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ON THE EQUIDISTANT DIMENSION OF HAMMING GRAPHS

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Abstract. In this paper, we examine the recently introduced concept of equidistant dimension eqdim(G) for Hamming graphs $H_{r,k}$. For hypercubes $Q_r = H_{r,2}$, exact values have been derived for $r \not\equiv 0 (\bmod 4)$. Finally, the exact value for eqdim$(H_{2,k})$ has been derived. We have shown that for Hamming graphs $H_{2,k}$, the equidistant dimension remains constant when $k \geq 5$, whereas for hypercubes, it grows linearly with the order of the graph.

1. Introduction and previous work

The concept of a resolving set (also known as a locating set) was introduced independently by Slater [8] and Harary and Melter [5]. This notion arises naturally in applications such as fault detection in computer networks modeled as graphs. Formally, a subset of vertices S is called a resolving set if every vertex in the graph can be uniquely identified by its vector of distances to the vertices in S. The metric dimension of a graph is then defined as the minimum cardinality of such a resolving set.

However, many authors have turned their attention precisely in the opposite direction – to resolvability, thus trying to study anonymization problems in networks instead of location aspects. Indeed, a subset of vertices A is a 2-antiresolving set if, for every vertex $v \notin A$, there exists another different vertex $w \notin A$ such that v and w have the same vector of distances to the vertices of A. The 2-metric antidimension of a graph is the minimum cardinality among all its 2-antiresolving sets.

Building upon this research direction, the work in [4] introduces novel graph-theoretic concept – distance-equalizer sets and equidistant dimension – with direct applications to network anonymization problems. Beyond their immediate utility for anonymization, these concepts prove valuable for several mathematical applications,

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including: (1) establishing improved bounds for doubly resolving sets of graphs, and (2) providing a new graph-theoretic formulation of a classical number theory problem.

The equidistant dimension problem is NP-hard in general [3]. Previous work has investigated the equidistant dimension of lexicographic graph products [3] and established key properties of distance-equalizer sets in arbitrary graphs [6, Lemma 1.6]. Additionally, exact values for the equidistant dimensions of Johnson and Kneser graphs have been determined [6]: $\mathrm{eqdim}(Jn, 2) = 3$, $\mathrm{eqdim}(J2k, k) = \frac{1}{2} \cdot \binom{2k}{k}$ for odd k and $\mathrm{eqdim}(Kn, 2) = 3$ are found. Moreover, it has been proved that $n-2$ is a tight upper bound for $\mathrm{eqdim}(Jn, 3)$.

The equidistant dimension of certain classes of convex polytopes has been studied in [7]. Exact value is equal to $2n$ for T_n, S_n with odd n and S''_n with even n. For R''_n with even n, exact value is equal to $3n$. Finally, for odd n, lower bounds of $3n$ and $2n$ have been found for R''_n and S''_n, respectively.

All graphs considered in this paper are connected, undirected, simple, and finite. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The order of G is $|V(G)|$. For any vertex $v \in V(G)$, its open neighborhood is the set $N(v) = \{w \in V(G) \mid vw \in E(G)\}$ and its closed neighborhood is $N[v] = N(v) \cup \{v\}$.

The degree of a vertex v, denoted by $\deg(v)$, is defined as the cardinality of $N(v)$. If $\deg(v) = 1$, then we say that v is a leaf, in which case the only vertex adjacent to v is called its support vertex. When $\deg(v) = |V(G)|-1$, we say that v is universal. The maximum degree of G is $\Delta(G) = \max\{\deg(v) \mid v \in V(G)\}$ and its minimum degree is $\delta(G) = \min\{\deg(v) \mid v \in V(G)\}$. The distance between two vertices $v, w \in V(G)$ is denoted by $d(v, w)$, and the diameter of G is $\mathrm{Diam}(G) = \max\{d(v, w) \mid v, w \in V(G)\}$.

A clique is a subset of pairwise adjacent vertices and the clique number of G, denoted by $\omega(G)$, is the maximum cardinality of a clique of G. An independent set of G is a subset of pairwise non-adjacent vertices and the independence number of G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set of G.

The set of vertices on equal distances from u and v is denoted in the literature by ${}_uW_v$ (see [1]). Formally, ${}_uW_v = \{w \in V(G) \mid d(u, w) = d(v, w)\}$.

Definition 1.1 ([4]). Let $x, y, w \in V(G)$. We say that w is equidistant from x and y if $d(x, w) = d(y, w)$.

Definition 1.2 ([4]). A subset S of vertices is called a distance-equalizer set for G if for every two distinct vertices $x, y \in V(G) \setminus S$ there exists a vertex $w \in S$ equidistant from x and y.

Definition 1.3 ([4]). The equidistant dimension of G, denoted by $\mathrm{eqdim}(G)$, is the minimum cardinality of a distance-equalizer set of G.

Theorem 1.4 ([4]). *For every graph G of order $n \geq 2$, the following statements hold.*

- $\mathrm{eqdim}(G) = 1$ *if and only if* $\Delta(G) = n - 1$;
- $\mathrm{eqdim}(G) = 2$ *if and only if* $\Delta(G) = n - 2$.

Corollary 1.5 ([4]). *If G is a graph of order n with $\Delta(G) < n-2$ then* $\mathrm{eqdim}(G) \geq 3$.

Lemma 1.6 ([6]). *Let G be a graph, and u and v be any vertices from $V(G)$. Then, S is a distance-equalizer set of G if and only if $S \cap (\{u, v\} \cup {}_uW_v) \neq \emptyset$.*

Corollary 1.7. *Let G be a graph, and u and v be any vertices from $V(G)$. If S is a distance-equalizer set of G and ${}_uW_v = \emptyset$ then $u \in S$ or $v \in S$.*

Table 1 presents the equidistant dimension of several well-known families of graphs from the literature. For comparison, the last two columns show the metric dimension and the doubly metric dimension of these graph families. The notation 'n.a.' indicates that a corresponding result is not available in the literature.

Graph G	Contraints	$\mathrm{eqdim}(G)$	$\dim(G)$	$\psi(G)$
P_n	$n \geq 2$	$n - r(\lceil \frac{n}{2} \rceil)$	1	2
	$n = 4k \geq 4$	$\frac{3n}{4} - 1$	2	3
C_n	$n = 4k + 2 \geq 6$	$\frac{n}{2}$	2	3
	$n = 4k + 1 \geq 5$	$n - r(\lceil \frac{n+1}{4} \rceil)$	2	2
K_n	$n \geq 3$	1	$n-1$	$n-1$
$K_{r,s}$	$2 \leq r \leq s = n - r$	r	$n-2$	$n-2$
$K_{1,n-1}$	$n \geq 4$	1	$n-2$	$n-1$
$K_2(r,s)$	$3 \leq r \leq s = n - r$	r	$n-4$	$n-2$
$K_{n_1,\ldots,n_p}$	$p \geq 3, n_1 + \cdots + n_p = n$	$\min\{3, n_1, \ldots, n_p\}$	$n-p$	$n-p$
$J_{n,2}$	$n \geq 6$	3	$\lceil \frac{2n}{3} \rceil$	n.a.
$J_{2k,k}$	odd k	$\frac{1}{2}\binom{2k}{k}$	$\leq 2k$	n.a.
$K_{n,2}$	$n \geq 6$	3	$\lceil \frac{2n}{3} \rceil$	n.a.
$J_{n,3}$	$n \geq 9$	$\leq n-2$	$\lfloor \frac{3n+3}{4} \rfloor$	n.a.
R''_n	even n	$3n$	3	n.a.
R''_n	odd n	$\geq 3n$	3	n.a.
S_n	odd n	$2n$	3	n.a.
S''_n	odd n	$\geq 2n$	3	3
S''_n	even n	$2n$	3	3
T_n	$n \geq 8$	$2n$	3	3

Table 1: Equidistant dimension and some other parameters of some families of graphs

2. New results

The *Hamming graph* $H_{r,k}$ is the Cartesian product [2]:

$$H_{r,k} = \underbrace{K_k \square K_k \square \ldots \square K_k}_{r} \tag{1}$$

where K_k denotes the complete graph with k vertices. The vertices of Hamming graphs can be considered also as r-dimensional vectors, where every coordinate has a value from the set $\{0, 1, \ldots, k-1\}$.

Obviously, $H_{r,k}$ has k^r vertices. Furthermore, every vertex has the r-dimensional neighborhood with $k-1$ neighbors with respect to each coordinate, so the overall number of edges is $k^r \cdot r \cdot (k-1)/2$. Hypercubes are Hamming graphs such that $k=2$, i.e., $Q_r = H_{r,2}$.

Since $Q_1 \cong P_2$, then next result holds.

RESULT 2.1. $\mathrm{eqdim}(Q_1) = 1$.

All remaining cases ($r \geq 2$) are fully or partially resolved in the following theorem.

THEOREM 2.2. *For $r \geq 2$ and $r \not\equiv 0 \pmod 4$, it holds that* $\mathrm{eqdim}(Q_r) = 2^{r-1}$.

Proof. **Step 1.** $\mathrm{eqdim}(Q_r) \geq 2^{r-1}$

Let S be a distance-equalizer set of Q_r, and u_0 and u_1 be arbitrary vertices, where $u_0 = (q_1, q_2, \ldots, q_{r-1}, 0)$ and $u_1 = (q_1, q_2, \ldots, q_{r-1}, 1)$, $q_i \in \{0,1\}$, $0 \leq i \leq r-1$.

Therefore, for arbitrary vertex $w \in V(Q_r)$ it holds that either $d(u_0, w) < d(u_1, w)$, or $d(u_0, w) > d(u_1, w)$, so $d(u_0, w) \neq d(u_1, w)$ implying that ${}_{u_0}W_{u_1} = \emptyset$. Using Corollary 1.7, either $u_0 \in S$ (the last coordinate of w is 0) or $u_1 \in S$ (the last coordinate of w is 1) holds. Let us divide vertices $V(Q_r)$ into the sets $\{(q_1, q_2, \ldots, q_{r-1}, 0), (q_1, q_2, \ldots, q_{r-1}, 1) \,|\, q_i \in \{0,1\}, 0 \leq i \leq r-1\}$. These sets are obviously mutually disjoint, their union is equal to $V(Q_r)$ and there are 2^{r-1} of them, so $|S| \geq 2^{r-1}$.

Step 2. $\mathrm{eqdim}(Q_r) \leq 2^{r-1}$

If we introduce the function $nz(v) : V(Q) \to N$ as the sum of non-zero coordinates of v, we can now color the vertices in chess board style. Vertices are "white" or "black" depending on the value of function nz: if the value is odd, those vertices are "white" and if the value is even, those vertices are "black". We will denote by $\oplus$ binary summation by coordinates of two vertices.

Let $S = \{v \in V(Q_r) \mid nz(v) \textit{ is even}\}$ or S be the set consisting of all black vertices, and u and v be arbitrary vertices from $V(Q_r) \setminus S$.

Case 1. r is odd.

It is obvious that $nz(u)$, $nz(v)$ are odd, i.e., they are white vertices. This means that they differ in even number of coordinates. Let us now construct vertex x in the following way: from those coordinates that are different in vertices u and v, we will take half of them from vertex u and half from vertex v and all coordinates which are equal in both u and v. If we denote $c = d(u,v)$, then $d(u,x) = d(v,x) = \frac{c}{2}$. Since c is an even number, it implies the existence of x. If x is a black vertex then $x \in S$, otherwise let us construct vertex x' from x by changing one of the $r-c$ coordinates that are the same in both u and v from 0 to 1 or vice versa. Now, x' is a black vertex and belongs to S. Therefore, since $|S| = 2^r/2 = 2^{r-1}$, the property holds.

Case 2. $r \equiv 2 \pmod 4$ and $nz(u \oplus v) < r$.

Vertices u and v are white and not diagonal, so it can be reduced to **Case 1**.

Case 3. $r \equiv 2(\bmod 4)$ and $nz(u \oplus v) = r$.

Vertices u and v are white and diagonal, which means that u is the complement of v, i.e., all their coordinates are different. Then vertex x can be constructed, without loss of generality, by taking the first half of u-coordinates ($r/2$ of them), and the second half of v-coordinates ($r/2$ of them). Since $r \equiv 2\,(\bmod 4)$ implies that $r/2$ is an odd number and since $d(u, x) = d(v, x) = r/2$ is an odd number, and taking in account that both u and v are white vertices, it follows that x is a black vertex and must belong to S. It can be concluded that the property holds.

Finally, in all cases, both x and x' are equidistant from u and v and $x, x' \in S$, so S is an equidistant set for Q_r, and $\mathrm{eqdim}(Q_r) \le |S| = 2^{r-1}$ for $r \not\equiv 0\,(\bmod 4)$. □

It should be noted that **Step 1.** holds for any $r \ge 2$. For better understanding the notions applied in Theorem 2.2, two illustrative examples are given:

Example 2.3. Let $r = 5$, $u = (1, 0, 1, 0, 1)$ and $v = (0, 1, 1, 1, 0)$, $nz(u) = nz(v) = 3$, $u \oplus v = (1, 1, 0, 1, 1)$, $c = nz(1, 1, 0, 1, 1) = 4$. Let $x = (1, 0, 1, 1, 0)$. It follows that $nz(u \oplus x) = nz(v \oplus x) = \frac{c}{2}$.

Example 2.4. Let $r = 6$, $u = (0, 1, 0, 1, 0, 1)$ and $v = (0, 0, 1, 1, 1, 0)$, $nz(u) = nz(v) = 3$, $u \oplus v = (0, 1, 1, 0, 1, 1)$, $c = nz(0, 1, 1, 0, 1, 1) = 4$. Let $x = (0, 1, 0, 1, 1, 0)$. It follows that $nz(u \oplus x) = nz(v \oplus x) = \frac{c}{2}$.

It is interesting to find the equidistant dimension of $H_{2,k}$. First three values, for $k = 3, 4$ and 5 are obtained by simple enumeration and are presented in Result 2.5.

Result 2.5. By a simple enumeration, it is found that

(I) $\mathrm{eqdim}(H_{2,3}) = 3$ with the corresponding distance-equalizer set $S = \{\{0, 0\}, \{0, 1\}, \{0, 2\}\}$;

(II) $\mathrm{eqdim}(H_{2,4}) = 4$ with the corresponding distance-equalizer set $S = \{\{0, 0\}, \{0, 1\}, \{0, 2\}, \{0, 3\}\}$;

(III) $\mathrm{eqdim}(H_{2,5}) = 5$ with the corresponding distance-equalizer set $S = \{\{0, 0\}, \{0, 1\}, \{0, 2\}, \{1, 0\}, \{2, 0\}\}$.

All other cases, for $k \ge 6$, are resolved by Theorem 2.6.

Theorem 2.6. *For $k \ge 6$ it holds that* $\mathrm{eqdim}(H_{2,k}) = 5$.

Proof. Since the graph $H_{2,k}$ is $(2k-2)$-regular, then $\Delta(H_{2,k}) = 2k - 2$. The order of graph $H_{2,k}$ is k^2, and then sufficient condition of Result 1.5 holds, so $\mathrm{eqdim}(H_{2,k}) \ge 3$. From Result 2.5 (III) it follows that for $H_{2,5}$ there is no distance-equalizer set S with cardinality in $\{3, 4\}$, i.e., $(\exists u, v \in V(H_{2,5}) \setminus S))(\forall x \in S) d(u, x) \ne d(v, x))$.

Let $S = \{(i_1, j_1), (i_2, j_2), (i_3, j_3), (i_4, j_4)\}$. Let A and B be arbitrary sets of indices such that $|A| = 5$, $|B| = 5$, $i_1, i_2, i_3, i_4 \in A$ and $j_1, j_2, j_3, j_4 \in B$. The Cartesian product $A \times B$ is a subset of $V(H_{2,k})$. Let G' be the induced subgraph of $H_{2,k}$ such that $V(G') = A \times B$.

It should be noted that all distances in the induced subgraph G' are equal to distances from graph $H_{2,k}$ and $G' \cong H_{2,5}$. Let $f : H_{2,5} \to G'$ be an isomorphism.

Since eqdim$(H_{2,5}) = 5$ by Result 2.5 (III), there is no equalizer set S' in $H_{2,5}$ with cardinality four, i.e., $(\exists u, v \in V(H_{2,5}))(\forall x \in S')(d(u,x) \neq d(v,x))$.

If $S' = f(S)$, then $f(u), f(v) \in V(G') \subset V(H_{2,k})$ $f(x) \in f(S) \subset V(G')$ such that $d(f(u), f(x)) \neq d(f(v), f(x))$ for each $x \in S'$. Therefore, S is not an equalizer set for G' so it is not an equalizer set for $H_{2,k}$.

Using the same arguments there is no equalizer set with 3 vertices.

Therefore, there is no distance-equalizer set S' of $H_{2,k}, k \geq 6$ with cardinality in $\{3,4\}$, i.e., $(\exists u, v \in V(H_{2,k}) \setminus S'))(\forall x \in S')d(u,x) \neq d(v,x))$. Therefore, eqdim$(H_{2,k}) \geq 5$.

Now, let eqdim$(H_{2,k}) \leq 5$. Let $S = \{\{0,0\},\{0,1\},\{0,2\},\{1,0\},\{2,0\}\}$ and $u = (i_u, j_u)$ and $v = (i_v, j_v)$. When checking S as a distance-equalizer set of $H_{2,k}$ there are two possible cases:

Case 1. $0 \leq i_u, i_v, j_u, j_v \leq 4$

Since the vertex-induced subgraph $H_{2,5}$ "inherits" the distance relationship from $H_{2,k}$, $k \geq 6$, then from (III) it directly follows that S is a distance-equalizer set for $H_{2,5}$.

Case 2. Otherwise.

Without loss of generality we can assume that $i_u \leq i_v$.

$u = (i_u, j_u)$		$v = (i_v, j_v)$		x	$d(u,x)$	$d(v,x)$
$i_u = 0$	$j_u \geq 3$	$i_v = 0$	$j_v \geq 3$	$(0,0)$	1	1
$i_u = 0$	$j_u \geq 3$	$i_v \in \{1,2\}$	$j_v \geq 3$	$(3 - i_v, 0)$	2	2
$i_u = 0$	$j_u \geq 3$	$i_v \geq 3$	$j_v = 0$	$(0,0)$	1	1
$i_u = 0$	$j_u \geq 3$	$i_v \geq 3$	$j_v \in \{1,2\}$	$(0, j_v)$	1	1
$i_u = 0$	$j_u \geq 3$	$i_v \geq 3$	$j_v \geq 3$	$(1,0)$	2	2
$i_u \in \{1,2\}$	$j_u \in \{1,2\}$	$i_v \geq 3$	$j_v \geq 3$	$(0,0)$	2	2
$i_u \in \{1,2\}$	$j_u \geq 3$	$i_v \in \{1,2\}$	$j_v \geq 3$	$(0,0)$	2	2
$i_u \in \{1,2\}$	$j_u \geq 3$	$i_v \geq 3$	$j_v \geq 3$	$(0,0)$	2	2
$i_u \geq 3$	$j_u = 0$	$i_v \geq 3$	$j_v = 0$	$(0,0)$	1	1
$i_u \geq 3$	$j_u = 0$	$i_v \geq 3$	$j_v \in \{1,2\}$	$(0, 3 - j_v)$	2	2
$i_u \geq 3$	$j_u \in \{1,2\}$	$i_v \geq 3$	$j_v = 0$	$(0, 3 - j_u)$	2	2
$i_u \geq 3$	$j_u \in \{1,2\}$	$i_v \geq 3$	$j_v \in \{1,2\}$	$(0,0)$	2	2
$i_u \geq 3$	$j_u \geq 3$	$i_v \geq 3$	$j_v = 0$	$(0,1)$	2	2
$i_u \geq 3$	$j_u \geq 3$	$i_v \geq 3$	$j_v \in \{1,2\}$	$(0,0)$	2	2
$i_u \geq 3$	$j_u \geq 3$	$i_v \geq 3$	$j_v \geq 3$	$(0,0)$	2	2

Table 2: Possible vertices $u, v \in V(H_{2,k})$ for $k \geq 6$.

All subcases are given in Table 2. In the first and second column, i_u and j_u coordinates of vertex u are presented. In the same way, in the following two columns, coordinates of vertex v are given. the fifth column presents vertex x that is equidistant from u and v. The last two columns present distances from vertices u and v to x, respectively. It can be observed from Table 2 that, in all subcases, the equality

$d(u, x) = d(v, x)$ holds. Combined with **Case 1.**, this confirms that the set S is a distance-equalizer set for $H_{2,k}$. Therefore, $\mathrm{eqdim}(H_{2,k}) = 5$ for $k \geq 6$. □

3. Conclusions

In this paper, we have investigated the equidistant dimensions of certain Hamming graphs. For the Hamming graphs $H_{2,k}$, the equidistant dimension is constant and equal to 5, for $k \geq 5$. In contrast, for hypercubes, the equidistant dimension increases linearly with the order of the graph. Specifically, when $r \not\equiv 0 \pmod 4$, we have $\mathrm{eqdim}(Q_r) = 2^{r-1}$, and when $r \equiv 0 \pmod 4$, $\mathrm{eqdim}(Q_r) \geq 2^{r-1}$.

Future work could investigate the equidistant dimension for broader graph families, along with developing exact algorithms and efficient heuristics for its computation.

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