

ON THE SPACES OF IDEMPOTENT PROBABILITY MEASURES ON SUPERPARACOMPACT SPACES

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Abstract. For a Tychonoff space X , we consider the space $I_f(X)$ that is a subspace of the space of all idempotent probability measures on X . Using a cover γ_X of the origin space X we build an open cover γ_0 of $I_f(X)$. Then we show that if γ_X is a finite-component family, then γ_0 is also a finite-component family. We obtain that the space $I_f(\beta X)$ of idempotent probability measures on the Stone-Čech compactification βX of X is a perfect compactification of $I_f(X)$. Finally, applying the superparacompactness criterion, we establish that $I_f(X)$ is superparacompact if and only if the given Tychonoff space X is superparacompact.

1. Introduction

The study of topological spaces and their properties has been a central topic in mathematics, with particular interest in understanding completeness properties and compactifications. Among these, the concept of superparacompactness, investigated in [10], has emerged as a significant property that generalizes classical notions of compactness in topology. In [11], several types of such spaces, which are called compact type spaces, were explored. Actions by the hyperspace functor were studied in [14]. To establish their results, the authors of the latter paper applied the criterion of superparacompactness of Tychonoff spaces given in [3]. In [8], the hyperspace of a compact Hausdorff space was equipped with a metric different from the well-known Hausdorff metric. We use the notion of density function considered in [1], where it was shown that each idempotent probability measure is determined by a unique density function. Using this property, we were able to define the concept of a support of an idempotent probability measure in a completely new way.

It should be noted that idempotent mathematics is increasingly used in various areas of modern research. For example, it plays an important role in artificial intelligence. In [17], some important topological properties of the space of idempotent

2020 Mathematics Subject Classification: 28C10, 46G12, 54A05, 54D20

Keywords and phrases: Superparacompact space; perfect compactification; idempotent probability measure; finite-component cover.

probability measures on a compact Hausdorff space were obtained. In particular, normality properties considered in [12] were proved for the functor I of idempotent probability measures in the category $Comp$ of compact Hausdorff spaces and their continuous mappings. Using a construction suggested in [4], the author of [7] has extended this functor to the category $Tych$ of Tychonoff spaces and their continuous mappings. Other types of extensions of the functor I were studied in [9, 13].

In this paper, we continue the investigation of superparacompactness of topological spaces. We show that the superparacompactness of the space $I_f(X)$ of idempotent probability measures is equivalent to the superparacompactness of a given Tychonoff space X . To obtain this result, we use the superparacompactness criterion for Tychonoff spaces. To ensure that this criterion is met, we first show that the space $I_f(\beta X)$ is a perfect compactification of $I_f(X)$. It is worth recalling that set systems play a key role in the study of classes of compact type spaces. For a given system γ_X of subsets of a Tychonoff space X , we construct a set system γ_0 in $I_f(X)$. We show that the openness of γ_X implies the openness of γ_0 . Moreover, we obtain that if γ_X is a finite-component family, then γ_0 is also a finite-component family.

2. Preliminaries

In the present paper, by a space we mean a topological T_1 -space, and by a mapping a continuous mapping. If A is a subset of a space X , then $\text{Int}_X(A)$, $\text{Cl}_X(A)$ and $\text{Fr}_X(A)$ denote the interior, closure and boundary of A in X , respectively. In general, topological terminology and notation follow the book [5].

We also put $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with the operations $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$. This so-called max-plus algebra (in fact, an idempotent semifield) is an important structure in idempotent mathematics [9].

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of a space X , and $x \in X$. Then:

- (i) $\mathcal{A}$ is said to be *star-countable* (respectively, *star-finite*) if each element of $\mathcal{A}$ intersects at most countably (respectively, finitely) many elements of $\mathcal{A}$;
- (ii) $\mathcal{A}$ *refines* $\mathcal{B}$ (or $\mathcal{A}$ is a *refinement* of $\mathcal{B}$) if for each element $A \in \mathcal{A}$ there is an element $B \in \mathcal{B}$ such that $A \subset B$;
- (iii) We set $[\mathcal{A}]_X := \{\text{Cl}_X(A) : A \in \mathcal{A}\}$.

DEFINITION 2.1 ([11]). A space X is said to be *superparacompact* if every open cover $\mathcal{U}$ of X has a finite-component open cover that refines $\mathcal{U}$.

For a Tychonoff space X let βX be its Stone-Ćech compactification (i. e., the maximal compact extension of X). The following criterion plays a key role in investigation of the class of superparacompact spaces.

THEOREM 2.2 ([10]). A Tychonoff space X is superparacompact if and only if for every closed set F lying in the remainder $\beta X \setminus X$ there exists a finite-component open cover η of X with $F \cap (\bigcup [\eta]_{\beta X}) = \emptyset$.

Let vX be a compact extension of a Tychonoff space X . If $H \subset X$ is an open set in X , then by $O(H)$ (or by $O_{vX}(H)$) we denote the maximal (by inclusion) open set in vX satisfying $O_{vX}(H) \cap X = H$. It is easy to see that

$$O_{vX}(H) = \bigcup_{\substack{\Gamma \in \tau_{vX}, \\ \Gamma \cap X = H}} \Gamma,$$

where τ_{vX} is the topology of the space vX .

A compactification vX of a Tychonoff space X is called *perfect with respect to an open set H in X* if the equality $\text{Cl}_{vX}(\text{Fr}_X(H)) = \text{Fr}_{vX}(O_{vX}(H))$ holds. If vX is perfect for every open set in X , then it is called a *perfect compactification* of the space X (see [2, p. 232]).

A compactification vX of space X is perfect if and only if for any two disjoint open sets U_1 and U_2 in X the equality $O(U_1 \cup U_2) = O(U_1) \cup O(U_2)$ holds [2]. The Stone-Čech compactification βX of a Tychonoff space X is a perfect compactification of X . The equality $O(U_1 \cup U_2) = O(U_1) \cup O(U_2)$ holds for every pair of open sets U_1 and U_2 in X if and only if X is normal, and the compactification vX coincides with the Stone-Čech compactification βX , i.e., $vX \cong \beta X$.

The following criterion of superparacompactness is proven in [3, Theorem 1.3].

THEOREM 2.3. *A Tychonoff space X is superparacompact if and only if there exists a perfect compactification bX of X such that for every compact $B \subset bX \setminus X$ the space X has a finite-component open cover η such that $B \cap (\bigcup [\eta]_b X) = \emptyset$.*

COROLLARY 2.4. *A Tychonoff space X is superparacompact if and only if for every compact $B \subset \beta X \setminus X$ the space X has a finite-component open cover η such that $B \cap (\bigcup [\eta]_\eta X) = \emptyset$.*

The real line is a strongly paracompact space, but it is not a superparacompact space. Any infinite discrete space is superparacompact but not compact. So, the class of superparacompact spaces lies strictly between the class of compact Hausdorff spaces and the class of strongly paracompact spaces.

Let us present the following property of superparacompact spaces (see [10, Proposition 1.4]):

(*) A closed subspace of a superparacompact space is superparacompact.

The author of [17] considered the functor $l : \mathfrak{Comp} \rightarrow \mathfrak{Comp}$ and showed that it is normal. Then, in [7], using the construction suggested by A. Ch. Chigogidze [4], it was obtained an extension $l_\beta : \mathfrak{Tych} \rightarrow \mathfrak{Tych}$. Here, the sign $\mathfrak{Comp}$ means the category of compact Hausdorff spaces and their continuous mappings, and $\mathfrak{Tych}$ the category of Tychonoff spaces and their continuous mappings.

In [17] on a compact Hausdorff space X an idempotent probability measure is defined as a functional $\mu : C(X) \rightarrow \mathbb{R}$ that meets the following conditions:

- (i) $\mu(c_X) = c$ for every constant function $c_X : X \rightarrow \mathbb{R}$, $c \in \mathbb{R}$. Here $c_X(x) = c$;
- (ii) $\mu(c \odot \varphi) = c \odot \mu(\varphi)$, $c \in \mathbb{R}$, $\varphi \in C(X)$. Here $c \odot \varphi = c + \varphi$;

(iii) $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$, $\varphi, \psi \in C(X)$. Here $\varphi \oplus \psi = \max\{\varphi, \psi\}$.

A set of all idempotent probability measures in X is denoted by $\mathbf{l}(X)$. It is endowed with the topology τ_p of pointwise convergence. For $\mu \in \mathbf{l}(X)$ sets

$$\langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle = \{ \nu \in \mathbf{l}(X) : |\nu(\varphi_i) - \mu(\varphi_i)| < \theta, i = 1, \dots, n \}$$

forms a base of $\mathbf{l}(X)$ at μ . Here $\varphi, \dots, \varphi_n \in C(X)$, $\theta > 0$.

Note that a function $f: X \rightarrow [-\infty, +\infty)$ is said to be *upper semi-continuous* if for each $x \in X$ and for every real number r that satisfies $f(x) < r$, there exists an open neighborhood $U \subset X$ of x such that $f(x') < r$ for all $x' \in U$.

Now we consider a compact Hausdorff space X , and put

$$USC_0(X) = \left\{ f: X \rightarrow [-\infty, 0] : f \text{ is upper semi-continuous such that} \right. \\ \left. \text{there exists } x \in X \text{ with } f(x) = 0 \right\}.$$

For a point $x \in X$ the Dirac measure $\delta_x: C(X) \rightarrow \mathbb{R}$ is defined by the equality $\delta_x(\varphi) = \varphi(x)$ at $\varphi \in C(X)$. It is easy to see that $\text{supp } \delta_x = \{x\}$. For each idempotent probability measure $\nu \in \mathbf{l}(X)$ there exists [1] a unique upper semi-continuous function $\chi_\nu \in USC_0(X)$ such that $\nu = \bigoplus_{x \in X} \chi_\nu(x) \odot \delta_x$. Consequently [15],

$$\mathbf{l}(X) = \left\{ \bigoplus_{x \in X} \lambda(x) \odot \delta_x : \lambda \in USC_0(X) \right\}.$$

For an idempotent measure $\mu = \bigoplus_{x \in X} \chi_\mu(x) \odot \delta_x$ the set $\text{supp } \mu = \{x \in X : \chi_\mu(x) > -\infty\}$ will be called the *support* of μ [15].

For a positive integer n let $\mathbf{l}_n(X)$ denote the set of all idempotent probability measures whose support consists of no more than n points. Put $\mathbf{l}_\omega(X) = \bigcup_{n=1}^{\infty} \mathbf{l}_n(X)$. An idempotent probability measure $\mu \in \mathbf{l}_\omega(X)$ is said to be an idempotent probability measure with finite support.

For a compact Hausdorff space X we put

$$\mathbf{l}_f(X) = \left\{ \mu = \bigoplus_{i=1}^n \chi_\mu(x_i) \odot \delta_{x_i} \in \mathbf{l}_\omega(X) : \exists x_{i_0} \in \text{supp } \mu = \{x_1, \dots, x_n\} \right. \\ \left. \text{such that } \chi_\mu(x_{i_0}) = 0 \text{ and } \chi_\mu(x_i) \leq -\frac{n}{n+1} \text{ at } i \neq i_0 \right\}.$$

In [3] it was considered inequality $\chi_\mu(x_j) \leq -\ln(n+1)$ instead of $\chi_\mu(x_j) \leq -\frac{n}{n+1}$. For a compact Hausdorff space X the built space $\mathbf{l}_f(X)$ is also a compact Hausdorff space.

For a point $x \in X$ let us define the following set

$$\mathbf{l}_f^x(X) = \left\{ \mu = \bigoplus_{i=1}^n \chi_\mu(x_i) \odot \delta_{x_i} \in \mathbf{l}_f(X) : x \in \text{supp } \mu \text{ and } \chi_\mu(x) = 0 \right\}.$$

It is easy to show that the following statement holds.

PROPOSITION 2.5. *For a compact Hausdorff space X we have:*

- (a) $I_f^x(X) \cap I_f^y(X) = \emptyset$ if and only if $x \neq y$ for each pair of points $x, y \in X$;
 (b) $I_f(X) = \bigcup_{x \in X} I_f^x(X)$.

Consider a subset $M \subset X$ and put

$$\langle M \rangle = \{ \mu \in I_f(X) : \text{there is a point } x \in M \text{ such that } \mu \in I_f^x(X) \}.$$

This construction immediately gives the following property.

PROPOSITION 2.6. *Let M be a nonempty subset of a Hausdorff compact space X . Then $I_f^x(X) \subset \langle M \rangle$ if and only if $x \in M$.*

PROPOSITION 2.7. *For any couple M and N of nonempty subsets of a Hausdorff compact space X we have $\langle M \rangle \cap \langle N \rangle \neq \emptyset$ if and if $M \cap N \neq \emptyset$.*

Proof. Let $M \cap N = \emptyset$. Then for an arbitrary $\mu \in \langle M \rangle$ and every $y \in N$ we have $\chi_\mu(y) \neq 0$. Indeed, if $y \in N \setminus \text{supp } \mu$, then $\chi_\mu(y) = -\infty$; if $y \in N \cap \text{supp } \mu$, then $\chi_\mu(y) \leq -\frac{|\text{supp } \mu|}{|\text{supp } \mu|+1}$. Therefore, $\chi_\mu(y) < 0$. Consequently, $\mu \notin \langle N \rangle$, that is, $\langle M \rangle \cap \langle N \rangle = \emptyset$.

Assume $M \cap N \neq \emptyset$. For a point $x \in M \cap N$ Proposition 2.6 provides $I_f^x(X) \subset \langle M \rangle$ and $I_f^x(X) \subset \langle N \rangle$. So, $\langle M \rangle \cap \langle N \rangle \neq \emptyset$.

Proposition 2.7 is proved. $\square$

THEOREM 2.8. *Let M be a nonempty subset of a Hausdorff compact space X . Then $\langle M \rangle$ is open in $I_f(X)$ if and only if M is open in X . Similarly, $\langle M \rangle$ is closed in $I_f(X)$ if and only if M is closed in X .*

Proof. Let $M = U$ be an open set in X . Clearly, $\delta_x \in \langle U \rangle$ for every $x \in U$. Indeed, $\delta_x \in I_f^x(X)$ for each $x \in X$, in particular, for $x \in U$.

Fix x_0 and $\mu_0 \in I_f^{x_0}(X)$ with $|\text{supp } \mu_0| \geq 2$. Consider a function $\varphi \in C(X)$, $\varphi : X \rightarrow [0, 1]$, such that $\varphi(x_0) = 1$ and $\varphi^{-1}\left(\left(\frac{1}{2}, 1\right]\right) = U$. We have $\chi_{\mu_0}(x_0) = 0$ and $\chi_{\mu_0}(x) \leq -\frac{2}{3}$ at any $x \in \text{supp } \mu_0 \setminus \{x_0\}$. Consequently, $\chi_{\mu_0}(x_0) \odot \varphi(x_0) = 0 \odot 1 = 1$ and $\chi_{\mu_0}(x) \odot \varphi(x) \leq -\frac{2}{3} \odot 1 = \frac{1}{3}$. That is why $\mu_0(\varphi) = 1$.

Let $\nu \in \langle \mu_0; \varphi; \frac{1}{2} \rangle \cap I_f(X)$. If $\text{supp } \nu = \{z\}$ and $z \notin U$, then $\nu = \delta_z$ and $\nu(\varphi) = \varphi(z) \leq \frac{1}{2}$. Adding $1 \leq \mu_0(\varphi) \leq 1$ and $-\frac{1}{2} \leq -\nu(\varphi) \leq 0$ we get the contradiction $\frac{1}{2} \leq \mu_0(\varphi) - \nu(\varphi) \leq 1$. So, in the case $\nu = \delta_z$ the affiliation $\nu \in \langle \mu_0; \varphi; \frac{1}{2} \rangle \cap I_f(X)$ yields $z \in U$. That means $\nu \in \langle U \rangle$.

Consider the case $|\text{supp } \nu| \geq 2$. Let $\nu \in I_f^y(X)$. Then $\chi_\nu(y) = 0$ and $\chi_\nu(x) \leq -\frac{2}{3}$ at any $x \in \text{supp } \nu \setminus \{y\}$. Suppose $y \notin U$. Then $\chi_\nu(y) \odot \varphi(y) \leq 0 \odot \frac{1}{2} = \frac{1}{2}$ and $\chi_\nu(x) \odot \varphi(x) \leq -\frac{2}{3} \odot 1 = \frac{1}{3}$. Hence, $\nu(\varphi) \leq \frac{1}{2}$, or the same $-\frac{1}{2} \leq -\nu(\varphi) \leq 0$. Adding $1 \leq \mu_0(\varphi) \leq 1$ and $-\frac{1}{2} \leq -\nu(\varphi) \leq 0$, we again received the same contradiction $\frac{1}{2} \leq \mu_0(\varphi) - \nu(\varphi) \leq 1$. So, $y \in U$ which implies $\nu \in \langle U \rangle$.

Thus, $\langle \mu_0; \varphi; \frac{1}{2} \rangle \cap I_f(X) \subset \langle U \rangle$.

Consider the case when $M = F$ is closed. Note that Proposition 2.5 provides that

$$\bigcup_{x \in F} I_f^x(X) \bigcup \bigcup_{x \in X \setminus F} I_f^x(X) = I_f(X) \quad \text{and} \quad \bigcup_{x \in F} I_f^x(X) \bigcap \bigcup_{x \in X \setminus F} I_f^x(X) = \emptyset.$$

Consequently, $\langle X \setminus F \rangle = \mathbf{l}_f(X) \setminus \langle F \rangle$. Since the left side of last equality is open, the set $\langle F \rangle$ should be closed.

Theorem 2.8 is proved. $\square$

Theorem 2.8 gives the following properties.

COROLLARY 2.9. *For a subset M of a Hausdorff compact space X we have*

- $\langle \text{Cl}_X M \rangle = \text{Cl}_{\mathbf{l}_f(X)} \langle M \rangle$;
- $\langle \text{Int}_X M \rangle = \text{Int}_{\mathbf{l}_f(X)} \langle M \rangle$.

3. A perfect compactification and superparacompactness of the space of idempotent probability measures

For an element $\alpha \in \mathbb{R}_{\max}$ and a subset $A \subset \mathbf{l}(X)$ we put $\alpha \odot A = \{\alpha \odot \mu : \mu \in A\}$. Note that if $\alpha < 0$, then the set $\alpha \odot A$ is not a subset of $\mathbf{l}(X)$ but it is a subset of the “parallel” space $\alpha \odot \mathbf{l}(X)$. Let

$$\alpha \odot A \oplus \beta \odot B = \{\alpha \odot \mu \oplus \beta \odot \nu : \mu \in A, \nu \in B\}$$

for some $\alpha, \beta \in \mathbb{R}_{\max}$ and $A, B \subset \mathbf{l}(X)$. Evidently, $\alpha \odot A \oplus \beta \odot B \subset \mathbf{l}(X)$ if and only if $\alpha \oplus \beta = 0$.

Now we note the following remarkable property.

REMARK 3.1. For every pair of $\alpha, \beta \in \mathbb{R}_{\max}$ with $\alpha \oplus \beta = 0$, we have

$$\alpha \odot \mathbf{l}(X) \oplus \beta \odot \mathbf{l}(X) = \mathbf{l}(X).$$

This property can be demonstrated by direct verification. So, for $\mu, \nu \in \mathbf{l}(X)$, and elements $\alpha, \beta \in \mathbb{R}_{\max}$ with $\alpha \oplus \beta = 0$, a max-plus combination $\alpha \odot \mu \oplus \beta \odot \nu$ is an idempotent probability measure on X .

PROPOSITION 3.2. *Let $\langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle$ and $\langle \nu; \psi_1, \dots, \psi_k; \delta \rangle$ be neighborhoods of idempotent probability measures μ and ν , respectively. Then the set*

$$\begin{aligned} & \alpha \odot \langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle \oplus \beta \odot \langle \nu; \psi_1, \dots, \psi_k; \delta \rangle \\ &= \{\alpha \odot \mu' \oplus \beta \odot \nu' : \mu' \in \langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle \text{ and } \nu' \in \langle \nu; \psi_1, \dots, \psi_k; \delta \rangle\} \end{aligned}$$

is open in $\mathbf{l}(X)$.

Proof. Denote $\Xi = \alpha \odot \langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle \oplus \beta \odot \langle \nu; \psi_1, \dots, \psi_k; \delta \rangle$. Three cases are possible.

- 1) $-\infty \leq \alpha < \beta = 0$. Then $\Xi \cap \mathbf{l}(X) = \langle \nu; \psi_1, \dots, \psi_k; \delta \rangle$; in this case the set $\alpha \odot \langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle$ is a subset of the “parallel” space $\alpha \odot \mathbf{l}(X)$ as an open subset.
- 2) $-\infty \leq \beta < \alpha = 0$. Then $\Xi \cap \mathbf{l}(X) = \langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle$, and $\beta \odot \langle \nu; \psi_1, \dots, \psi_k; \delta \rangle$ is open in the “parallel” space $\beta \odot \mathbf{l}(X)$.

So, in the above two cases, in general, the set $\alpha \odot \langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle$ is open in the space $\alpha \odot \mathbf{l}(X)$, and $\beta \odot \langle \nu; \psi_1, \dots, \psi_k; \delta \rangle$ is in $\beta \odot \mathbf{l}(X)$. Consequently, the set Ξ is open in $\alpha \odot \mathbf{l}(X) \oplus \beta \odot \mathbf{l}(X) = \mathbf{l}(X)$.

3) $\alpha = \beta = 0$. Then $\Xi \cap \mathbf{l}(X) = \langle \mu; \varphi_1, \dots, \varphi_n; \theta \rangle \oplus \langle \nu; \psi_1, \dots, \psi_k; \delta \rangle$ is open in $\mathbf{l}(X) \oplus \mathbf{l}(X) = \mathbf{l}(X)$. Proposition 3.2 is proved. $\square$

REMARK 3.3. Consider any finite set of elements $\alpha_i \in \mathbb{R}_{\max}$ with $\bigoplus_{i=1}^n \alpha_i = 0$. Then the equality in Remark 3.1 can be rewritten as

$$\bigoplus_{i=1}^n \alpha_i \bigodot \mathbf{l}(X) = \mathbf{l}(X).$$

Consequently, Proposition 3.2 can be reformulated for any finite summands.

Since $USC_0(X)$ is a subspace in $\mathbb{R}_{\max}^X$, its neighborhood system at an upper semi-continuous function $\lambda \in USC_0(X)$ has a shape

$$(\lambda; x_1, \dots, x_n; \theta) = \{\gamma \in USC_0(X) : |\gamma(x_i) - \lambda(x_i)| < \theta, i = 1, \dots, n\}.$$

Here $x_i \in X, i = 1, \dots, n; \theta > 0$.

For points $x_i \in X$ and their open neighborhoods $Ux_i, i = 1, \dots, n$, it is easy to see that the set

$$(\lambda; Ux_1, \dots, Ux_n; \theta) = \{\gamma \in USC_0(X) : \text{there is } y_i \in Ux_i \text{ such that } |\gamma(y_i) - \lambda(x_i)| < \theta, i = 1, \dots, n\}$$

is also open as well. Indeed, take any $\gamma \in (\lambda; Ux_1, \dots, Ux_n; \theta)$. For $i = 1, \dots, n$, let $|\gamma(y_i) - \lambda(x_i)| = a_i < \theta$ for some $y_i \in Ux_i$. Put $a = \max\{a_1, \dots, a_n\}$. Then for each $\xi \in (\gamma; y_1, \dots, y_n; \theta - a)$ we have

$$\begin{aligned} |\xi(y_i) - \lambda(x_i)| &= |\xi(y_i) - \gamma(y_i) + \gamma(y_i) - \lambda(x_i)| \\ &\leq |\xi(y_i) - \gamma(y_i)| + |\gamma(y_i) - \lambda(x_i)| < \theta - a + a = \theta, \end{aligned}$$

i.e., $\xi \in (\lambda; Ux_1, \dots, Ux_n; \theta)$. So, $(\gamma; y_1, \dots, y_n; \theta - a) \subset (\lambda; Ux_1, \dots, Ux_n; \theta)$ which shows the openness of $(\lambda; Ux_1, \dots, Ux_n; \theta)$.

To prove the next result we need the following statement.

LEMMA 3.4 ([14]). *Let γX be a compact extension of a space X and, V, W be disjoint open sets in γX . Let $V \cap X = V_\gamma$, and $W \cap X = W_\gamma$. Then the following equality is true:*

$$\text{Cl}_{\gamma X}(X \setminus V_\gamma) \cap \text{Cl}_{\gamma X}(X \setminus W_\gamma) = \text{Cl}_{\gamma X}(X \setminus (V_\gamma \cup W_\gamma))$$

Obviously, $\mathbf{l}(\beta X)$ is one of compact extensions of $\mathbf{l}_\beta(X)$. Hence, $\mathbf{l}_f(\beta X)$ is a compact extension of $\mathbf{l}_f(X) \equiv \mathbf{l}_f(\beta X) \cap \mathbf{l}_\beta(X)$. The following result clarifies this situation.

THEOREM 3.5. *For a Tychonoff space X the space $\mathbf{l}_f(\beta X)$ is a perfect compactification of the space $\mathbf{l}_f(X)$.*

Proof. Let U_1 and U_2 be disjoint open sets in X . Since βX is the perfect compactification of X we have $O_{\beta X}(U_1 \cup U_2) = O_{\beta X}(U_1) \cup O_{\beta X}(U_2)$. Consider open sets $\langle U_1 \rangle, \langle U_2 \rangle$ in $\mathbf{l}_f(X)$. It is clear that $\langle U_1 \rangle \cap \langle U_2 \rangle = \emptyset$. We will show that

$$O_{\mathbf{l}_f(\beta X)}(\langle U_1 \rangle \cup \langle U_2 \rangle) = O_{\mathbf{l}_f(\beta X)}(\langle U_1 \rangle) \cup O_{\mathbf{l}_f(\beta X)}(\langle U_2 \rangle).$$

The inclusion $\supset$ straightly follows from the definition of sets of the type $O(H)$. Therefore, we have to show the inverse inclusion. Let $\mu' \in \mathcal{I}(\beta X)$ be an idempotent probability measure such that $\mu' \notin O_{\mathcal{I}_f(\beta X)}(\langle U_1 \rangle) \cup O_{\mathcal{I}_f(\beta X)}(\langle U_2 \rangle)$. Then $\mu' \in \mathcal{I}_f(\beta X) \setminus O_{\mathcal{I}_f(\beta X)}(\langle U_i \rangle)$, $i = 1, 2$. From [2, p. 234] we have

$$\mathcal{I}_f(\beta X) \setminus O_{\mathcal{I}_f(\beta X)}(\langle U_i \rangle) = \text{Cl}_{\mathcal{I}_f(\beta X)}(\mathcal{I}_f(X) \setminus \langle U_i \rangle), \quad i = 1, 2.$$

Hence, $\mu' \in \text{Cl}_{\mathcal{I}_f(\beta X)}(\mathcal{I}_f(X) \setminus \langle U_i \rangle)$, $i = 1, 2$. Since $\langle U_1 \rangle \cap \langle U_2 \rangle = \emptyset$ by Lemma 3.4 we have

$$\text{Cl}_{\mathcal{I}_f(\beta X)}(\mathcal{I}_f(X) \setminus \langle U_1 \rangle) \cap \text{Cl}_{\mathcal{I}_f(\beta X)}(\mathcal{I}_f(X) \setminus \langle U_2 \rangle) = \text{Cl}_{\mathcal{I}_f(\beta X)}(\mathcal{I}_f(X) \setminus (\langle U_1 \rangle \cup \langle U_2 \rangle)).$$

Thus, $\mu' \in \text{Cl}_{\mathcal{I}_f(\beta X)}(\mathcal{I}_f(X) \setminus (\langle U_1 \rangle \cup \langle U_2 \rangle))$, i.e., $\mu' \in \mathcal{I}_f(\beta X) \setminus O_{\mathcal{I}_f(\beta X)}(\langle U_1 \rangle \cup \langle U_2 \rangle)$. Consequently, $\mu' \notin O_{\mathcal{I}_f(\beta X)}(\langle U_1 \rangle \cup \langle U_2 \rangle)$. Thus, we have established that the inclusion $O_{\mathcal{I}_f(\beta X)}(\langle U_1 \rangle \cup \langle U_2 \rangle) \subset O_{\mathcal{I}_f(\beta X)}(\langle U_1 \rangle) \cup O_{\mathcal{I}_f(\beta X)}(\langle U_2 \rangle)$ is also true. Theorem 3.5 is proved. $\square$

Now we can state the main result of the paper.

THEOREM 3.6. *For a Tychonoff space X , the space $\mathcal{I}_f(X)$ is superparacompact if and only if X is superparacompact.*

Proof. If $\mathcal{I}_f(X)$ is superparacompact, then Property (*) of superparacompact spaces (see, Preliminaries, page 3) implies the superparacompactness of the closed subset $X \subset \mathcal{I}_f(X)$.

Let now X be a superparacompact space, $B \subset \mathcal{I}_f(\beta X) \setminus \mathcal{I}_f(X)$ any compact set. Then $\mu \in \mathcal{I}_f(\beta X) \setminus \mathcal{I}_f(X)$ for every $\mu \in B$. Due to Theorem 3.5, $\mathcal{I}_f(\beta X)$ is a perfect compactification of $\mathcal{I}_f(X)$. Hence, we have to show the existence of an open finite-component cover η of $\mathcal{I}_f(X)$, that pricks out B in $\mathcal{I}_f(\beta X)$.

Let $\mu \in B$ be an arbitrary element. Then $\text{supp } \mu \not\subset X$ since $\mu \in \mathcal{I}_f(\beta X) \setminus \mathcal{I}_f(X)$. Owing to Corollary 2.4 for every point $x \in \text{supp } \mu \setminus X \subset \beta X \setminus X$ there exists an open finite-component cover ξ_x pricking out x in βX , i. e., $x \notin \bigcup [\xi_x]_{\beta X}$. Fix a point $x_0 \in \text{supp } \mu \setminus X$. Then $\text{supp } \mu \not\subset \text{Cl}_{\beta X}(U)$ for every $U \in \xi_{x_0}$ and $\mu \in B$. Hence, $\mu \notin \langle \text{Cl}_{\beta X}(U) \rangle$ for any $U \in \xi_{x_0}$. Now the equality in Corollary 2.9 gives

$$\mu \notin \bigcup [\eta]_{\mathcal{I}_f(\beta X)} = \bigcup \{ \text{Cl}_{\mathcal{I}_f(\beta X)}(\langle U \rangle) : U \in \xi_{x_0} \}.$$

Arbitrariness of $x_0 \in \text{supp } \mu \setminus X$ and $\mu \in B$ yield that $B \cap \bigcup [\eta]_{\mathcal{I}_f(\beta X)} = \emptyset$. Finally, applying Theorem 2.3 and Proposition 2.7 we complete the proof. Theorem 3.6 is proved. $\square$

ACKNOWLEDGEMENT. The authors are grateful to the referees for comments and suggestions which helped us to improve the exposition of the paper.

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(received 09.06.2025; in revised form 20.02.2026; available online 12.10.2026)

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