a_1 + \dots + t_n a_n, x) := \exp(t_1 A_1 + \dots + t_n A_n)x, \quad t_1 a_1 + \dots + t_n a_n \in M, \quad x \in \mathbb{K}^m, \quad (5)$$

is a semiflow over M , provided that $A_1, \dots, A_n$ belong to $\mathbb{K}^{m,m}$, the algebra of all $\mathbb{K}$ -matrices of format $m \times m$, and $A_i A_j = A_j A_i$ for $1 \leq i, j \leq n$. If for each $i \in \mathbb{N}_n$ we have $A_i \geq 0$, resp. $A_i > 0$, and $\Omega = [0, +\infty)^n$, resp. $\Omega = (0, +\infty)^n$, then the expression $\varphi : M \times \Omega \rightarrow \Omega$, given by (5) for all $x \in \Omega$, is a semiflow over M .

2. If $(T(t))_{t \in \mathbb{R}^n}$ is a strongly continuous group of operators and $T(t)\Omega \subseteq \Omega$ for all $t \in \mathbb{R}^n$, resp. $t \in [0, +\infty)^n$, then the mapping $\varphi : \mathbb{R}^n \times \Omega \rightarrow \Omega$, resp. $\varphi : [0, +\infty)^n \times \Omega \rightarrow \Omega$, given by $\varphi(t, x) := T(t)x$, $t \in \mathbb{R}^n$, resp. $t \in [0, +\infty)^n$, $x \in \Omega$ is a semiflow over M . For example, if $E = L^p(\mathbb{R}^n)$, where $1 \leq p < +\infty$ and $n \in \mathbb{N}$, then the expression $[T(t_1, \dots, t_n)f](x_1, \dots, x_n) := f(t_1 + x_1, \dots, t_n + x_n)$, $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, $(x_1, \dots, x_n) \in \mathbb{R}^n$, $f \in L^p(\mathbb{R}^n)$ is a strongly continuous group of operators on E whose infinitesimal generator is the tuple $(\partial/\partial x_1, \dots, \partial/\partial x_n)$, where each partial derivative is taken with its maximal distributional domain. If $\Omega = \{f \in L^p(\mathbb{R}^n) : f \geq 0 \text{ a.e.}\}$, then we have $T(t)\Omega \subseteq \Omega$ for all $t \in \mathbb{R}^n$.

(ii) If $n \in \mathbb{N}$, Ω_i is an open non-empty subset of $\mathbb{R}^{m_i}$, where $m_i \in \mathbb{N}$, $M_i = [0, +\infty)$, $M_i = (-\infty, 0]$ or $M_i = \mathbb{R}$, $\varphi_i : M_i \times \Omega_i \rightarrow \Omega_i$ is a semiflow over M_i ($1 \leq i \leq n$), $M := M_1 \times \dots \times M_n$ and $\Omega := \Omega_1 \times \dots \times \Omega_n$, then the mapping $\varphi : M \times \Omega \rightarrow \Omega$, given by

$$\varphi(t_1, \dots, t_n, x_1, \dots, x_n) := \left(\varphi_1(t_1, x_1), \dots, \varphi_n(t_n, x_n) \right),$$

for any $t = (t_1, \dots, t_n) \in M$ and $x = (x_1, \dots, x_n) \in \Omega$, is a semiflow over M . For example, the mapping $\varphi : [0, +\infty)^n \times (0, 1)^n \rightarrow (0, 1)^n$, given by

$$\varphi(t_1, \dots, t_n, x_1, \dots, x_n) := \left(\frac{x_1}{x_1 + (1 - x_1)e^{-t_1}}, \dots, \frac{x_n}{x_n + (1 - x_n)e^{-t_n}} \right),$$

for any $t = (t_1, \dots, t_n) \in [0, +\infty)^n$ and $x = (x_1, \dots, x_n) \in (0, 1)^n$, is a semiflow over $[0, +\infty)^n$; cf. also [16, Example 3.20].

(iii) Suppose that $M = [0, +\infty)^n$, $\varphi_i : [0, +\infty) \times \Omega \rightarrow \Omega$ is a semiflow given in the formulation of [16, Theorem 3.9] with the locally Lipschitz continuous vector field

$F_i(\cdot) = (F_{i,1}(\cdot), \dots, F_{i,d}(\cdot))$ on Ω , and $(T_{\varphi_i}(t_i))_{t_i \geq 0}$ is the induced strongly continuous semigroup on the space $E = L^p(\Omega, \mu)$, where we assume that μ is admissible for each semiflow φ_i ($1 \leq i \leq n$). Set

$$T(t_1, \dots, t_n) := T_{\varphi_1}(t_1) \cdot \dots \cdot T_{\varphi_n}(t_n), \quad t = (t_1, \dots, t_n) \in [0, +\infty)^n. \quad (6)$$

Then the function $u(t) := T(t)f$, $t \in [0, +\infty)^n$ can be viewed as the solution semigroup to the following n -parameter Cauchy problem

$$(ACP) : \begin{cases} u \in C([0, +\infty)^n : E) \cap C^1((0, +\infty)^n : E), \\ u_{t_i}(t, x) = \sum_{j=1}^d F_{i,j}(x) \frac{\partial}{\partial x_j} u(t, x), \quad t \in (0, +\infty)^n, \quad 1 \leq i \leq n, \quad x \in \Omega, \\ u(0, x) = f(x), \quad x \in \Omega; \end{cases}$$

see [16, Theorem 3.9] for more details. It is also clear that, if the semigroup $(T_{\varphi_i}(t_i))_{t_i \geq 0}$ is hypercyclic (chaotic) for some $1 \leq i \leq n$, then the semigroup $(T(t_1, \dots, t_n))_{(t_1, \dots, t_n) \in [0, +\infty)^n}$ is also hypercyclic (chaotic); the converse statement is not true, in general.

For example, if $1 \leq p < +\infty$, $M := [0, +\infty)^n$, $\Omega := \mathbb{R}^d$,

$$\varphi_n(t, x) := e^{t_1 + \dots + t_n} x, \quad t = (t_1, \dots, t_n) \in [0, +\infty)^n, \quad x \in \mathbb{R}^d$$

and $\rho(x) := (1 + |x|^2)^{-1}$, $x \in \mathbb{R}^d$, then $\varphi_n(\cdot, \cdot)$ is a semiflow and the semigroup $(T_{\varphi_n}(t))_{t \in [0, +\infty)^n}$, given by the equation (8) below, is strongly continuous on the space $L^p_\rho(\mathbb{R}^d)$ [$C_{0,\rho}(\mathbb{R}^d)$], which can be simply proved with the help of the identity

$$T_{\varphi_n}(t_1, \dots, t_n) = T_{\varphi_1}(t_1) \cdot \dots \cdot T_{\varphi_1}(t_n), \quad t = (t_1, \dots, t_n) \in [0, +\infty)^n \quad (7)$$

and the argumentation from [16, Example 3.19]. In the one-dimensional setting, we know that $(T_{\varphi_1}(t))_{t \geq 0}$ is chaotic on $L^p_\rho(\mathbb{R}^d)$ as well as that $(T_{\varphi_1}(t))_{t \geq 0}$ is not hypercyclic on $C_{0,\rho}(\mathbb{R}^d)$; cf. [16, Example 3.19, Example 5.5]. Using (7) and the above observation, it readily follows that the multiparameter semigroup $(T_{\varphi_n}(t))_{t \in [0, +\infty)^n}$ is chaotic on $L^p_\rho(\mathbb{R}^d)$. But, since $\varphi_n(t, 0) = 0$ for all $t \in [0, +\infty)^n$, $(T_{\varphi_n}(t))_{t \in [0, +\infty)^n}$ cannot be hypercyclic on $C_{0,\rho}(\mathbb{R}^d)$.

(iv) Further study of dynamical properties of semigroup $(T(t))_{t \in [0, +\infty)^n}$ of form (6) is without scope of this paper if there exist indexes $1 \leq i, j \leq n$ such that $\varphi_i(\cdot, \cdot) \neq \varphi_j(\cdot, \cdot)$. In this case, the function $\varphi(\cdot, \cdot)$ given by the equation (10) below with $(a_1, \dots, a_n) = (e_1, \dots, e_n)$ is not a semiflow over $[0, +\infty)^n$. On the other hand, if $\varphi : [0, +\infty) \times \Omega \rightarrow \Omega$ is a semiflow and $T(t_1, \dots, t_n) := T_\varphi(t_1) \cdot \dots \cdot T_\varphi(t_n)$, $t = (t_1, \dots, t_n) \in [0, +\infty)^n$, then $(T(t))_{t \in [0, +\infty)^n}$ is a multiparameter semigroup induced by the semiflow $\psi : [0, +\infty)^n \times \Omega \rightarrow \Omega$, given by

$$\psi(t, x) := \varphi_1\left(t_1, \varphi_1\left(t_2, \dots, \varphi_1\left(t_{n-1}, \varphi_1(t_n, x)\right)\right)\right), \quad t = (t_1, \dots, t_n) \in [0, +\infty)^n, \quad x \in \Omega,$$

which can be easily shown by induction.

In the remainder of paper, we analyze dynamical properties of multiparameter operator semigroups induced by semiflows. We always assume henceforth that the monoid $M \subseteq \mathbb{R}^n$ has the form (2), where $(a_1, \dots, a_n)$ is a normed basis of $\mathbb{R}^n$ and D_i is either $[0, +\infty)$, $(-\infty, 0]$ or $\mathbb{R}$ for $1 \leq i \leq n$. If $r > 0$, then we set $M(r) := \{t \in M : |t| \leq r\}$.

2.1 Main results and proofs

Fix an element $t \in M$, a function $f : \Omega \rightarrow \mathbb{K}$ and a semiflow $\varphi : M \times \Omega \rightarrow \Omega$. Define $T_\varphi(t)f : \Omega \rightarrow \mathbb{K}$ by

$$(T_\varphi(t)f)(x) := f(\varphi(t, x)), \quad x \in \Omega. \quad (8)$$

Then $T_\varphi(0)f = f$, $T_\varphi(t)T_\varphi(s)f = T_\varphi(s)T_\varphi(t)f = T_\varphi(t+s)f$, $t, s \in M$; furthermore, if Ω is an open non-empty subset of $\mathbb{R}^n$, then the Brouwer theorem implies $C_c(\Omega) \subseteq T_\varphi(t)(C_c(\Omega))$. We refer the reader to [16, Theorem 2.1], resp. [16, Theorem 2.2], for the necessary and sufficient conditions ensuring that the composition operator $T_\varphi(t) : L_\rho^p(\Omega) \rightarrow L_\rho^p(\Omega)$, resp. $T_\varphi(t) : C_{0,\rho}(\Omega) \rightarrow C_{0,\rho}(\Omega)$, is well-defined and continuous.

We proceed with the following illustrative example.

EXAMPLE 2.4. Suppose that $n = d$, $M + \Omega \subseteq \Omega$ and $p : \Omega \rightarrow \Omega$ is a continuous bijection together with its inverse mapping $p^{-1} : \Omega \rightarrow \Omega$. Define

$$\varphi(t, x) := p^{-1}(t + p(x)), \quad t \in M, \quad x \in \Omega. \quad (9)$$

Then $\varphi : M \times \Omega \rightarrow \Omega$ is a semiflow over M .

In [22, Example 2.2], we have $n = d = 1$ and $M = \Omega = \mathbb{R}$; we have constructed therein a strongly continuous, positively supercyclic group (i.e., S-hypercyclic group with $S = [0, +\infty)$) induced by a semiflow $\varphi(\cdot, \cdot)$ of the form (9), where $p : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing real polynomial of odd degree; the pivot Banach space in our analysis was $E = L_{\rho_1}^p(\mathbb{R})$, where $\rho_1(x) := f(p(x))$, $x \in \mathbb{R}$ for some admissible weight function $f : \mathbb{R} \rightarrow (0, +\infty)$ in the sense of [5]. If we denote the corresponding semigroup by $(T_\varphi(t))_{t \geq 0}$, then $(T_\varphi(t))_{t \geq 0}$ is not hypercyclic on $L_{\rho_1}^p(\mathbb{R})$.

Let us define now the semiflow $\Psi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ by $\Psi(t_1, \dots, t_n, x) := \varphi(t_n, x)$, $(t_1, \dots, t_n) \in \mathbb{R}^n$, $x \in \mathbb{R}$ and the multiparameter strongly continuous semigroup $(T(t))_{t \in \mathbb{R}^n}$ on $L_{\rho_1}^p(\mathbb{R})$ by $T(t_1, \dots, t_n) := T_\varphi(t_n)$, $t = (t_1, \dots, t_n) \in \mathbb{R}^n$. Then it is clear that $(T(t))_{t \in \mathbb{R}^n}$ is positively supercyclic but not hypercyclic on $L_{\rho_1}^p(\mathbb{R})$; moreover, $(T(t))_{t \in \mathbb{R}^n}$ is induced by the semiflow $\Psi(\cdot, \cdot)$ since $(T(t))_{t \in \mathbb{R}^n}$ is given by (8), with $M = \mathbb{R}^n$, $\Omega = \mathbb{R}$ and the semiflow $\varphi(\cdot, \cdot)$ replaced with the semiflow $\Psi(\cdot, \cdot)$ therein.

Further on, if $\varphi : M \times \Omega \rightarrow \Omega$ is a semiflow over M , then we define $\varphi_i : D_i \times \Omega \rightarrow \Omega$ by

$$\varphi_i(t_i, x) := \varphi(t_i a_i, x), \quad t_i \in D_i, \quad x \in \Omega \quad (1 \leq i \leq n).$$

Then it is clear that $\varphi_i(\cdot, \cdot)$ is a semiflow over D_i ($1 \leq i \leq n$) and

$$\varphi(t, x) = \varphi_1\left(t_1, \varphi_2\left(t_2, \dots, \varphi_{n-1}\left(t_{n-1}, \varphi_n(t_n, x)\right)\right)\right), \quad t = t_1 a_1 + \dots + t_n a_n \in M, \quad x \in \Omega. \quad (10)$$

If $f : \Omega \rightarrow \mathbb{K}$, then we define

$$(T_{\varphi_i}(t_i)f)(x) := f(\varphi_i(t_i, x)), \quad t_i \in D_i, \quad x \in \Omega \quad (1 \leq i \leq n).$$

The semigroup $(T_\varphi(t))_{t \in M}$ can be written as the composition of semigroups $(T_{\varphi_i}(t_i))_{t_i \geq 0}$ for $1 \leq i \leq n$, since

$$T_\varphi(t_1 a_1 + \dots + t_n a_n) = T_\varphi(t_1 a_1) \cdot \dots \cdot T_\varphi(t_n a_n) = T_{\varphi_1}(t_1) \cdot \dots \cdot T_{\varphi_n}(t_n),$$

for any $t = t_1 a_1 + \dots + t_n a_n \in M$. Furthermore, if $E = L_\rho^p(\Omega)$ or $E = C_{0,\rho}(\Omega)$, then the directional infinitesimal generator of $(T_\varphi(t))_{t \in M}$ is the tuple $A = (A_1, \dots, A_n)$, where $D(A_i)$ is the set of all functions $f \in E$ for which the limit $\lim_{t_i \rightarrow 0} (T_{\varphi_i}(t_i)f - f)/t_i =: A_i f$

exists in E ; cf. (3)-(4).

Suppose now that $\varphi_i(t_i, x) = (\varphi_{i,1}(t_i, x), \dots, \varphi_{i,d}(t_i, x))$ for all $t_i \in D_i$ and $x \in \Omega$, as well as that the functions $\varphi_{i,j}(t_i, x)$ have the continuous partial derivatives $(\partial\varphi_{i,j}/\partial t_i)(t_i, x)$ for all $t_i \in D_i$ and $x \in \Omega$. Using L'Hospital's rule, it seems very plausible that $D(A_i)$ contains all functions $f \in C^1(\Omega) \cap E$ such that the mapping

$$(A_i f)(x) := \sum_{j=1}^d \frac{\partial f}{\partial u_j}(x) \cdot \frac{\partial \varphi_{i,j}}{\partial t_i}(0, x), \quad x \in \Omega \quad (11)$$

belongs to E since the informal calculus shows that:

$$\begin{aligned} \lim_{t_i \rightarrow 0} \frac{T_i(t_i)f - f}{t_i}(x) &= \lim_{t_i \rightarrow 0} \frac{d}{dt_i} [T_i(t_i)f](x) = \lim_{t_i \rightarrow 0} \frac{d}{dt_i} [f(\varphi_i(t_i, x))] \\ &= \lim_{t_i \rightarrow 0} \sum_{j=1}^d \frac{\partial f}{\partial u_j}(\varphi_i(t_i, x)) \cdot \frac{\partial \varphi_{i,j}}{\partial t_i}(t_i, x) \\ &= \sum_{j=1}^d \frac{\partial f}{\partial u_j}(x) \cdot \frac{\partial \varphi_{i,j}}{\partial t_i}(0, x), \quad x \in \Omega \quad (1 \leq i \leq n); \end{aligned}$$

cf. also [16, Theorem 3.9]. The interested reader may try to examine the question whether the operator B_i whose domain consists of all functions $f \in C^1(\Omega) \cap E$ such that the mapping on the right hand side of (11) belongs to E is closable and the question whether the directional infinitesimal generator A_i is equal to $\bar{B}_i$, provided that the answer to the first question is affirmative and the functions $\varphi_{i,j}(t_i, x)$ have the continuous partial derivatives $(\partial\varphi_{i,j}/\partial t_i)(t_i, x)$ for all $t_i \in D_i$ and $x \in \Omega$ ($1 \leq i \leq n$). This enables one to explore dynamics of strong solutions to problem $(ACP)_M$ by means of Theorem 1.2 and results established in remainder of this subsection; cf also the last paragraph of Section 1.

Now we will examine the strong continuity of semigroup $(T_\varphi(t))_{t \in M}$. If $M \in \{[0, \infty), \mathbb{R}, \mathbb{C}\}$ or $M = \Delta(\alpha)$ for some angle $\alpha \in (0, \pi/2]$, where $\Delta(\alpha) := \{re^{i\theta} : r \geq 0, \theta \in [-\alpha, \alpha]\}$, then the strong continuity of semigroup $(T_\varphi(t))_{t \in M}$ in $L_\rho^p(\Omega)$, resp. $C_{0,\rho}(\Omega)$, has been analyzed in [16, Theorem 3.2, Theorem 3.4] and [22, Theorem 2.5, Theorem 2.6]. Concerning the monoids $M \subseteq \mathbb{R}^n$ of form (2), we will first clarify the following auxiliary result (the proof is almost the same as the proof of [22, Lemma 2.1] and therefore omitted).

LEMMA 2.5. *Suppose $\varphi : M \times \Omega \rightarrow \Omega$ is a semiflow. Then, for every compact set $K \subseteq \Omega$ and for every $\delta > 0$ with $K + B(0, \delta) \subseteq \Omega$, there exists $r > 0$ such that*

$$K \cap \varphi(t, (\Omega \setminus (K + B(0, \delta)))) = \emptyset \quad \text{for all } t \in M(r).$$

Here, $B(0, \delta) = \{x \in \mathbb{R}^d : |x| \leq \delta\}$ and $K + B(0, \delta) = \{x + y : x \in K, y \in B(0, \delta)\}$.

Keeping in mind Lemma 2.5 as well as [22, Theorem 2.1, Lemma 2.2] and the proof of [22, Theorem 2.2], we can clarify the following result.

THEOREM 2.6. *Suppose $\varphi : M \times \Omega \rightarrow \Omega$ is a semiflow and $\varphi(t, \cdot)$ is a locally Lipschitz continuous function for all $t \in M$. Then (ii) implies (i), where:*

- (i) $(T_\varphi(t))_{t \in M}$ is a strongly continuous semigroup in $L_\rho^p(\Omega)$, and
- (ii) $\exists M, \omega \in \mathbb{R} \forall t \in M : \rho(\cdot) \leq M e^{\omega|t|} \rho(\varphi(t, \cdot)) |\det D\varphi(t, \cdot)|$ a.e.
- If, additionally, $\varphi(t, \cdot)^{-1}$ is locally Lipschitz continuous for all $t \in M$, then (i) and (ii) are equivalent.

Furthermore, using Lemma 2.5 and the proof of [15, Theorem 3.4], we can clarify the following slight extension of [22, Theorem 2.3].

THEOREM 2.7. *Assume that $\varphi : M \times \Omega \rightarrow \Omega$ is a semiflow. Then $(T_\varphi(t))_{t \in M}$ is a strongly continuous semigroup in $C_{0,\rho}(\Omega)$ if and only if the following holds:*

- (i) $\exists M, \omega \in \mathbb{R} \forall t \in M \forall x \in \Omega : \rho(x) \leq M e^{\omega|t|} \rho(\varphi(t, x))$, and
- (ii) for every compact set $K \subseteq \Omega$ and, for every $\delta > 0$ and $t \in M$, we have
- $$\varphi(t, \cdot)^{-1}(K) \cap \{x \in \Omega : \rho(x) \geq \delta\} \text{ is a compact subset of } \Omega.$$

We will divide our further exposition into three separate parts:

Part 1. S-hypercyclic multiparameter strongly continuous semigroups induced by semiflows. Since the single operators of semigroup $(T_\varphi(t))_{t \in M}$ have dense range and commute with each other, we can apply [7, Theorem 1, Proposition 1] in order to see that the S-hypercyclicity of $(T_\varphi(t))_{t \in M}$ is equivalent with its S-topological transitivity. Using this observation, [16, Theorem 4.3, Theorem 4.5] and the proof of [22, Theorem 2.4], we can state the following slight extension of [21, Theorem 18].

THEOREM 2.8. *Let $\varphi : M \times \Omega \rightarrow \Omega$ be a semiflow.*

- (i) *Suppose that $(T_\varphi(t))_{t \in M}$ is a strongly continuous semigroup in $L_\rho^p(\Omega)$. Then the following statements are equivalent.*
- (i.1) *$(T_\varphi(t))_{t \in M}$ is S-hypercyclic in $L_\rho^p(\Omega)$.*
- (i.2) *For every compact set $K \subseteq \Omega$ there exist a sequence (L_k) of measurable subsets of K , a sequence (t_k) in M and a sequence (c_k) in $\mathbb{S} \setminus \{0\}$ such that:*

$$\lim_{k \rightarrow \infty} \int_{K \setminus L_k} \rho(x) dx = 0 \quad (12)$$

$$\text{and} \quad \lim_{k \rightarrow \infty} c_k \int_{\varphi(t_k, \cdot)^{-1}(L_k)} \rho(x) dx = \lim_{k \rightarrow \infty} \frac{1}{c_k} \int_{\varphi(t_k, L_k)} \rho(x) dx = 0.$$

In case $S = [0, \infty)$, the above is also equivalent with:

- (i.3) *For every compact set $K \subseteq \Omega$, there exist a sequence (L_k) of measurable subsets of K and a sequence (t_k) in M such that (12) holds and*

$$\lim_{k \rightarrow \infty} \left[\int_{\varphi(t_k, \cdot)^{-1}(L_k)} \rho(x) dx \cdot \int_{\varphi(t_k, L_k)} \rho(x) dx \right] = 0.$$

- (ii) *Suppose that $(T_\varphi(t))_{t \in M}$ is a strongly continuous semigroup in $C_{0,\rho}(\Omega)$ and, for every compact set $K \subseteq \Omega$, we have $\inf_{x \in K} \rho(x) > 0$. Then the following statements are equivalent:*

- (ii.1) *$(T_\varphi(t))_{t \in M}$ is S-hypercyclic in $C_{0,\rho}(\Omega)$.*

(ii.2) For every compact set $K \subseteq \Omega$, there exist a sequence (t_k) in M and a sequence (c_k) in $S \setminus \{0\}$ such that:

$$\lim_{k \rightarrow \infty} c_k \sup_{x \in \varphi(t_k, \cdot)^{-1}(K)} \rho(x) = \lim_{k \rightarrow \infty} \frac{1}{c_k} \sup_{x \in \varphi(t_k, K)} \rho(x) = 0.$$

In case $S = [0, \infty)$, the above is also equivalent with:

(ii.3) For every compact set $K \subseteq \Omega$, there exists a sequence (t_k) in M such that:

$$\lim_{k \rightarrow \infty} \left[\sup_{x \in \varphi(t_k, \cdot)^{-1}(K)} \rho(x) \cdot \sup_{x \in \varphi(t_k, K)} \rho(x) \right] = 0.$$

Part 2. Chaotic multiparameter strongly continuous semigroups induced by semiflows. In this part, we will only state the following result, which can be deduced with the help of arguments used in the proofs of [16, Theorem 5.3, Theorem 5.7] and the analysis from [16, Remark 5.4]:

THEOREM 2.9. Let $\varphi : M \times \Omega \rightarrow \Omega$ be a semiflow.

(i) Suppose that $(T_\varphi(t))_{t \in M}$ is a strongly continuous semigroup in $L_\rho^p(\Omega)$, N is a Lebesgue measurable set such that its Lebesgue measure $m(N)$ is equal to zero and $\varphi(t, N) = N$, $t \in M$. Suppose, further, that for each compact subset K of $\Omega \setminus N$ there exists a real number $r_K > 0$ such that $\varphi(t, K) \cap K = \emptyset$ for every $t \in M$ with $|t| \geq r_K$. Then $(T_\varphi(t))_{t \in M}$ is chaotic in $L_\rho^p(\Omega)$ if and only if for every compact set $K \subseteq \Omega \setminus N$ there exist a sequence (L_k) of measurable subsets of K and a sequence (t_k) in M such that $\lim_{k \rightarrow +\infty} |t_k| = +\infty$, (12) holds and

$$\lim_{k \rightarrow \infty} \left[\sum_{l=1}^{+\infty} \int_{\varphi(t_k, \cdot)^{-1}(L_k)} \rho(x) dx + \sum_{l=1}^{+\infty} \int_{\varphi(t_k, L_k)} \rho(x) dx \right] = 0.$$

(ii) Suppose that $(T_\varphi(t))_{t \in M}$ is a strongly continuous semigroup in $C_{0,\rho}(\Omega)$ and, for every compact set $K \subseteq \Omega$, we have $\inf_{x \in K} \rho(x) > 0$ and the existence of a real number $r_K > 0$ such that $\varphi(t, K) \cap K = \emptyset$ for every $t \in M$ with $|t| \geq r_K$. Then $(T_\varphi(t))_{t \in M}$ is chaotic in $C_{0,\rho}(\Omega)$ if and only if for every compact set $K \subseteq \Omega$ there exists a point $t_K \in M$ such that

$$\lim_{k \rightarrow \infty} \left[\sup_{x \in \varphi(kt_K, \cdot)^{-1}(K)} \rho(x) + \sup_{x \in \varphi(kt_K, K)} \rho(x) \right] = 0.$$

Part 3. (f, S) -topologically mixing multiparameter strongly continuous semigroups induced by semiflows. We will first clarify the following simple consequence of [23, Proposition 3.13].

PROPOSITION 2.10. Suppose that $f : M \rightarrow [0, +\infty)$ and $\sup_{t \in M} f(t) = +\infty$. Then $(T_\varphi(t))_{t \in M}$ is (f, S) -topologically mixing if and only if $(T_\varphi(t_k))_{k \in \mathbb{N}}$ is S -topologically transitive for every sequence $(t_k)_{k \in \mathbb{N}}$ in M such that $\lim_{k \rightarrow +\infty} f(t_k) = +\infty$.

Keeping in mind [16, Theorem 4.3, Theorem 4.5] and Proposition 2.10 (cf. also Theorem 2.8), we immediately get the following result.

THEOREM 2.11. Let $\varphi : M \times \Omega \rightarrow \Omega$ be a semiflow, $f : M \rightarrow [0, +\infty)$ and $\sup_{t \in M} f(t) = +\infty$.

(i) Suppose that $(T_\varphi(t))_{t \in M}$ is a strongly continuous semigroup in $L^p_\rho(\Omega)$. Then $(T_\varphi(t))_{t \in M}$ is (f, S) -topologically mixing if and only if for every compact set $K \subseteq \Omega$ and for every sequence $(t_k)_{k \in \mathbb{N}}$ in M such that $\lim_{k \rightarrow +\infty} f(t_k) = +\infty$ there exist a sequence (L_l) of measurable subsets of K , a subsequence of (t_{k_l}) of (t_k) and a sequence (c_l) in $S \setminus \{0\}$ such that

$$\lim_{l \rightarrow \infty} \int_{K \setminus L_l} \rho(x) dx = 0 \quad \text{and} \quad \lim_{l \rightarrow \infty} c_l \int_{\varphi(t_{k_l}, \cdot)^{-1}(L_l)} \rho(x) dx = \lim_{l \rightarrow \infty} \frac{1}{c_l} \int_{\varphi(t_{k_l}, L_l)} \rho(x) dx = 0.$$

(ii) Suppose that $(T_\varphi(t))_{t \in M}$ is a strongly continuous semigroup in $C_{0,\rho}(\Omega)$ and that $\inf_{x \in K} \rho(x) > 0$ for every compact set $K \subseteq \Omega$. Then $(T_\varphi(t))_{t \in M}$ is (f, S) -topologically mixing if and only if for every sequence $(t_k)_{k \in \mathbb{N}}$ in M such that $\lim_{k \rightarrow +\infty} f(t_k) = +\infty$ and for every compact set $K \subseteq \Omega$ there exist a subsequence of (t_{k_l}) of (t_k) and a sequence (c_l) in $S \setminus \{0\}$ such that:

$$\lim_{l \rightarrow \infty} c_l \sup_{x \in \varphi(t_{k_l}, \cdot)^{-1}(K)} \rho(x) = \lim_{l \rightarrow \infty} \frac{1}{c_l} \sup_{x \in \varphi(t_{k_l}, K)} \rho(x) = 0.$$

It is worth noting that the assumption on strong continuity of operator semigroup $(T_\varphi(t))_{t \in M}$ can be neglected from the formulations of Theorem 2.8, Theorem 2.9 and Theorem 2.11.

We close the paper with the following illustrative example (cf. also [16, Example 3.20, Example 3.21] and [22, Example 3.1, Theorem 3.2, Example 3.2]).

EXAMPLE 2.12. Suppose that $1 \leq p < +\infty$, $M := [0, +\infty)^n$ and $\Omega := (0, +\infty)^n$; then it is clear that $M + \Omega \subseteq \Omega$. Let $f : (0, +\infty) \rightarrow (0, +\infty)$ be the mapping defined in [22, Example 2.1], with $(a_s)_{s \in \mathbb{N}}$ being a sequence of positive real numbers such that the set $\{1/a_s : s \in \mathbb{N}\}$ is bounded; then $f(\cdot)$ is a strictly increasing, bijective, locally Lipschitz continuous mapping and the inverse mapping $f^{-1} : (0, +\infty) \rightarrow (0, +\infty)$ has the same properties. Define $M := \sup\{1/a_s : s \in \mathbb{N}\}$ and $p : (0, +\infty)^n \rightarrow (0, +\infty)^n$ by $p(x_1, \dots, x_n) := (f(x_1), \dots, f(x_n))$, $(x_1, \dots, x_n) \in (0, +\infty)^n$; then $p : (0, +\infty)^n \rightarrow (0, +\infty)^n$ is bijective, locally Lipschitz continuous and the inverse mapping $p^{-1} : (0, +\infty)^n \rightarrow (0, +\infty)^n$, given by

$$p^{-1}(x_1, \dots, x_n) := (f^{-1}(x_1), \dots, f^{-1}(x_n)), \quad (x_1, \dots, x_n) \in (0, +\infty)^n,$$

has the same properties. After that, we define the semiflow $\varphi : M \times \Omega \rightarrow \Omega$ by (9); then it is straightforward to see that

$$\varphi(t_1, \dots, t_n, x_1, \dots, x_n) := \left(f^{-1}(t_1 + f(x_1)), \dots, f^{-1}(t_n + f(x_n)) \right), \quad t \in M, \quad x \in \Omega,$$

as well as that $\varphi : M \times \Omega \rightarrow \Omega$ is a locally Lipschitz continuous semiflow over M . Set $\rho_1(x) := (1 + f(x))^{-1}$, $x > 0$,

$$\rho_n(x_1, \dots, x_n) := \rho_1(x_1) \cdot \dots \cdot \rho_1(x_n), \quad (x_1, \dots, x_n) \in (0, +\infty)^n,$$

and $E := L^p_\rho((0, +\infty)^n)$.

Using the estimate given on [22, l. 3, p. 8] and Theorem 2.6, it readily follows that $(T_\varphi(t))_{t \in M}$ is strongly continuous. We will prove here that $(T_\varphi(t))_{t \in M}$ is strongly topologically mixing (i.e., (f, S) -topologically mixing with $f(t_1, \dots, t_n) = t_1 + \dots + t_n$, $(t_1, \dots, t_n) \in M$ and $S = \{1\}$) as well as that $(T_\varphi(t))_{t \in M}$ is chaotic provided that $n \geq 2$. Towards this end, let us fix a compact set $K = [a_1, b_1] \times \dots \times [a_n, b_n] \subseteq (0, +\infty)^n$ and an integer $l \in \mathbb{N}$. If $(t_k)_{k \in \mathbb{N}} = ((t_{k,1}, \dots, t_{k,n}))_{k \in \mathbb{N}}$ is any sequence in M such that $t_{k,1} + \dots + t_{k,n} \rightarrow +\infty$ as $k \rightarrow +\infty$, then the computation given on [22, p. 8] and the Fubini theorem together imply

$$\begin{aligned} & \int_{\varphi(lt_k, K)} \rho_1(x_1) \cdot \dots \cdot \rho_1(x_n) dx_1 \dots dx_n \\ &= \int_{f^{-1}(lt_{k,1} + [a_1, b_1])} \rho_1(x_1) dx_1 \cdot \dots \cdot \int_{f^{-1}(lt_{k,n} + [a_n, b_n])} \rho_1(x_n) dx_n \\ &\leq M \frac{f(b_1) - f(a_1)}{f(a_1) + lt_{k,1} + 1} \cdot \dots \cdot \frac{f(b_n) - f(a_n)}{f(a_n) + lt_{k,n} + 1} \\ &\leq \frac{M_K}{(1 + lt_{k,1}) \cdot \dots \cdot (1 + lt_{k,n})}, \end{aligned}$$

where $M_K := M(f(b_1) - f(a_1)) \cdot \dots \cdot (f(b_n) - f(a_n))$. Keeping in mind the fact that $\varphi(t, K)^{-1} = \emptyset$ for all $t \in M$ with $|t| \geq r_K$, where the real number $r_K > 0$ is suitable chosen, and Theorem 2.11(i), it follows that $(T_\varphi(t))_{t \in M}$ is strongly topologically mixing. Since

$$\frac{M_K}{(1 + lt_{k,1}) \cdot \dots \cdot (1 + lt_{k,n})} \leq M_K l^{-n} (t_{k,1} \cdot \dots \cdot t_{k,n})^{-1}, \quad k \in \mathbb{N},$$

we similarly obtain from Theorem 2.9(i) that $(T_\varphi(t))_{t \in M}$ is chaotic provided that $n \geq 2$.

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