MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 02.09.2026

research paper
оригинални научни рад
DOI: 10.57016/MV-MnZO6936

HAMMING ENERGY OF SOME GRAPH CLASSES AND CORONA PRODUCTS OF PARTICULAR GRAPHS BASED ON EQUITABLE PARTITIONING

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Abstract. For a simple graph G, the Hamming matrix $H(G)$ is defined in terms of vertex degrees, and its eigenvalues constitute the Hamming spectrum of G. The sum of the absolute values of these eigenvalues is called the Hamming energy, denoted by $HE(G)$. In this paper, using equitable partitions, the Hamming spectrum and Hamming energy are explicitly determined for several classes of graphs, including double-wheel graphs, Dutch windmill graphs D_5^m, bistar graphs, and the corona product $K_3 \circ \overline{K}_m$. In each family, the vertex set is partitioned so that the quotient matrix has small dimension (at most 4×4), which enables direct computation of eigenvalues. The results contribute new closed forms of $HE(G)$ and extend the collection of graphs whose complete Hamming spectrum is known.

1. Introduction

Graphs are among the most extensively studied mathematical structures in modern science due to their rich theoretical foundations and wide range of practical applications. The branch of mathematics that deals with the study of graphs and their properties is known as graph theory. To gain clearer insights into the complex interconnections within graphs, they are often represented in matrix form, which has drawn significant attention from researchers. This representation has made spectral analysis of graphs a key tool for uncovering their structural properties and has found notable applications in fields such as chemistry, physics, and computer science.

Among the various types of matrices used to describe graphs, the adjacency matrix holds a special place, as it captures the connectivity between vertices. From this matrix, an important numerical indicator known as graph energy is derived. The concept of graph energy was introduced by I. Gutman in [3], where it was defined as the sum of the absolute values of the eigenvalues of the adjacency matrix. Due to this definition, graph energy is studied within the framework of spectral graph theory.

2020 Mathematics Subject Classification: 05C09, 05C50
Keywords and phrases: Hamming energy; double-wheel; Dutch windmill; corona product; bistar graph.

Due to its correspondence with the total π-electron energy of conjugated hydrocarbons (computed using Hückel's method), this concept has acquired significant chemical relevance [4], becoming a cornerstone of mathematical chemistry and applied mathematics [9, 10]. Since graphs can be represented not only by an adjacency matrix but also by other matrix forms, numerous alternative types of graph energy have been developed, expanding its range of potential applications [5, 6].

In line with the aforementioned considerations, a new graph invariant called Hamming energy was recently introduced in [14], based on the so-called Hamming matrix. This energy is defined as the sum of the absolute values of the eigenvalues of the Hamming matrix. The core contribution of the cited paper lies in establishing the chemical relevance of this measure, along with explicit computations of the Hamming energy for specific graph classes and estimates (bounds) for this energy in selected graph families. Further significance was attributed to Hamming energy in [13], where the authors emphasized its value by comparing it with the classical graph energy.

Given the inherent complexity of accurately computing graph energy, researchers in graph theory have increasingly focused on deriving reliable spectral bounds. In this context, Das and Gutman formulated upper and lower bounds for the graph energy in terms of the number of vertices and edges [2]. Ramane et al. [12] investigated the eigenvalues of the Hamming matrix and the resulting Hamming energy for several graph families, providing bounds and explicit spectra in specific cases.

Several studies have advanced this field by introducing new types of graph energies and investigating specific classes of graphs. For instance, [15] examines graph structures such as Dutch windmill and double-wheel graphs and establishes results for several energy types, including classical, Seidel, Laplacian, chromatic, distance-based, and Harary energy. Furthermore, the computation of various energies for bistar graphs, corona products of graphs, and other graph classes considered in [7, 8] has motivated the present study. In this paper, we focus on determining the Hamming energy of these particular graph families by employing the equitable partition technique.

Following the introductory part, which outlines the main motivation behind this research, the paper is structured as follows. Section 2 presents the fundamental concepts and relevant results from matrix spectral theory that form the basis of the applied method for computing the Hamming energy of the considered graph classes. Section 3 contains the main results of the study, specifically concerning the determination of the Hamming spectra and energies for the following graph families: double-wheel graphs, Dutch windmill graphs D_5^m, the corona product of graphs $K_3 \circ \overline{K}_m$, and bistar graphs. For each of these classes, we employ the method of equitable vertex partitioning to efficiently compute the corresponding Hamming energy.

2. Preliminaries

Consider a simple graph $G = (V(G), E(G))$ with n vertices and m edges, where the vertex set is given by $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set by $E(G) =$

$\{e_1, e_2, \ldots, e_m\}$. Two vertices v_i and v_j are said to be adjacent if there exists an edge connecting them, in which case we write $v_i \sim v_j$; otherwise, we write $v_i \not\sim v_j$. An edge is said to be incident with each of its endpoints. The degree of a vertex v in the graph G, denoted $d_G(v)$ or simply $d(v)$, represents the number of edges incident to v.

Based on the structure of G and the relationships between its vertices, we can define a square matrix whose entries indicate adjacency between vertices. This matrix, denoted by $A = A(G) = (a_{ij})_{n\times n}$, is referred to as the adjacency matrix of G. It is one of the key structural matrices associated with a graph. Another important matrix is the degree matrix, denoted by $D = D(G) = \mathrm{diag}(d(v_1), \ldots, d(v_n))$, which is a diagonal matrix that lists vertex degrees.

Throughout the remainder of the paper, the symbol $I = I_n$ will be used for the identity matrix of order n, and $J_{m\times n}$ will denote the $m \times n$ matrix consisting entirely of ones. In cases where $m = n$, we will write J_n or simply J, assuming the dimension is clear from the context. The matrix J_n satisfies the following important property.

Remark 2.1. The matrix J_n, whose entries are all equal to one, has the following spectral properties: it possesses a single non-zero eigenvalue equal to n, corresponding to the eigenvector $\mathbf{1} = (\underbrace{1, 1, \ldots, 1}_{n})^T$, and zero as an eigenvalue with multiplicity $n-1$. The eigenvectors associated with the zero eigenvalue are all orthogonal to the vector $\mathbf{1}$.

In addition to the adjacency matrix and the diagonal matrix of vertex degrees, which are square matrices, graph theory also frequently employs the incidence matrix, which is a rectangular matrix. Recall that the incidence matrix of a graph G is given by $B(G) = (b_{ij})_{n\times m}$, where

$$b_{ij} = \begin{cases} 1, & \text{if the vertex } v_i \text{ is incident with the edge } e_j, \\ 0, & \text{otherwise.} \end{cases}$$

The rows of this matrix, when interpreted as binary strings, correspond to the vertices of the graph. Denoting the binary string related to a vertex v by $s(v)$, the authors in [11] introduced the Hamming index of a graph G (abbreviated as the H-index), denoted by $H_B(G)$, defined as the sum of Hamming distances between all pairs of binary strings generated from the incidence matrix of G, i.e.

$$H_B(G) = \sum_{i<j} H_d(s(v_i), s(v_j)),$$

where $H_d(s(v_i), s(v_j))$ represents the standard Hamming distance between the strings $s(v_i)$ and $s(v_j)$, which means the number of coordinates in which they differ.

In the recent publication [14], the Hamming matrix $H(G) = (h_{ij})_{n\times n}$ of a graph G was defined, with entries given by

$$h_{ij} = H_d\big(s(v_i), s(v_j)\big), \quad i, j = 1, \ldots, n. \tag{1}$$

To clearly differentiate the eigenvalues of the adjacency matrix and the Hamming matrix of a graph G, the eigenvalues of $A(G)$ and $H(G)$ are termed as the A-eigenvalues and H-eigenvalues, respectively, while their corresponding sets are called the A-spectrum and the H-spectrum. The Hamming energy of the graph G is also referred to as the H-energy of G.

Since the matrix $H(G)$ is symmetric, all its eigenvalues are real numbers. Let these eigenvalues be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and denote the set of eigenvalues by $Spec(H(G))$, which represents the H-spectrum.

For convenience in expressing spectra with repeated eigenvalues, we use the notation $\lambda^{[k]}$ to indicate that the eigenvalue λ has multiplicity k.

A particularly useful result in the process of determining the Hamming matrix of a graph G is provided in the paper [11], which we will frequently reference throughout this paper.

Theorem 2.2 ([11]). *Let u and v be vertices of a graph G. Then*

$$H_d(G)(s(u), s(v)) = \begin{cases} d(u) + d(v) - 2, & \text{if } u \sim v, \\ d(u) + d(v), & \text{if } u \not\sim v, \\ 0, & \text{if } u = v. \end{cases}$$

The Hamming matrix $H(G)$ is a real symmetric non-negative matrix (i.e., a Hermitian matrix) with zeros along the main diagonal. Hence, its trace satisfies $\mathrm{tr}(H(G)) = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} h_{ii} = 0$. Furthermore, all properties that hold for Hermitian or non-negative matrices also apply to $H(G)$.

Having in mind Theorem 2.2 we conclude that all off-diagonal entries of $H(G)$ are strictly positive except in the graphs K_2 and nK_1. Consequently, $H(G)$ is a nonnegative irreducible matrix for every graph other than K_2 and nK_1.

In what follows, we present fundamental properties of irreducible matrices that will be utilized in the subsequent analysis.

Theorem 2.3 ([1]). *Let M be an irreducible symmetric matrix with non-negative entries. Then the largest eigenvalue λ_1 of M is simple, with a corresponding eigenvector whose entries are all positive (known as the Perron vector). Moreover, $|\lambda| \leq \lambda_1$ for all eigenvalues λ of M.*

Theorem 2.4 ([1]). *Let A be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Given a partition $\{1, 2, \ldots, n\} = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_m$, with $|\Delta_i| = n_i > 0$, $\sum\limits_{i=1}^{m} n_i = n$, consider the corresponding blocking $A = (A_{ij})$ such that A_{ij} is a $n_i \times n_j$ block. Let e_{ij} be the sum of the entries in A_{ij} and set $B = (e_{ij}/n_i)$ (e_{ij}/n_i is the average row sum in A_{ij}). Then the spectrum of B is contained in the segment $[\lambda_n, \lambda_1]$.*

Theorem 2.5 ([1]). *Let A be any non-negative symmetric matrix partitioned into blocks as in Theorem 2.4. Let the blocks A_{ij} have constant row sums b_{ij} and $B = (b_{ij})$. Then the spectrum of B is contained in the spectrum of A (taking into account the multiplicities of the eigenvalues). Furthermore, the largest eigenvalue* (*index*) *of B is equal to the largest eigenvalue of A.*

Remark 2.6. The matrix B defined in Theorem 2.4 is called the quotient matrix of A. In the case of constant row sums of A_{ij} for each pair i and j, B is called equitable quotient matrix of A, and the corresponding partition of the matrix A into blocks A_{ij} is called equitable partition.

3. Main results

In this section, we systematically apply the method of equitable partitions to compute the H-spectrum and H-energy for several families of graphs. By grouping the vertex set into a small number of naturally defined classes, we obtain a much smaller quotient matrix B, whose eigenvalues belong to the spectrum of the complete Hamming matrix $H(G)$.

3.1 The H-spectrum of the double-wheel graph DW_n

Double-wheel graphs serve as important benchmark structures in the study of graceful labeling and are widely used for evaluating symmetry-breaking techniques in constraint programming. Due to their rich symmetrical properties, they are particularly relevant in the design and analysis of algorithms for graph labeling and related combinatorial optimization problems. A double-wheel graph DW_n of size n is the graph $2C_n \nabla K_1$, i.e., it consists of two disjoint cycles of size n whose vertices are connected to a common vertex.

Figure 1 shows a double-wheel graph of size 6.

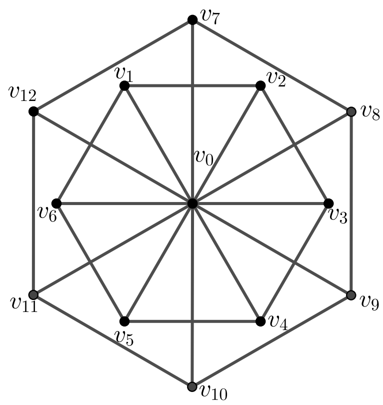

Figure 1: Double-wheel graph DW_6

In the following theorem, we present a characterization of the H-spectrum of the double-wheel graph DW_n, as well as the corresponding value of its H-energy.

Theorem 3.1. *The H-spectrum of the double-wheel graph DW_n is the set*

$$Spec(H(DW_n)) = \left\{6n-5 \pm \sqrt{(6n-5)^2 + 2n(2n+1)^2}\right\}$$
$$\bigcup \left\{ \left(-6 - 4\cos\frac{2k\pi}{n}\right)^{[2]} \mid k = 1, 2, \ldots, n-1 \right\} \bigcup \{-10\}.$$

In addition, the H-energy of DW_n can be expressed as

$$HE(DW_n) = 2\left(6n - 5 + \sqrt{(6n-5)^2 + 2n(2n+1)^2}\right).$$

Proof. If we denote by v_0 the central vertex of DW_n, and by $v_1, \ldots, v_{2n}$ the vertices of disjoint cycles, where vertices $v_1, v_2, \ldots, v_n$ belong to the first cycle, and the vertices $v_{n+1}, v_{n+2}, \ldots, v_{2n}$ belong to the second cycle, then the Hamming matrix $H(DW_n) = (h_{ij})$, $i, j = 0, 1, \ldots, 2n$, of double-wheel graph has the form

$$H(DW_n) = \left[\begin{array}{c|ccc|ccc} 0 & 2n+1 & \cdots & 2n+1 & 2n+1 & \cdots & 2n+1 \\ \hline 2n+1 & & & & & & \\ \vdots & & H & & & 6J_n & \\ 2n+1 & & & & & & \\ \hline 2n+1 & & & & & & \\ \vdots & & 6J_n & & & H & \\ 2n+1 & & & & & & \end{array}\right],$$

where H is the matrix that corresponds to each of the cycles and has the form

$$H = \begin{bmatrix} 0 & 4 & 6 & \cdots & 6 & 4 \\ 4 & 0 & 4 & \cdots & 6 & 6 \\ 6 & 4 & 0 & \cdots & 6 & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 4 & 6 & 6 & \cdots & 4 & 0 \end{bmatrix}.$$

The vertex set of DW_n can be partitioned into two classes (partition sets) as follows:

- $C_0 = \{v_0\}$, where the vertex v_0 has degree $2n$,
- $C_1 = \{v_1, v_2, \ldots, v_{2n}\}$ with all the vertices of degree 3,

thus implying block form of the matrix $H(DW_n)$ as

$$H(DW_n) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},$$

where $H_{11} = [0]_{1\times1}$ is the block corresponding to the central vertex v_0, $H_{12} = (h_{0i})_1^{2n} = (2n+1)J_{1\times(2n+1)}$ is the block representing connections between central and peripheral vertices, $H_{21} = H_{12}^T$ and $H_{22} = (h_{ij})_{(2n)\times(2n)}$ is the block that corresponds to the peripheral vertices from class C_2.

It is easy to see that the partition of $H(DW_n)$ into blocks is equitable with the quotient matrix B given as

$$B = \begin{bmatrix} 0 & 2n(2n+1) \\ 2n+1 & 2(6n-5) \end{bmatrix}.$$

The spectrum of B is obtained by solving the characteristic equation $\det(B - \lambda I) = 0$. Solving this equation yields the eigenvalues of B in the form $\lambda_{1,2}^B = 6n - 5 \pm \sqrt{(6n-5)^2 + 2n(2n+1)^2}$. According to Theorem 2.5 these eigenvalues belong to the spectrum of $H(DW_n)$, and the index (i.e., the maximum eigenvalue) of $H(DW_n)$ coincides with the index of B.

The remaining $2n-1$ eigenvalues of $H(DW_n)$ belong to the subspace which is orthogonal on vectors that are constant within each block. In fact, these eigenvalues correspond to disjoint cycles of length n (represented by the block matrix H_{22}) and can be determined by Fourier diagonalization. It can be verified easily that $H_{22} = 6\,(J_{2n} - I_{2n}) \;-\; 2\big(A(C_n) \oplus A(C_n)\big)$, where $A(C_n) \oplus A(C_n)$ is the direct sum of the two $n \times n$ adjacency matrices of C_n. Recall that for a single cycle C_n, the adjacency matrix satisfies $A(C_n)\,x^{(k)} = 2\cos\left(\frac{2\pi k}{n}\right) x^{(k)}$, $k = 0, 1, \dots, n-1$, where $x^{(k)} = (1, \omega^k, \omega^{2k}, \dots, \omega^{(n-1)k})^T$, $\omega = e^{2\pi i/n}$, are the classical Fourier eigenvectors.

Let $u^{(k)} = (1, \omega^k, \omega^{2k}, \dots, \omega^{(n-1)k})$. For each $k = 1, 2, \dots, n-1$, there are two linearly independent extensions of $x^{(k)}$, one for each cycle, defined with $y^{(k)} = (0, u^{(k)}, \underbrace{0, \dots, 0}_{n})^T$ and $z^{(k)} = (\underbrace{0, \dots, 0}_{n+1}, u^{(k)})^T$, which satisfy

$$H(DW_n)\,y^{(k)} = \big(-6 - 4\cos\tfrac{2\pi k}{n}\big)\,y^{(k)} \quad \text{and} \quad H(DW_n)\,z^{(k)} = \big(-6 - 4\cos\tfrac{2\pi k}{n}\big)\,z^{(k)}.$$

For $k = 0$, we can define the vector $x^{(0)} = (0, \underbrace{1, 1, \dots, 1}_{n}, \underbrace{-1, -1, \dots, -1}_{n})^T$ and a direct calculation shows that $H(DW_n)\,x^{(0)} = -10\,x^{(0)}$.

By previous considerations we conclude that the spectrum of $H(DW_n)$ is given by

$$\begin{aligned} Spec(H(DW_n)) = &\left\{6n{-}5 \pm \sqrt{(6n{-}5)^2 + 2n(2n+1)^2}\right\} \\ &\bigcup \left\{-6{-}4\cos\frac{2k\pi}{n} \;\mid\; k = 1, 2, \dots, n{-}1, (\text{counted twice, once per } C_n)\right\} \\ &\bigcup \{-10\}\,, \end{aligned}$$

which implies that the Hamming energy of DW_n is given as

$$HE(DW_n) = \sum_{i=1}^{n} |\lambda_i| = \underbrace{2\sqrt{(6n{-}5)^2{+}2n(2n{+}1)^2}}_{S} + |{-10}| + \sum_{k=1}^{n-1} 2\left|-6{-}4\cos\frac{2k\pi}{n{-}1}\right|$$

$$= S{+}10{+}2\cdot 6(n{-}1) + \sum_{k=1}^{n-1} 8\cdot\cos\frac{2k\pi}{n{-}1} = 2\left(\sqrt{(6n{-}5)^2{+}2n(2n{+}1)^2}{+}6n{-}5\right). \qquad \square$$

3.2 The H-spectrum of the Dutch windmill graph D_5^m

In [15] various types of energies were obtained for the Dutch windmill graph D_5^m. Building on these results, we here compute the H-spectrum and H-energy of D_5^m. Recall that D_5^m is obtained by coalescing m copies of the cycle C_5 at a single central vertex. As before, we partition the vertex set into classes based on which we obtain the corresponding equitable quotient matrix. The construction of D_5^m is illustrated in Figure 2.

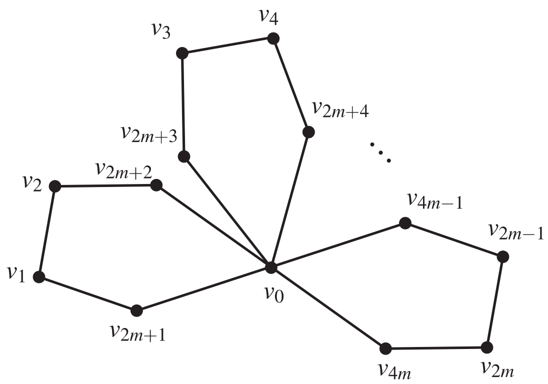

Figure 2: Dutch windmill graph D_5^m

THEOREM 3.2. *Let D_5^m be a Dutch windmill graph. Then, its H-spectrum is the set*

$$\left\{\lambda_1, \lambda_2, \lambda_3,\ (-5-\sqrt{5})^{[m-1]},\ (-5+\sqrt{5})^{[m-1]},\ (-3-\sqrt{5})^{[m]},\ (-3+\sqrt{5})^{[m]}\right\},$$

where $\{\lambda_1, \lambda_2, \lambda_3\}$ is the spectrum of the equitable quotient matrix B corresponding to the matrix $H(D_5^m)$.

Proof. The corresponding Hamming matrix of the Dutch windmill graph is given by

$$H(D_5^m)=\begin{bmatrix} 0 & 2(m{+}1) & 2(m{+}1) & \cdots & 2(m{+}1) & 2m & 2m & 2m & \cdots & 2m \\ 2(m{+}1) & 0 & 2 & \cdots & 4 & 2 & 4 & 4 & \cdots & 4 \\ 2(m{+}1) & 2 & 0 & \cdots & 4 & 4 & 2 & 4 & \cdots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(m{+}1) & 4 & 4 & \cdots & 0 & 4 & 4 & 4 & \cdots & 2 \\ 2m & 2 & 4 & \cdots & 4 & 0 & 4 & 4 & \cdots & 4 \\ 2m & 4 & 2 & \cdots & 4 & 4 & 0 & 4 & \cdots & 4 \\ 2m & 4 & 4 & \cdots & 4 & 4 & 4 & 0 & \cdots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2m & 4 & 4 & \cdots & 2 & 4 & 4 & 4 & \cdots & 0 \end{bmatrix}.$$

According to the structure of the graph D_5^m, the vertex set $V(D_5^m)$ can be partitioned into three classes:

- $C_1 = \{v_0\}$, which contains the central vertex,
- $C_2 = \{v_1, v_2, \ldots, v_{2m}\}$, containing vertices not adjacent to v_0,
- $C_3 = \{v_{2m+1}, v_{2m+2}, \ldots, v_{4m}\}$, which contains the remaining vertices adjacent to v_0.

In this partition, the Hamming matrix $H(D_5^m)$ takes the block form

$$H(D_4^m) = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix},$$

where H_{11}=$[0]_{1\times 1}$ is the block corresponding to the central vertex v_0, H_{12}=$2(m+1)J_{1\times 2m}$ is the block representing the connections between the central vertex and the vertices not adjacent to it, $H_{21} = H_{12}^T$, $H_{13} = 2mJ_{1\times 2m}$ is the block representing the connections between the central vertex and the vertices adjacent to it, $H_{31} = H_{13}^T$, H_{22} is the $2m \times 2m$ block corresponding to the vertices from the class C_2, H_{23} is the $2m \times 2m$ block representing the connections between the vertices from C_2 and C_3, $H_{32} = H_{23}^T$, and H_{33} is the $2m\times 2m$ block corresponding to vertices from the class C_3.

The quotient matrix $B = (b_{ij})_{3\times 3}$ of this equitable partition is given as

$$B = \begin{bmatrix} 0 & 4m(m+1) & 4m^2 \\ 2(m+1) & 2(4m-3) & 2(4m-1) \\ 2m & 2(4m-1) & 4(2m-1) \end{bmatrix},$$

and its eigenvalues are $\lambda_1, \lambda_2, \lambda_3$. These eigenvalues also belong to the H-spectrum of D_5^m.

To find the remaining eigenvalues of $H(D_5^m)$, we construct $4m-2$ linearly independent eigenvectors. Let $\{e_1, e_2, \ldots, e_{4m}\}$ be the standard orthonormal basis of $\mathbb{R}^{4m}$. At this point, let us note an interesting feature that appears in the obtained vectors. During the determination of linearly independent vectors, the golden ratio appears as a constant, i.e., the values $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = 1-\varphi = -\frac{1}{\varphi}$.

We partition our construction into four families:

1) For $i = 1, \ldots, m-1$, set

$$y_i^{(1)} = -\varphi\, e_1 - \varphi\, e_2 + \varphi\, e_{2m-2i+2} + \varphi\, e_{2m-2i+1} - e_{2m+1} - e_{2m+2} + e_{4m-2i+2} + e_{4m-2i+1},$$

and let $w_i^{(1)} = \big(0, y_i^{(1)}\big)^T$.

2) For $i = 1, \ldots, m-1$, set

$$y_i^{(2)} = -\psi\, e_1 - \psi\, e_2 + \psi\, e_{2m-2i+2} + \psi\, e_{2m-2i+1} - e_{2m+1} - e_{2m+2} + e_{4m-2i+2} + e_{4m-2i+1},$$

and let $w_i^{(2)} = \big(0, y_i^{(2)}\big)^T$.

3) For $i = 1, \ldots, m$, set

$$y_i^{(3)} = -\psi\, e_{2m-2i+2} + \psi\, e_{2m-2i+1} + e_{4m-2i+2} - e_{4m-2i+1}, \text{ and let } w_i^{(3)} = \big(0, y_i^{(3)}\big)^T.$$

4) For $i = 1, \ldots, m$, set

$$y_i^{(4)} = -\varphi\, e_{2m-2i+2} + \varphi\, e_{2m-2i+1} + e_{4m-2i+2} - e_{4m-2i+1}, \text{ and let } w_i^{(4)} = \big(0, y_i^{(4)}\big)^T.$$

One checks directly that the vectors

$$w_1^{(1)}, \ldots, w_{m-1}^{(1)},\ w_1^{(2)}, \ldots, w_{m-1}^{(2)},\ w_1^{(3)}, \ldots, w_m^{(3)},\ w_1^{(4)}, \ldots, w_m^{(4)}$$

are linearly independent. Moreover, a straightforward computation shows that

$$\begin{aligned} H(D_5^m) \cdot w_i^{(1)} &= (-5-\sqrt{5})\, w_i^{(1)},\ i = 1, \ldots, m-1, \\ H(D_5^m) \cdot w_i^{(2)} &= (-5+\sqrt{5})\, w_i^{(2)},\ i = 1, \ldots, m-1, \\ H(D_5^m) \cdot w_i^{(3)} &= (-3-\sqrt{5})\, w_i^{(3)},\ i = 1, \ldots, m, \\ H(D_5^m) \cdot w_i^{(4)} &= (-3+\sqrt{5})\, w_i^{(4)},\ i = 1, \ldots, m. \end{aligned}$$

Therefore, the spectrum of the matrix $H(D_5^m)$ is the set

$$\Big\{\lambda_1, \lambda_2, \lambda_3, (-5-\sqrt{5})^{[m-1]}, (-5+\sqrt{5})^{[m-1]}, (-3-\sqrt{5})^{[m]}, (-3+\sqrt{5})^{[m]}\Big\}. \qquad \square$$

Based on the previous theorem, we can easily determine the H-energy of the graph D_5^m.

Corollary 3.3. *The H-energy of the Dutch windmill graph D_5^m is given by*

$$HE(D_5^m) = |\lambda_1| + |\lambda_2| + |\lambda_3| + 16m - 10,$$

where λ_1, λ_2, λ_3 are the eigenvalues of the equitable quotient matrix B corresponding to the matrix $H(D_5^m)$.

3.3 The H-spectrum of the corona product $K_3 \circ \overline{K}_m$ of two graphs

The corona product of two graphs is a binary operation on graphs that produces a new graph by combining the two given graphs. This operation is often used in graph theory to construct more complex structures and to study their properties, such as spectral characteristics, graph energy, and other invariants. In the paper [7], the topological indices of families of the corona product of the graphs were investigated, which served as a motivation for determining the H-spectrum and the H-energy of the graph obtained by the corona product of two graphs. Specifically, in this subsection, we consider the graph $K_3 \circ \overline{K}_m$ formed by the corona product of the graphs K_3 and $\overline{K}_m = mK_1$. For clarity and to better illustrate the considered class of graphs, Figure 3 presents graph $K_3 \circ \overline{K}_6$.

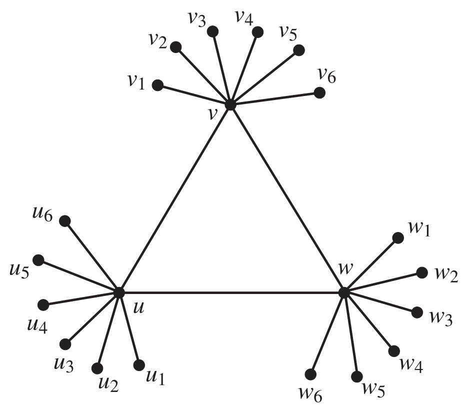

Figure 3: A representation of the corona product $K_3 \circ \overline{K}_6$ of two graphs

Theorem 3.4. *Let $K_3 \circ \overline{K}_m$ be the corona product of the graphs K_3 and $\overline{K}_m = mK_1$. Then, its H-spectrum is the set*

$$Spec(H(K_3 \circ \overline{K}_m)) = \{5m + 1 \pm \sqrt{(m+1)(9m^2 + 34m + 9)}, -2^{[3m-3]},$$
$$-(m + 2 + \sqrt{m^2 + 4m})^{[2]}, -(m + 2 - \sqrt{m^2 + 4m})^{[2]}\}.$$

Proof. The Hamming matrix of the considered graph is given as

$$H\big(K_3\circ\overline{K}_m\big)=\left[\begin{array}{c:c:c:c}(2m+2)\,(J_3-I_3) & M_{12} & M_{13} & M_{14}\\ \hdashline M_{12}^T & 2(J_m-I_m) & 2J_m & 2J_m\\ \hdashline M_{13}^T & 2J_m & 2(J_m-I_m) & 2J_m\\ \hdashline M_{14}^T & 2J_m & 2J_m & 2(J_m-I_m)\end{array}\right],$$

where

$$M_{12}=\begin{bmatrix}m{+}1 & \cdots & m{+}1\\ m{+}3 & \cdots & m{+}3\\ m{+}3 & \cdots & m{+}3\end{bmatrix},\ M_{13}=\begin{bmatrix}m{+}3 & \cdots & m{+}3\\ m{+}1 & \cdots & m{+}1\\ m{+}3 & \cdots & m{+}3\end{bmatrix},\ M_{14}=\begin{bmatrix}m{+}3 & \cdots & m{+}3\\ m{+}3 & \cdots & m{+}3\\ m{+}1 & \cdots & m{+}1\end{bmatrix}.$$

In accordance with the structure of the corona product $K_3\circ\overline{K}_m$, the vertex set $V(K_3\circ\overline{K}_m)$ can be divided into classes as follows:

- $C_1=\{u,v,w\}$, the three "central" vertices, each of degree $m+2$,
- $C_2=\{u_1,u_2,\dots,u_m,v_1,v_2,\dots,v_m,w_1,w_2,\dots,w_m\}$, the $3m$ pendant vertices.

Consequently, the Hamming matrix decomposes into a 2×2 block form given by

$$H\big(K_3\circ\overline{K}_m\big)=\begin{bmatrix}H_{11} & H_{12}\\ H_{12}^T & H_{22}\end{bmatrix},$$

where H_{11} is the block corresponding to the mutual relations among the vertices within class C_1, H_{12} is the $3\times 3m$ block representing the connections between the vertices of class C_1 and the vertices of class C_2, and H_{22} is the $3m\times 3m$ block corresponding to the vertices of class C_2.

The corresponding equitable quotient matrix $B=(b_{ij})_{2\times 2}$ has entries $b_{11}=4m{+}4$, $b_{12}=3m^2+7m$,$b_{21}=3m+7$, $b_{22}=6m-2$, so its characteristic equation is

$$det(B-\lambda I)=\begin{vmatrix}4m+4-\lambda & 3m^2+7m\\ 3m+7 & 6m-2-\lambda\end{vmatrix}=0.$$

Solving this equation gives two eigenvalues

$$\lambda_{1,2}=5m+1\pm\sqrt{(m+1)\,(9m^2+34m+9)},$$

which also belong to the H-spectrum of $K_3\circ\overline{K}_m$.

To obtain the remaining $3m{+}1$ eigenvalues, we provide $3m{+}1$ linearly independent eigenvectors. Let $\{e_1,e_2,\dots,e_{3m+3}\}$ be the standard orthonormal basis of $\mathbb{R}^{3m+3}$, and set $\alpha=\frac{1}{2}\big(m+\sqrt{m^2+4m}\big)$, $\beta=\frac{1}{2}\big(m-\sqrt{m^2+4m}\big)$.

We then split our construction into five families.

1) For $i=1,2$, define

$$y_i^{(1)}=-\alpha e_1+\alpha e_{4-i}+\sum_{k=(3-i)m+4}^{(4-i)m+3}e_k-\sum_{k=4}^{m+3}e_k,\quad \text{and set}\quad w_i^{(1)}=(y_i^{(1)})^T.$$

2) For $i=1,2$, define

$$y_i^{(2)}=-\beta e_1+\beta e_{4-i}+\sum_{k=(3-i)m+4}^{(4-i)m+3}e_k-\sum_{k=4}^{m+3}e_k,\quad \text{and let}\quad w_i^{(2)}=(y_i^{(2)})^T.$$

3) For $i = 1, \ldots, m-1$, set
$$y_i^{(3)} = e_{2m+4} - e_{3m+4-i}, \quad \text{and} \quad w_i^{(3)} = (y_i^{(3)})^T.$$

4) For $i = 1, \ldots, m-1$, set
$$y_i^{(4)} = e_{m+4} - e_{2m+4-i}, \quad \text{and} \quad w_i^{(4)} = (y_i^{(4)})^T.$$

5) For $i = 1, \ldots, m-1$, set
$$y_i^{(5)} = e_4 - e_{m+4-i}, \quad \text{and} \quad w_i^{(5)} = (y_i^{(5)})^T.$$

One checks directly that the vectors
$$\{\, w_1^{(1)}, w_2^{(1)},\, w_1^{(2)}, w_2^{(2)}\} \,\cup\, \{\, w_i^{(j)} \mid 1 \le i \le m-1,\ j = 3,4,5\}$$
are linearly independent. Moreover, a straightforward computation shows that
$$\begin{aligned} H(K_3 \circ \overline{K}_m) \cdot w_i^{(1)} &= -\big(m + 2 + \sqrt{m^2+4m}\big)\, w_i^{(1)},\ i = 1,2, \\ H(K_3 \circ \overline{K}_m) \cdot w_i^{(2)} &= -\big(m + 2 - \sqrt{m^2+4m}\big)\, w_i^{(2)},\ i = 1,2, \\ H(K_3 \circ \overline{K}_m) \cdot w_i^{(j)} &= -2\, w_i^{(j)},\ j = 3,4,5,\ i = 1,\ldots,m-1. \end{aligned}$$
Therefore, the spectrum of the matrix $H(K_3 \circ \overline{K}_m)$ is the set
$$\begin{aligned} &\{5m + 1 \pm \sqrt{(m+1)(9m^2+34m+9)}, -2^{[3m-3]}, \\ &-(m+2+\sqrt{m^2+4m})^{[2]}, -(m+2-\sqrt{m^2+4m})^{[2]}\}. \end{aligned}$$
□

Corollary 3.5. *The Hamming energy of the corona product $K_3 \circ \overline{K}_m$ is given by* $HE(K_3 \circ \overline{K}_m) = 2\left(5m + 1 + \sqrt{(m+1)(9m^2+34m+9)}\right)$.

3.4 The H-spectrum of the bistar graph $B(n, m)$

The bistar graph $B(n, m)$ is obtained by joining the centers of two star graphs $K_{1,n}$ and $K_{1,m}$ with an edge. Thus, the bistar graph has $n + m + 2$ vertices: two central vertices and n and m pendant vertices attached to them, respectively. The bistar graph is illustrated in Figure 4.

In [7], the energy of the bistar graph $B(n, m)$ was studied, which motivated us to determine the H-spectrum and H-energy for this class of graphs.

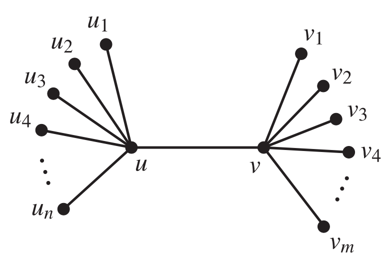

Figure 4: A representation of the bistar graph $B(n, m)$

Theorem 3.6. *Let $B(n,m)$ be a bistar graph. Then, its H-spectrum is the set $Spec(B(n,m)) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, -2^{[m+n-2]}\}$, where $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ is the spectrum of the equitable quotient matrix B corresponding to the matrix $H(B(n,m))$.*

Proof. The Hamming matrix of the bistar graph $B(n,m)$ has the form

$$H\big(B(n,m)\big) = \left[\begin{array}{c|c|c} (m+n)\,(J_2 - I_2) & M_{12} & M_{13} \\ \hline M_{12}^T & 2\,(J_n - I_n) & 2\,J_{n\times m} \\ \hline M_{13}^T & 2\,J_{m\times n} & 2\,(J_m - I_m) \end{array}\right],$$

where

$$M_{12} = \begin{bmatrix} n & \cdots & n \\ m+2 & \cdots & m+2 \end{bmatrix}_{2\times n} \quad \text{and} \quad M_{13} = \begin{bmatrix} n+2 & \cdots & n+2 \\ m & \cdots & m \end{bmatrix}_{2\times m}.$$

In accordance with the structure of the graph $B(n,m)$, the vertex set $V(B(n,m))$ can be divided into four classes (partition sets) as follows:

- $C_1 = \{u\}$ is the center of the star $K_{1,n}$,
- $C_2 = \{v\}$ is the center of the star $K_{1,m}$,
- $C_3 = \{u_1, u_2, \ldots, u_n\}$ are the pendant vertices attached to u,
- $C_4 = \{v_1, v_2, \ldots, v_m\}$ are the pendant vertices attached to v.

Accordingly, the Hamming matrix $H\big(B(n,m)\big)$ admits a 4×4 block decomposition

$$H\big(B(n,m)\big) = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{12}^T & H_{22} & H_{23} & H_{24} \\ H_{13}^T & H_{23}^T & H_{33} & H_{34} \\ H_{14}^T & H_{24}^T & H_{34}^T & H_{44} \end{bmatrix},$$

where the block H_{ij} represents the mutual connections between the vertices from the corresponding classes.

Based on the observed block matrix, we can form the quotient matrix $B = (b_{ij})_{4\times 4}$ of this equitable partition given as

$$B = \begin{bmatrix} 0 & n+m & n^2 & m(n+2) \\ n+m & 0 & n(m+2) & m^2 \\ n & m+2 & 2(n-1) & 2m \\ n+2 & m & 2n & 2(m-1) \end{bmatrix}.$$

By solving its characteristic equation $\det\big(B - \lambda I\big) = 0$ we obtain the four eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ which belong to the H-spectrum of $B(n,m)$.

To obtain the remaining $n+m-2$ eigenvalues (all equal to -2), we construct a corresponding set of linearly independent eigenvectors.

Let $\{e_1, \ldots, e_{n+m}\}$ be the standard orthonormal basis of $\mathbb{R}^{n+m}$. Define two families:

1) For $i = 1, \ldots, n-1$, set $y_i^{(1)} = e_{n+1-i} - e_1$ and $w_i^{(1)} = (0, 0, y_i^{(1)})^T$. These $n-1$ vectors satisfy $H(B(n,m)) \cdot w_i^{(1)} = -2\, w_i^{(1)}$.

2) For $i = 1, \ldots, m-1$, set $y_i^{(2)} = e_{m+n+1-i} - e_{n+1}$ and $w_i^{(2)} = (0, 0, y_i^{(2)})^T$. These $m-1$ vectors also satisfy $H(B(n,m)) \cdot w_i^{(2)} = -2\, w_i^{(2)}$.
It can be easily seen that these $n+m-2$ vectors are linearly independent. Hence,

$$Spec\big(H(B(n,m))\big) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4,\ -2^{[n+m-2]}\}. \qquad \square$$

Based on the obtained H-spectrum, we can determine the H-energy of an arbitrary bistar graph.

Corollary 3.7. *The H-energy of a bistar graph $B(n,m)$ is given by $HE(B(n,m)) = |\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4| + 2(m+n-2)$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of the equitable quotient matrix B corresponding to the matrix $H(B(n,m))$.*

4. Conclusions

In this paper, the Hamming energies of four important graph classes were determined: double-wheel graphs, Dutch windmill graphs D_5^m, the corona product of graphs $K_3 \circ \overline{K}_m$, and bistar graphs. By applying the method of equitable partitions, we obtained closed-form expressions for the Hamming spectra and energies of the considered graphs, thereby enriching the existing list of graphs for which exact values of these spectral invariants are known.

Possible directions for future research include the study of the Hamming energy in a broader range of graph classes, as well as its generalizations to more complex structures such as various types of graph products and their variants. Special attention should be given to analyzing the relationship between the Hamming energy of the corona product of graphs and the Hamming energies of its factor graphs. Such investigations could contribute to a deeper understanding of structural properties of graphs through spectral invariants and reveal potential patterns in energy behavior under different graph operations.

Acknowledgement. This work was supported by the Serbian Ministry of Science, Technological Development and Innovation (Agreement No. 451-03-137/2025-03/200122 and Agreement No. 451-03-34/2026-03/200122).

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(received 24.10.2025; in revised form 22.12.2025; available online 02.09.2026)

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