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BETTER APPROXIMATION AND THEORETICAL VALIDATION FOR INTEGRAL-TYPE OPERATORS

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Abstract. This study is on an innovative technique of integral-type operators that adopt the Baskakov basis function in recursion form and the Szász basis function, accentuating how well they approximate integrable functions. The study addresses the challenge of achieving more accurate function approximation, and mainly contributes to improving the theoretical aspects of the proposed operators. We examine the convergence properties of the proposed operators by employing Peetre's K-functional, second-order modulus of smoothness, and modulus of continuity. Additionally, we derive the Voronovskaja-type asymptotic formula and establish approximation results in weighted spaces. Finally, we show that the proposed operators significantly enhance the approximation accuracy through various examples and graphs.

1. Introduction

In 1912, S.N. Bernstein [8] unveiled the seminal Bernstein polynomials, which provide elegant and intuitive proof of the Weierstrass approximation theorem when applied to the interval $[0,1]$. Building on this foundation, Szász [22] later expanded these operators to the infinite interval $[0,\infty)$. This pioneering work has inspired the development of numerous linear positive operators over unbounded intervals, including the Baskakov operators [7] defined on $[0,\infty)$. Subsequently, Gupta and Srivastava [14] employed Baskakov and Szász basis functions to present a sequence of operators for the integrable function μ on $[0,\infty)$ as follows:

$$\mathcal{S}_{\eta,\ell}^{\mu,P}(\mu;x) = \eta \sum_{\ell=0}^{\infty} P_{\eta,\ell}(x) \int_0^{\infty} s_{\eta,\ell}^*(z)\mu(z)dz, \quad \eta \in \mathbb{N}, \tag{1}$$

where $P_{\eta,\ell}(x) = \binom{\eta+\ell-1}{\ell} x^{\ell}(1+x)^{-\eta-\ell}$ and $s_{\eta,\ell}^*(x) = e^{-\eta x}\frac{(\eta x)^{\ell}}{\ell!}$.

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Many researchers have recently concentrated on integral-type modifications of operators to find better approximations on unbounded intervals (see [1, 5, 9, 10, 12, 20, 21, 25]). To enhance the approximation, Khosravian-Arab et al. [18] introduced new modified Bernstein operators. If $\mu \in \mathcal{C}[0,1]$, then sequence of the operators are given by

$$B^{\mu, J^{M,1}}_{\eta,\ell}(\mu;x) = \sum_{\ell=0}^{\eta} J^{M,1}_{\eta,\ell}\,\mu\left(\frac{\ell}{\eta}\right),\ x \in [0,1], \tag{2}$$

$$J^{M,1}_{\eta,\ell} = \zeta(x,\eta)J_{\eta-1,\ell}(x) + \zeta(1-x,\eta)J_{\eta-1,\ell-1}(x), 1 \le \ell \le \eta-1, \tag{3}$$

$$J^{M,1}_{\eta,0} = \zeta(x,\eta)(1-x)^{\eta-1}, J^{M,1}_{\eta,\eta} = \zeta(1-x,\eta)^{\eta-1}, \text{ and}$$

$$\zeta(-x,\eta) = \zeta_1(\eta)(-x) + \zeta_0(\eta), \text{ and where } \zeta_0(\eta),\ \zeta_1(\eta) \text{ are unknown sequences.}$$

Recently, some researchers constructed integral-type operators by using a modified Bernstein basis function (3) to find a better approximation. To see relevant work in this area, one may refer to [2, 3, 6, 15, 17, 23].

In the same way, very recently, Acu et al. [4] introduced a sequence of modified Baskakov-type operators as follows:

$$Q^{\mu, P^{M,1}}_{\eta,\ell}(\mu;x) = \sum_{\ell=0}^{\infty} P^{M,1}_{\eta,\ell}\,\mu\left(\frac{\ell}{\eta}\right), \quad x \in [0,\infty). \tag{4}$$

The fundamental polynomials satisfy the recursion

$$P^{M,1}_{\eta,\ell} = \zeta(-x,\eta)P_{\eta+1,\ell}(x) + \zeta(1+x,\eta)P_{\eta+1,\ell-1}(x), \quad \ell \ge 1,$$

$$P^{M,1}_{\eta,0} = \zeta(-x,\eta)(1+x)^{-\eta-1}, \text{ and}$$

$$\zeta(-x,\eta) = \zeta_1(\eta)(-x) + \zeta_0(\eta), \text{ where } \zeta_0(\eta),\ \zeta_1(\eta) \text{ are unknown sequences.}$$

2. Construction of the operators

We construct a new class of operators for contribute to improving the theoretical aspects of the proposed operators. If μ is a integrable function on $[0,\infty)$ such that $\int_0^\infty s^*_{\eta,\ell}(z)\mu(z)dz < \infty$, then

$$\mathcal{H}^{\mu, P^{M,1}}_{\eta,\ell}(\mu;x) = \eta \sum_{\ell=0}^{\infty} P^{M,1}_{\eta,\ell} \int_0^\infty s^*_{\eta,\ell}(z)\mu(z)dz, \quad \eta \in \mathbb{N}. \tag{5}$$

In particular, if $\zeta_0(\eta) = -1$, $\zeta_1(\eta) = 1$, then (5) reduces to (1). Note that throughout the paper, we consider the relation $\zeta_0(\eta) + \zeta_1(\eta) = 1 - \zeta_0(\eta)$.

The structure of the paper is outlined as follows. We create an innovative technique of integral-type operators that adopt the Baskakov basis function in recursion form and the Szász basis function in Section 2. We obtain some basic results for the operators in Section 3. Section 4 explores the convergence properties of the proposed operators by employing Peetre's K-functional, second-order modulus of smoothness, and modulus of continuity. Additionally, we derive the Voronovskaja-type asymptotic formula. We establish approximation results in weighted spaces in Section 5. In the final section, we show how the proposed operators significantly enhance the approximation accuracy.

3. Basic results

We present several lemmas that will be used in the proofs of the main results.

LEMMA 3.1. *For $e_r(z) = z^r$ and $r = 0, 1, 2, 3, 4$, then the moments of the operators* (5) *as follows:*

$$\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(e_0(z);x) = 1,$$

$$\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(e_1(z);x) = x + \frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta},$$

$$\begin{aligned}\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(e_2(z);x) &= x^2 + \frac{x^2(\eta+4(\eta+1)(1-\zeta_0(\eta)))}{\eta^2}\\ &+ \frac{2x(2\eta+(\eta+5)(1-\zeta_0(\eta)))+4(1-\zeta_0(\eta))+2}{\eta^2},\end{aligned}$$

$$\begin{aligned}\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(e_3(z);x) &= x^3 + \frac{x^3(\eta(3\eta+2)+6(\eta+1)(\eta+2)(1-\zeta_0(\eta)))}{\eta^3}\\ &+ \frac{3(\eta+1)x^2(3\eta+(\eta+14)(1-\zeta_0(\eta)))}{\eta^3}\\ &+ \frac{18x(\eta+(\eta+3)(1-\zeta_0(\eta)))+18(1-\zeta_0(\eta))+6}{\eta^3},\end{aligned}$$

$$\begin{aligned}\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(e_4(z);x) &= x^4 + \frac{x^4(\eta(6\eta^2+11\eta+6)+8(\eta+1)(\eta+2)(\eta+3)(1-\zeta_0(\eta)))}{\eta^4}\\ &+ \frac{4(\eta+1)(\eta+2)x^3(4\eta+(\eta+27)(1-\zeta_0(\eta)))}{\eta^4}\\ &+ \frac{24(\eta+1)x^2(3\eta+2(\eta+8)(1-\zeta_0(\eta)))}{\eta^4}\\ &+ \frac{48x(2\eta+(3\eta+7)(1-\zeta_0(\eta)))+96(1-\zeta_0(\eta))+24}{\eta^4}.\end{aligned}$$

Proof. We have

$$I_{\eta,\ell}(r) = \int_0^\infty z^r s_{\eta,\ell}^*(z)dz = \frac{\eta^{-r-1}(\ell+r)!}{\ell!},$$

$$\begin{aligned}\sum_{\ell=0}^{\infty} I_{\eta,\ell}(r)\zeta(-x,\eta)P_{\eta+1,\ell}(x) &= \eta^{-r-1}r!(1+x)^{-\eta-1}(1+x-(1+2x)(1-\zeta_0(\eta))\\ &\times {}_2F_1\left(r+1,\eta+1;1;\frac{x}{1+x}\right),\end{aligned}$$

and

$$\sum_{\ell=1}^{\infty} I_{\eta,\ell}(r)\zeta(1+x,\eta)P_{\eta+1,\ell-1}(x) = \eta^{-r-1}(r+1)!(1+x)^{-\eta-1}((1+2x)(1-\zeta_0(\eta))-x)$$
$$\times\ {}_2F_1\left(r+2,\eta+1;2;\frac{x}{1+x}\right),$$

where hypergeometric function ${}_2F_1(a,b;c;x) = \sum\limits_{i=0}^{\infty} \frac{(a)_i(b)_i}{(c)_i i!}x^i,\ (a)_i = \prod\limits_{j=0}^{i-1}(a+j)$.
Using the relations above, we can easily compute the moments for the proposed operators. □

Lemma 3.2. *Let* $\alpha_{\eta,r}(x) = \mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}((e_1(z)-xe_0(z))^r;x)$ *and* $r=1,2,4$, *we calculate easily central moments by using Lemma (3.1), we get*

$$\alpha_{\eta,1}(x) = \frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta},$$
$$\alpha_{\eta,2}(x) = \frac{x^2(\eta+4(1-\zeta_0(\eta)))+2x(\eta+5(1-\zeta_0(\eta)))+4(1-\zeta_0(\eta))+2}{\eta^2}, \tag{6}$$
$$\begin{aligned}\alpha_{\eta,4}(x) = {}& \frac{x^4(3\eta(\eta+2)+8(5\eta+6)(1-\zeta_0(\eta)))}{\eta^4}\\ &+\frac{4x^3(\eta(3\eta+8)+(41\eta+54)(1-\zeta_0(\eta)))}{\eta^4}\\ &+\frac{12x^2(\eta(\eta+6)+2(9\eta+16)(1-\zeta_0(\eta)))}{\eta^4}\\ &+\frac{24x(3\eta+(3\eta+14)(1-\zeta_0(\eta)))+96(1-\zeta_0(\eta))+24}{\eta^4}.\end{aligned} \tag{7}$$

In this paper, (6) *and* (7) *are referred to as* $\alpha_{\eta,2}(x)$ *and* $\alpha_{\eta,4}(x)$ *in the Theorems 4.2, 4.3, 4.4, 4.5, and 5.2.*

Now, we define a normed space given by $\mathcal{S}_{\mathcal{B}^d}[0,\infty) = \{\mu \in \mathcal{C}[0,\infty) : \mu$ is bounded over $[0,\infty)$ and $\int_0^\infty s^*_{\eta,\ell}(z)\ \mu(z)dz < \infty\}$ that has the norm

$$\|\mu\| = \sup_{x\in[0,\infty)} |\mu(x)|. \tag{8}$$

Lemma 3.3. *If* $\rho_0(\eta)$ *is a bounded sequence, and* $\mu \in \mathcal{S}_{\mathcal{B}^d}[0,\infty)$, *then we get*

$$|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)| \le \|\mu\|. \tag{9}$$

4. Direct result and asymptotic formula

In this section, we explore some important results.

THEOREM 4.1. *Let* $\mu \in \mathcal{S}_{\mathcal{B}^d}[0,\infty)$. *If* $\lim\limits_{\eta\to\infty} \zeta_0(\eta)$ *exists, then* $\lim\limits_{\eta\to\infty} \mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x) = \mu(x)$ *holds uniformly on compact subsets of* $[0,\infty)$.

Proof. Lemma 3.1 allows for easy to establish that $\lim\limits_{\eta\to\infty} \mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(e_r(z);x) = x^r$ for $r = 0,1,2,3,4$, and hence the well known Bohman-Korovkin's theorem due to [19], operators $\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}$ converge uniformly on every compact subset of $[0,\infty)$ to $\mu(x)$. □

For $\mu \in \mathcal{S}_{\mathcal{B}^d}[0,\infty)$, the Peetre's K-functional is given by

$$K_2^f(\mu,\delta) = \inf_{\psi\in W^*_{\mathcal{C}^2([0,\infty))}} \{\|\mu-\psi\| + \delta\|\psi''\|\}, \text{ where } \delta>0, \text{ and}$$

$$W^*_{\mathcal{C}^2([0,\infty))} = \{\psi \in \mathcal{S}_{\mathcal{B}^d}[0,\infty) : \psi', \psi'' \in \mathcal{S}_{\mathcal{B}^d}[0,\infty)\}.$$

$$K_2^f(\mu,\delta) \le \mathcal{M}_1\omega_2(\mu,\sqrt{\delta}), \tag{10}$$

where $\omega_2(\mu,\sqrt{\delta}) = \sup\limits_{0<\epsilon^*\le\sqrt{\delta}} \left(\sup\limits_{x,x+\epsilon^*,x+2\epsilon^*\in[0,\infty)} |\mu(x+2\epsilon^*) - 2\mu(x+\epsilon^*) + \mu(x)| \right)$ is called the second order of modulus of continuity of μ. The expression

$$\omega(\mu,\delta) = \sup_{|z-x|\le\delta} |\mu(z)-\mu(x)| \tag{11}$$

is commonly referred to as the modulus of continuity of μ, where $x,z\in[0,\infty)$.

THEOREM 4.2. *Let* $\zeta_0(\eta)$ *be a bounded sequence. If* $\mu \in \mathcal{S}_{\mathcal{B}^d}[0,\infty)$, *and* $\delta = \alpha_{\eta,2}(x) + \left(\frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}\right)^2$, *then*

$$|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x) - \mu(x)| \le \mathcal{M}_1\omega_2(\mu,\sqrt{\delta/2}) + \omega\left(\mu, \frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}\right),$$

where $\mathcal{M}_1 > 0$ *is constant.*

Proof. Firstly, we introduce

$$\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x) = \mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x) - \mu\left(x + \frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}\right) + \mu(x). \tag{12}$$

Let $\psi \in W^*_{\mathcal{C}^2([0,\infty))}$, then by Taylor's theorem

$$\psi(z) = \psi(x) + (z-x)\psi'(x) + \frac{1}{2}\int_x^z (z-w)\psi''(w)dw. \tag{13}$$

Applying $\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}$ on (13),

$$\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\psi;x) = \psi(x) + \psi'(x)\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(z-x;x) + \frac{1}{2}\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}\left(\int_x^z (z-w)\psi''(w)dw;x\right).$$

By Lemma 3.2 and (12), $\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(z-x;x)=0$. Therefore,

$$|\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\psi;x)-\psi(x)| \le \mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}\left(\int\limits_x^z (z-w)|\psi^{''}(w)|dw;x\right)$$
$$\le \|\psi^{''}\|\left|\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}\left(\int\limits_x^z (z-w)dw;x\right)\right|.$$

By (12),

$$|\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\psi;x)-\psi(x)| \le \|\psi^{''}\|\left(\left|(\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}\left(\int\limits_x^z (z-w)dw;x\right)\right|\right.$$
$$\left.+\left|\int_x^{x+\frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}}\left(x+\frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}-w\right)dw\right|\right)$$
$$\le \|\psi^{''}\|\left(\alpha_{\eta,2}(x)+\left(\frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}\right)^2\right)=\delta\|\psi^{''}\|.$$

We have

$$\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)=\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\mu-\psi;x)-(\mu-\psi)(x)+\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\psi;x)-\psi(x)$$
$$+\mu\left(x+\frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}\right)-\mu(x).$$
$$|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)| \le |\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\mu-\psi;x)-(\mu-\psi)(x)|+|\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\psi;x)-\psi(x)|$$
$$+\left|\mu\left(x+\frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}\right)-\mu(x)\right|$$
$$\le 2\|\mu-\psi\|+\delta\|\psi^{''}\|+\omega\left(\mu,\frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}\right).$$

Taking infimum of ψ on $W^*_{\mathcal{C}^2([0,\infty))}$ of the right hand side of the inequality,

$$|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)| \le K_2^f\left(\mu,\frac{\delta}{2}\right)+\omega\left(\mu,\frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}\right).$$

Using (10), we get

$$|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)| \le \mathcal{M}_1\omega_2(\mu,\sqrt{\delta/2})+\omega\left(\mu,\frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}\right). \qquad \square$$

Now, we address Voronovskaja-type asymptotic formula [24] for the operators $\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}$.

Theorem 4.3. *If* $\lim\limits_{\eta\to\infty}\zeta_0(\eta)=a_0\in\mathbb{R}$, *and functions* $\mu,\mu^{'},\mu^{''}\in\mathcal{S}_{\mathcal{B}^d}[0,\infty)$, *then*

$$\lim_{\eta\to\infty}\eta[\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)]=((2x+1)(1-a_0)+1)\mu^{'}(x)+x(x+2)\frac{\mu^{''}(x)}{2}.$$

Proof. By the Taylor's theorem

$$\mu(z) = \mu(x) + (z-x)\mu^{'}(x) + \frac{(z-x)^2}{2}\mu^{''}(x) + h_{\mathfrak{B}}(x;z)(z-x)^2, \tag{14}$$

where $h_{\mathfrak{B}}(x;z)$ is the Peano form of the remainder, $h_{\mathfrak{B}}(x;z) \in \mathcal{S}_{\mathcal{B}^d}[0,\infty)$, and $\lim_{z\to x} h_{\mathfrak{B}}(x;z) = 0$.

Applying $\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}$ on (14),

$$\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x) = \mu(x) + \mu^{'}(x)\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}((z-x);x) + \frac{\mu^{''}(x)}{2}\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}((z-x)^2;x)$$
$$+ \mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(h_{\mathfrak{B}}(x;z)(z-x)^2;x).$$

$$\lim_{\eta\to\infty} \eta[\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x) - \mu(x)] = \mu^{'}(x)\lim_{\eta\to\infty}(\eta\alpha_{\eta,1}(x)) + \frac{\mu^{''}(x)}{2}\lim_{\eta\to\infty}(\eta\alpha_{\eta,2}(x))$$
$$+ \lim_{\eta\to\infty}(\eta\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(h_{\mathfrak{B}}(x;z)(z-x)^2;x))$$
$$= ((2x+1)(1-a_0)+1)\mu^{'}(x) + x(x+2)\frac{\mu^{''}(x)}{2} + \mathcal{E}_\eta,$$

where $\mathcal{E}_\eta = \lim_{\eta\to\infty}(\eta\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(h_{\mathfrak{B}}(x;z)(z-x)^2;x))$.

Using Cauchy-Bunyakovsky-Schwarz inequality, we get

$$\eta\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(h_{\mathfrak{B}}(x;z)(z-x)^2;x) \le \sqrt{\eta^2\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(h_{\mathfrak{B}}^2(x;z);x)}\sqrt{\alpha_{\eta,4}(x)}.$$

We observe that if $\eta\to\infty$, then $z\to x$ and $\lim_{z\to x} h_{\mathfrak{B}}(x;z) = 0$. It follows that

$$\lim_{\eta\to\infty}(\eta^2\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(h_{\mathfrak{B}}^2(x;z);x)) = 0 \text{ uniformly with respect to } x\in[0,\infty).$$

So, $\mathcal{E}_\eta = \lim_{\eta\to\infty}(\eta\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(h_{\mathfrak{B}}(x;z)(z-x)^2;x)) = 0$. This completes the proof. □

Here, we examine how to assess the degree of approximation by using the Ditzian-Toitik moduli of smoothness. By [11], let

$$\omega_{\theta^\lambda}^2(\mu,\delta) = \sup_{0<\epsilon^*\le\delta}\left(\sup_{x,x+\epsilon^*\theta^\lambda,x-2\epsilon^*\theta^\lambda\in[0,\infty)}\left|\mu(x+\epsilon^*\theta^\lambda) - 2\mu(x) + \mu(x-\epsilon^*\theta^\lambda)\right|\right),$$

and the K-functional is given by

$$K_{2,\theta^\lambda}^f(\mu,\delta^2) = \inf_{\psi^{'}\in A.C._{loc}([0,\infty))}\{\|\mu-\psi\| + \delta^2\|\theta^{2\lambda}\psi^{''}\|\},$$

and $\mathfrak{E}_\lambda^2 = \{\psi\in\mathcal{S}_{\mathcal{B}^d}[0,\infty) : \psi^{'}\in A.C._{loc}([0,\infty)), \|\theta^{2\lambda}\psi^{''}\| < \infty\}$, where $\theta^2(x) = x$, $0\le\lambda\le 1$. We have $K_{2,\theta^\lambda}(\mu,\delta^2) \sim \omega_{\theta^\lambda}^2(\mu,\delta)$.

Theorem 4.4. *Let $\zeta_0(\eta)$ be a bounded sequence. If $\mu\in\mathcal{S}_{\mathcal{B}^d}[0,\infty)$ and $x\in[0,\infty)$, then*

$$|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x) - \mu(x)| \le 4\omega_{\theta^\lambda}^2\left(\mu, \frac{\delta_\eta^{(1-\lambda)}(x)}{\sqrt{2n}}\right) + \omega\left(\mu, \frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}\right).$$

Proof. Consider

$$\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)=\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu\left(x+\frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}\right)+\mu(x). \quad (15)$$

Let $\psi\in\mathfrak{E}_\lambda^2$, by Taylor's theorem

$$\psi(z)=\psi(x)+(z-x)\psi^{'}(x)+\frac{1}{2}\int_x^z(z-w)\psi^{''}(w)dw. \quad (16)$$

Applying $\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}$ on (16),

$$\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\psi;x)=\psi(x)+\psi^{'}(x)\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(z-x;x)+\frac{1}{2}\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}\left(\int_x^z(z-w)\psi^{''}(w)dw;x\right). \quad (17)$$

By Lemma 3.2 and (15), $\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(z-x;x)=0$. We have

$$|\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)|\le 3\|\mu\|, \quad (18)$$

and

$$\alpha_{\eta,2}(x)\le\frac{1}{\eta}\delta_\eta^2(x),$$

where $\delta_\eta^2(x)=\dfrac{\theta^2(x)(\eta+4(1-\zeta_0(\eta)))+2\theta(x)(\eta+5(1-\zeta_0(\eta)))+4(1-\zeta_0(\eta))+2}{\eta}$.

From [11, p. 141], for $z<w<x$, we have

$$\frac{|z-w|}{\theta^{2\lambda}(w)}\le\frac{|z-x|}{\theta^{2\lambda}(x)}\quad\text{and}\quad\frac{|z-w|}{\delta_\eta^{2\lambda}(w)}\le\frac{|z-x|}{\delta_\eta^{2\lambda}(x)}. \quad (19)$$

By (15) and (17),

$$\begin{aligned}|\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\psi;x)-\psi(x)|&\le\left|\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}\left(\int_x^z(z-w)\psi^{''}(w)dw;x\right)\right|\\&\le\left|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}\left(\int_x^z(z-w)\psi^{''}(w)dw;x\right)\right|\\&\quad+\left|\int_x^{x+\frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}}\left(x+\frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}-w\right)\psi^{''}(w)dw\right|,\end{aligned}$$

using (19),

$$\begin{aligned}|\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\psi;x)-\psi(x)|&\le\|\delta_\eta^{2\lambda}\psi^{''}\|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}\left(\frac{(z-x)^2}{\delta_\eta^{2\lambda}(x)};x\right)+\frac{\|\delta_\eta^{2\lambda}\psi^{''}\|}{\delta_\eta^{2\lambda}(x)}\left(\frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}\right)^2\\&\le\delta_\eta^{-2\lambda}(x)\|\delta_\eta^{2\lambda}\psi^{''}\|\left(\alpha_{\eta,2}(x)\right)+\delta_\eta^{-2\lambda}(x)\|\delta_\eta^{2\lambda}\psi^{''}\|\Big(\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}((z-x);x)\Big)^2\\&\le\delta_\eta^{-2\lambda}(x)\|\delta_\eta^{2\lambda}\psi^{''}\|\frac{\delta_\eta^2(x)}{\eta}+\delta_\eta^{-2\lambda}(x)\|\delta_\eta^{2\lambda}\psi^{''}\|\frac{\delta_\eta^2(x)}{\eta}.\end{aligned}$$

Hence,

$$|\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\psi;x)-\psi(x)|\le \frac{2\delta_\eta^{2(1-\lambda)}(x)}{\eta}\|\delta_\eta^{2\lambda}\psi''\|. \tag{20}$$

Using (8), (18), and (20),

$$\begin{aligned}|\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)| &\le |\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\mu-\psi;x)|+|\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\psi;x)-\psi(x)|+|\mu(x)-\psi(x)|\\ &\le 4\|\mu-\psi\|+|\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\psi;x)-\psi(x)|\\ &\le 4\|\mu-\psi\|+\frac{2\delta_\eta^{2(1-\lambda)}(x)}{\eta}\|\delta_\eta^{2\lambda}\psi''\|.\end{aligned}$$

Hence,

$$\begin{aligned}|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)| &\le |\mathcal{N}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)|+\left|\mu\left(x+\frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}\right)-\mu(x)\right|\\ &\le 4\omega_{\theta^\lambda}^2\left(\mu,\frac{\delta_\eta^{(1-\lambda)}(x)}{\sqrt{2\eta}}\right)+\omega\left(\mu,\frac{(2x+1)(1-\zeta_0(\eta))+1}{\eta}\right).\end{aligned}$$ □

Theorem 4.5. *For $\mu\in\mathcal{S}_{\mathcal{B}^d}[0,\infty)$ and bounded sequence $\zeta_0(\eta)$, then*

$$|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)|\le 2\omega(\mu,\delta),\ \textit{where}\ \delta=\sqrt{\alpha_{\eta,2}(x)}.$$

Proof. We use the property of modulus of continuity,

$$|\mu(z)-\mu(x)|\le\omega(\mu,|z-x|)\le\left(1+\frac{1}{\delta}|z-x|\right)\omega(\mu,\delta).$$

$$\begin{aligned}|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)| &\le a_{\eta,\ell}(x)\int_0^\infty s_{\eta,\ell}^*(z)\,|\mu(z)-\mu(x)|\,dz\\ &\le\left(1+\frac{1}{\delta}*a_{\eta,\ell}(x)\int_0^\infty s_{\eta,\ell}^*(z)\,|z-x|\,dz\right)\omega(\mu,\delta),\end{aligned}$$

where $a_{\eta,\ell}(x)=\eta\sum\limits_{\ell=0}^{\infty}P_{\eta,\ell}^{M,1}(x)$.

By Cauchy-Bunyakovsky-Schwarz inequality,

$$\begin{aligned}&|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)|\\ &\le\left(1+\left(\frac{1}{\delta}\sqrt{a_{\eta,\ell}(x)\int_0^\infty s_{\eta,\ell}^*(z)dz}\sqrt{a_{\eta,\ell}(x)\int_0^\infty s_{\eta,\ell}^*(z)(z-x)^2dz}\right)\right)\omega(\mu,\delta)\\ &=\left(1+\frac{1}{\delta}\sqrt{(\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}((z-x)^2;x)}\right)\omega(\mu,\delta)=\left(1+\frac{1}{\delta}\sqrt{\alpha_{\eta,2}(x)}\right)\omega(\mu,\delta)=2\omega(\mu,\delta).\end{aligned}$$ □

5. Weighted approximation

The purpose of this section is to determine approximation properties of the operators $\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}$ using weighted spaces of continuous functions.

Let $\mathcal{S}_{y_w}^{K}([0,\infty))$ be a normed space specified by

$$\mathcal{S}_{y_w}^{K}([0,\infty)) = \{\mu : |\mu(x)| \le \mathcal{K}_\mu y_w(x),\ x\in[0,\infty)\},$$

endowed with the norm
$$\|\mu\|_2 = \sup_{x\in[0,\infty)} \frac{|\mu(x)|}{y_w(x)},$$

where $y_w(x) = 1 + x^2$ and constant $\mathcal{K}_\mu > 0$ depends on function μ.

Moreover, we define following spaces,

(i) $\mathcal{A}_{y_w}([0,\infty)) = \{\mu \in \mathcal{S}_{y_w}^{K}([0,\infty)) : \mu$ is continuous function on $[0,\infty)\}$,

(ii) $\mathcal{A}_{y_w}^{*}([0,\infty)) = \{\mu \in \mathcal{A}_{y_w}([0,\infty)) : \lim\limits_{x\to\infty} \dfrac{\mu(x)}{y_w(x)}$ exists in $\mathbb{R}\}$.

It was shown in [16] that for any $\mu \in \mathcal{A}_{y_w}^{*}([0,\infty))$, the weighted modulus of continuity is denoted by

$$\Omega(\mu;\delta) = \sup_{0\le e^*<\delta, x\in[0,\infty)} \frac{|\mu(x+e^*) - \mu(x)|}{(1+e^{*2})y_w(x)}. \tag{21}$$

Lemma 5.1. *If $\mu \in \mathcal{A}_{y_w}([0,\infty))$, then*
(i) $\Omega_1(\mu;\delta)$ is a monotonically increasing function of δ,

(ii) $\Omega_1(\mu;\beta\delta) \le 2(1+\beta)(1+\delta^2)\Omega_1(\mu;\delta), \beta > 0$,

(iii) $\Omega_1(\mu;\delta) \to 0$ as $\delta \to 0$.

The definition of weighted modulus of continuity enables us to write

$$|\mu(z) - \mu(x)| \le y_w(x)(1 + (z-x)^2)\Omega_1(\mu; |z-x|). \tag{22}$$

Theorem 5.2. *If $\zeta_0(\eta)$ is a bounded sequence, and $\mu \in \mathcal{S}_{\mathcal{B}^d}[0,\infty)$, then*

$$\begin{aligned}&|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x) - \mu(x)|\\ \le& 2\left(1+\frac{1}{\eta}\right) y_w(x)\Omega_1\left(\mu;\frac{1}{\sqrt{\eta}}\right)\left(1 + \frac{x(x+2)}{\eta}(\mathcal{E}_1 + \sqrt{\mathcal{E}_1\mathcal{E}_2 x(x+2)}) + \sqrt{\mathcal{E}_1 x(x+2)}\right).\end{aligned}$$

Proof. For $x, z \in [0,\infty)$. From (22) and Lemma 5.1, we get

$$\begin{aligned}|\mu(z) - \mu(x)| &\le (1 + (z-x)^2)y_w(x)\Omega_1\left(\mu; \frac{|z-x|\delta}{\delta}\right)\\ &\le 2(1+\delta^2)y_w(x)\left(1 + \frac{|z-x|}{\delta}\right)(1+(z-x)^2)\Omega_1(\mu;\delta). \end{aligned}\tag{23}$$

Applying $\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}$ on (23),

$$|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x) - \mu(x)| \le 2(1+\delta^2)y_w(x)\Omega_1(\mu;\delta)\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}\left[\left(1+\frac{|z-x|}{\delta}\right)(1+(z-x)^2);x\right]$$

$$=2(1+\delta^2)y_w(x)\Omega_1(\mu;\delta)\left[1+\alpha_{\eta,2}(x)+\frac{1}{\delta}\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(|z-x|;x)+\frac{1}{\delta}\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(|z-x|(z-x)^2;x)\right].$$

By Cauchy-Bunyakovsky-Schwarz inequality, we get

$$\begin{aligned}|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)| &\leq 2(1+\delta^2)y_w(x)\Omega_1(\mu;\delta)\,[1+\alpha_{\eta,2}(x)\\ &\quad+\frac{1}{\delta}(\alpha_{\eta,2}(x))^{1/2}+\frac{1}{\delta}\left(\alpha_{\eta,2}(x)\right)^{1/2}\left(\alpha_{\eta,4}(x)\right)^{1/2}\Big].\end{aligned} \tag{24}$$

From (6) and (7), we have $\alpha_{\eta,2}(x)\leq\frac{\mathcal{E}_1 x(x+2)}{\eta}$ and $\alpha_{\eta,4}(x)\leq\frac{\mathcal{E}_2 x^2(x+2)^2}{\eta^2}$, where $\mathcal{E}_1>1$ and $\mathcal{E}_2>1$ are constants.

Using the above inequalities in (24) and choosing $\delta=\frac{1}{\sqrt{\eta}}$, we get desired result. □

Theorem 5.3. *If $\zeta_0(\eta)$ is a bounded sequences, then for each $\mu\in\mathcal{A}^*_{y_w}([0,\infty))$ and $x\in[0,\infty)$, we have*

$$\lim_{\eta\to\infty}\|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;.)-\mu\|_2=0.$$

Proof. Using [13], it is sufficient to verify the following conditions to show this theorem

$$\lim_{\eta\to\infty}\|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(e_r;x)-x^r\|_2=0,\ r=0,1,2. \tag{25}$$

We have $\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(e_0;x)=1$, so for $r=0$ (25) holds. By Lemma 3.1,

$$\begin{aligned}\|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(e_1;x)-x\|_2 &= \sup_{x\in[0,\infty)}\frac{|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(e_1;x)-x|}{y_w(x)}\\ &=\frac{1}{\eta}\sup_{x\in[0,\infty)}\left(\frac{(2x+1)(1-\zeta_0(\eta))+1}{y_w(x)}\right)\to 0 \text{ as } \eta\to\infty.\end{aligned}$$

For $r=1$, the condition (25) is satisfied. Again by Lemma 3.1,

$$\|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(e_2;x)-x^2\|_2=\sup_{x\in[0,\infty)}\frac{|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(e_2;x)-x^2|}{y_w(x)}$$

$$=\frac{1}{\eta^2}\sup_{x\in[0,\infty)}\left(\frac{x^2(\eta+4(\eta+1)(1-\zeta_0(\eta)))+2x(2\eta+(\eta+5)(1-\zeta_0(\eta)))}{y_w(x)}+\frac{4(1-\zeta_0(\eta))+2}{y_w(x)}\right).$$

Evidently, $\|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(e_2;x)-x^2\|_2\to 0$ as $\eta\to\infty$, for $r=2$, the condition (25) is satisfied. Hence, the theorem proved. □

Corollary 5.4. *For bounded sequence $\zeta_0(\eta)$, and $\mu\in\mathcal{A}_{y_w}([0,\infty))$, then*

$$\lim_{\eta\to\infty}\sup_{x\in[0,\infty)}\frac{|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)|}{(y_w(x))^{s+1}}=0,\ \textit{where } s>0.$$

Proof. For any fixed $x_0>0$,

$$\sup_{x\in[0,\infty)}\frac{|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)|}{(y_w(x))^{s+1}}\leq\sup_{x\leq x_0}\frac{|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)|}{(y_w(x))^{s+1}}+\sup_{x>x_0}\frac{|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)|}{(y_w(x))^{s+1}}$$

$$\leq \|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu\|_{C[0,x_0]}+\|\mu\|_2 \sup_{x>x_0} \frac{|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(1+z^2;x)|}{(y_w(x))^{s+1}}+\sup_{x>x_0} \frac{|\mu(x)|}{(y_w(x))^{s+1}}.$$

According to Theorem 4.1, the second term from the leftmost in the above inequality tends to 0 as $\eta \to \infty$, and for the fixed x_0, terms $\|\mu\|_2 \sup_{x>x_0} \frac{|\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(1+z^2;x)|}{(y_w(x))^{s+1}}$ and $\sup_{x>x_0} \frac{|\mu(x)|}{(y_w(x))^{s+1}}$ can be made sufficiently small if we choose x_0 large enough. Thus, the goal of proof is proved. □

6. Graphical and numerical analysis

This section evaluates the performance of the operators through Examples 6.1 and 6.2. The graphs in Figures 1, 2, and 3 present a comparison of the operators $\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}$ and $\mathcal{S}_{\eta,\ell}^{\mu,P}$ for various choices of η and $\zeta_0(\eta)$. These comparisons reveal that operators $\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}$ performs better than $\mathcal{S}_{\eta,\ell}^{\mu,P}$. We compute the absolute error $Er_{\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}} = |\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}(\mu;x)-\mu(x)|$ for different values of x over the interval $[0,10]$. The results are presented in Tables 1, 2, and 3. All computations were carried out using Wolfram Mathematica, version 12.0.

Example 6.1. Let us consider test function $\mu(x) = x^3 - 2x^2 + 3$, and sequence $\zeta_0(\eta)$ such that $\zeta_0(\eta) + \zeta_1(\eta) = 1 - \zeta_0(\eta)$.

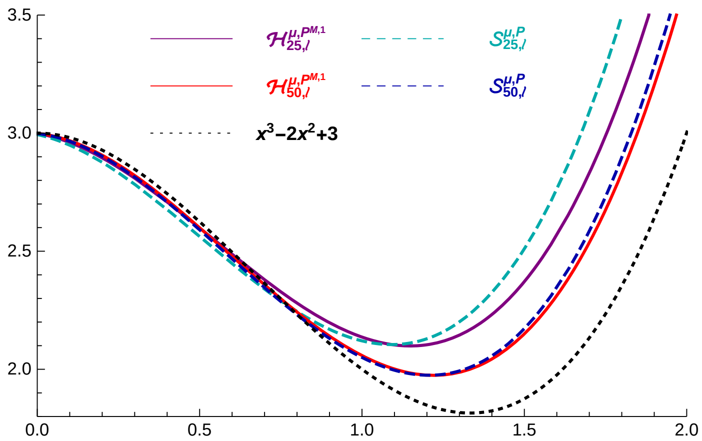

Figure 1: Convergence behaviour of the operators $\mathcal{H}_{\eta,\ell}^{\mu,P^{M,1}}$ and $\mathcal{S}_{\eta,\ell}^{\mu,P}$, when $\zeta_0(\eta) = 1 + \frac{10}{\eta}$.

x	$Er_{\mathcal{H}^{\mu,PM,1}_{25,\ell}}$	$Er_{\mathcal{S}^{\mu,P}_{25,\ell}}$	$Er_{\mathcal{H}^{\mu,PM,1}_{50,\ell}}$	$Er_{\mathcal{S}^{\mu,P}_{50,\ell}}$
2	1.065600000	1.574784000	0.650624000	0.753648000
4	5.667609600	11.42438400	4.283129600	5.576048000
6	14.54256640	35.45638400	12.59293440	17.38404800
8	28.42836480	79.58438400	27.27639680	39.09604800
10	48.06289920	149.7219840	50.02987520	73.63044800

Table 1: Absolute error table, when $\zeta_0(\eta) = 1 + \frac{10}{\eta}$.

Example 6.2. Let us consider test function $\mu(x) = e^{-5x}x^2$, and sequence $\zeta_0(\eta)$ such that $\zeta_0(\eta) + \zeta_1(\eta) = 1 - \zeta_0(\eta)$.

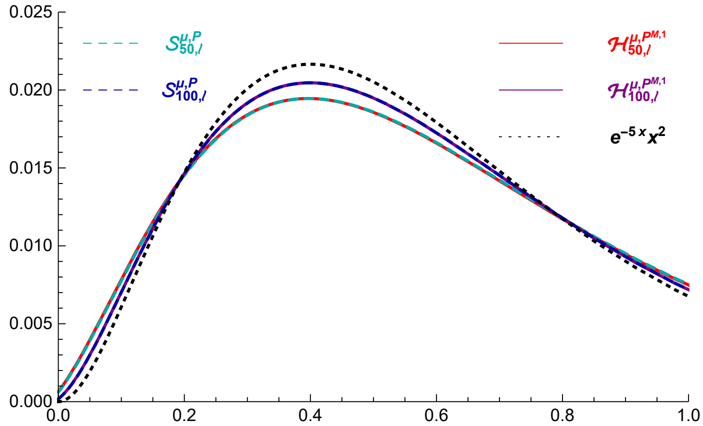

Figure 2: Convergence behaviour of the operators $\mathcal{H}^{\mu,PM,1}_{\eta,\ell}$ and $\mathcal{S}^{\mu,P}_{\eta,\ell}$, when $\zeta_0(\eta) = \frac{\eta-1}{\eta+1}$.

x	$Er_{\mathcal{H}^{\mu,PM,1}_{50,\ell}}$	$Er_{\mathcal{S}^{\mu,P}_{50,\ell}}$	$Er_{\mathcal{H}^{\mu,PM,1}_{100,\ell}}$	$Er_{\mathcal{S}^{\mu,P}_{100,\ell}}$
0.2	0.000125888	0.000185218	0.000066347	0.000083740
0.4	0.002201000	0.002200320	0.001191800	0.001191750
0.6	0.001329850	0.001290670	0.000679063	0.000667798
0.8	0.000006117	0.000042608	0.000049574	0.000063174
1.0	0.000738063	0.000780769	0.000432683	0.000444103

Table 2: Absolute error table, when $\zeta_0(\eta) = \frac{\eta-1}{\eta+1}$.

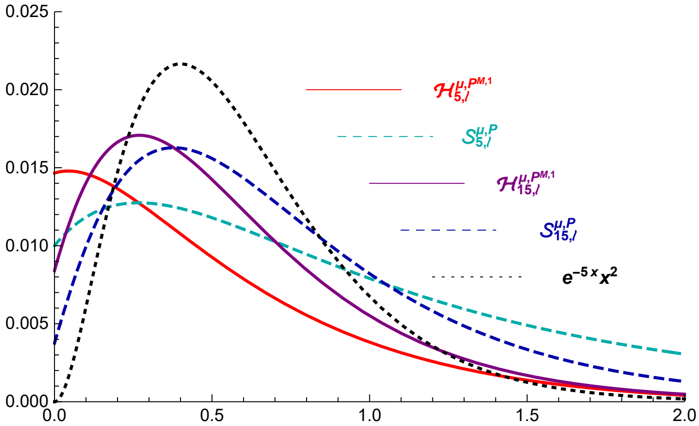

Figure 3: Convergence behaviour of the operators $\mathcal{H}^{\mu,PM,1}_{\eta,\ell}$ and $\mathcal{S}^{\mu,P}_{\eta,\ell}$, when $\zeta_0(\eta)=\frac{2}{\eta^2+4}$.

x	$Er_{\mathcal{H}^{\mu,PM,1}_{5,\ell}}$	$Er_{\mathcal{S}^{\mu,P}_{5,\ell}}$	$Er_{\mathcal{H}^{\mu,PM,1}_{15,\ell}}$	$Er_{\mathcal{S}^{\mu,P}_{15,\ell}}$
2	0.000228556	0.002865275	0.000298000	0.001111522
4	0.000118996	0.000589816	0.000000779	0.000036130
6	0.000060832	0.000165405	0.000000324	0.000001787
8	0.000027798	0.000059520	0.000000045	0.000000141
10	0.000013539	0.000025397	0.000000006	0.000000016

Table 3: Absolute error table, when $\zeta_0(\eta)=\frac{2}{\eta^2+4}$.

7. Conclusion

The present study explores an innovative technique of integral-type operators that adopt the Baskakov basis function in recursion form and the Szász basis function, emphasizing how well they approximate integrable functions over the interval $[0,\infty)$. A vital component of the investigation is to assess the flexibility and convergence of the proposed operators, which count on the choice of the sequence $\zeta_0(\eta)$ and η, respectively. It examines how variations in these traits influence the performance of the proposed operators. In addition, the performance of the operators under various choices η and sequence $\zeta_0(\eta)$ is also visualized using the graphs. The graphs offer a simpler overview of how these traits influence the behavior of the proposed operators, which can be especially valuable when communicating findings to others.

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