MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 01.07.2025

research paper
оригинални научни рад
DOI: 10.57016/MV-WdCO2702

FIXED POINTS OF A FAMILY OF GENERAL MAPPINGS

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Abstract. In this paper, we establish new results on the existence, uniqueness, and convergence for a one-parameter family of mappings, where each member has a unique fixed point that is also the unique common fixed point of the entire family. These results apply to both contractive-type mappings and those that do not adhere to any contractive conditions. Our theorem includes, as special cases, the results of [B. D. Gel'man, *Caristi's inequality and α-contraction mappings*, Funct. Anal. Appl., **53** (2019), 224–228], [R. P. Pant, V. Rakočević, D. Gopal, A. Pant, M. Ram, *A general fixed point theorem*, Filomat, **35** (2021), 4061–4072], as well as the well-known fixed-point theorems of Banach, Kannan, Chatterjea, and Ćirić.

1. Introduction

The following Caristi or Caristi-Kirk fixed point theorem was proved in [3, 4].

THEOREM 1.1. *Let f be a self-mapping of a complete metric space (X, d). Suppose there exists a lower semi-continuous function $\varphi : X \to \mathbb{R}^+$ such that for each $x \in X$*

$$d(x, fx) \leq \varphi(x) - \varphi(fx).$$

Then f possesses a fixed point.

The Caristi fixed point theorem is fundamental in fixed point theory, influencing areas like convex minimization, variational inequalities, and control theory through Ekeland's approach [8]. For more details and generalizations, see [9] and the references therein.

In [9], Gel'man established several generalizations of Caristi's theorem in both single-valued and set-valued settings by augmenting the contraction condition with a function of two vector arguments, which is neither required to be a metric nor a continuous function. Let (X, d) be a complete metric space. Suppose $f : X \to X$ is a continuous mapping, and $\alpha : X \times X \to \mathbb{R}$ is a given function. A mapping $f : X \to X$ is called an *α-contraction* (contraction with respect to a function α) if it is continuous

2020 Mathematics Subject Classification: 47H10, 37C25
Keywords and phrases: Fixed points; Ćirić contraction; asymptotic continuity.

and there exists a constant $k \in (0,1)$ such that $\alpha(fx, fy) \leq k\alpha(x,y)$, for all $x, y \in X$. Not every α-contraction mapping necessarily has a fixed point [9].

THEOREM 1.2 ([9]). *Let f be a continuous self-mapping of a complete metric space (X,d). If there exists a constant $c > 0$ such that*

$$\alpha(fx, fy) + cd(x,y) \leq \alpha(x,y),$$

for all $x, y \in X$, then for any initial point $x_0 \in X$, the sequence of successive approximations $x_{n+1} = fx_n$ converges to a point x^*, which is the unique fixed point of f. Furthermore, the following estimate holds:

$$d(x^*, x_0) \leq \frac{\alpha(x_0, fx_0) - \gamma_0}{c},$$

where α is bounded from below and $\gamma_0 = \inf_{(x,y)\in X\times X} \alpha(x,y)$.

Let (X,d) be a metric space and $f : X \to X$ be a mapping. The orbit of a point $x_0 \in X$ under f is the set $O(f, x_0) = \{x_0, f(x_0), f^2(x_0), \ldots, f^n(x_0), \ldots\}$.

DEFINITION 1.3. Let (X,d) be a metric space. A mapping $f : X \to X$ is called:
1. *orbitally continuous* [7] if $\lim_{i\to\infty} f^{m_i}x = u$ for some $u \in X$ implies $\lim_{i\to\infty} f(f^{m_i}x) = f(u)$ for each $x \in X$;

2. *weakly orbitally continuous* [14] if the set $\{y \in X : \lim_{i\to\infty} f^{m_i}y = u$, for some $u \in X$ implies $\lim_{i\to\infty} f(f^{m_i}y) = f(u)\}$ is nonempty whenever the set $\{x \in X : \lim_{i\to\infty} f^{m_i}x = u\}$ is nonempty;

3. *k-continuous* [13], for $k = 1, 2, 3, \ldots$, if $\lim_{n\to\infty} f^k(x_n) = f(z)$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f^{k-1}(x_n) = z$ for some $z \in X$;

4. *asymptotically k-continuous* or *asymptotically continuous* [15] if $\lim_{k,n\to\infty} f(f^k(x_n)) = f(z)$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{k,n\to\infty}(f^k(x_n)) = z$ for some $z \in X$.

For $k > 1$, the continuity conditions for f^k and the k-continuity of f are independent. While k-continuity of f ensures that f is asymptotically k-continuous, the converse does not hold [15]. A review by the first author [2] contrasts various weaker forms of continuity and their significance in fixed point theory.

The following theorem generalizes several well-known fixed point theorems and is due to Pant et al. [12] (see also [1] for the case of a pair of mappings).

THEOREM 1.4. *Let f be a self-mapping of a complete metric space (X,d). Suppose $\varphi : X \to [0,\infty)$ is such that for all $x, y \in X$ we have*

$$d(fx, fy) \leq \varphi(x) - \varphi(fx) + \varphi(y) - \varphi(fy). \tag{1}$$

If f is weakly orbitally continuous or f is orbitally continuous or f is k-continuous, then f has a unique fixed point.

REMARK 1.5. It is pertinent to mention that the condition (1) includes both contractive and non-contractive mappings [12].

In this paper, motivated by the work of Pant et al. [11], we establish fixed point theorems for a one-parameter family of self-mappings, extending the work of Gel'man [9] and Pant et al. [12], who studied various classes of single self-mappings in the context of a complete metric space. Each member of this family has a unique fixed point, which is also the unique common fixed point of the entire family. Our main theorem yields several well-known fixed point results for families of mappings as corollaries.

2. Main results

We begin this section with the following result.

THEOREM 2.1. *Let $\{f_r : 0 \le r \le 1\}$ be a family of self-mappings on a complete metric space (X, d). Suppose there exists a constant $c > 0$ and a function $\alpha : X \times X \to \mathbb{R}$ that is bounded from below, with $\gamma_0 = \inf_{(x,y)\in X\times X} \alpha(x, y)$, such that*

$$\alpha(f_r x, f_r y) + cd(x, y) \le \alpha(x, y), \tag{2}$$

for all $x, y \in X$. If each f_r is k-asymptotically continuous for some integer $k \ge 1$, or if each f_r is weakly orbitally continuous, then each f_r has a unique fixed point. Moreover, if every pair of mappings (f_r, f_s) satisfies the condition

$$\alpha(f_r x, f_s y) + cd(x, y) \le \alpha(x, y), \tag{3}$$

for all $x, y \in X$, then the family of mappings has a unique common fixed point, which is also the unique fixed point of each f_r.

Proof. Let $x_0 \in X$, and construct a sequence $\{x_n\}$ iteratively by $x_{n+1} = f_r(x_n)$, that is, $x_{n+1} = f_r^n(x_0)$ for some fixed $r \in [0, 1]$. Define $u_n = \alpha(x_n, x_{n+1})$ for all $n = 0, 1, 2, 3, \ldots$. From (2), we have $\alpha(f_r x_n, f_r x_{n+1}) + cd(x_n, x_{n+1}) \le \alpha(x_n, x_{n+1})$, which implies $cd(x_n, x_{n+1}) \le u_n - u_{n+1}$. Thus, the sequence $\{u_n\}$ is non-increasing and bounded below by γ_0, meaning that $\{u_n\}$ converges.

To show that $\{x_n\}$ is a Cauchy sequence, we sum the inequality

$$d(x_n, x_{n+1}) \le \frac{u_n - u_{n+1}}{c}$$

over n and obtain

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \le \frac{u_0 - \lim_{n\to\infty} u_n}{c} \le \frac{u_0 - \gamma_0}{c}.$$

This implies that the sequence $\{x_n\}$ is Cauchy. By the completeness of X, there exists a limit point $z_r \in X$ such that $\lim_{n\to\infty} x_n = f_r^n x_0 = z_r$. Also, $\lim_{n\to\infty} f_r^k x_n = z_r$ for each integer $k \ge 1$ and $\lim_{k,n\to\infty} f_r^k x_n = z_r$.

Suppose f_r is k-asymptotically continuous. Since $\lim_{k,n\to\infty} f_r^k x_n = z_r$, k-asymptotic continuity of f_r implies $\lim_{k,n\to\infty} f_r(f_r^k x_n) = f_r(z_r)$. This implies $f_r(z_r) = z_r$, since $\lim_{k,n\to\infty} f_r(f_r^k x_n) = \lim_{k,n\to\infty} f_r^{k+1}(z_r) = z_r$. Therefore, z_r is a fixed point of f_r.

Next, suppose that f_r is weakly orbitally continuous. Since $f_r^n x_0$ converges for each $x_0 \in X$, weak orbital continuity of f_r implies that there exists $y_0 \in X$ such that $\lim_{n\to\infty} f_r^n y_0 = z_r$ and $\lim_{n\to\infty} f_r(f_r^n y_0) = f_r z_r$ for some z_r in X. This implies $z_r = f_r(z_r)$, so z_r is a fixed point of f_r.

Suppose u and v are both fixed points of f_r. Using the inequality (2), we get $\alpha(f_r u, f_r v) + cd(u,v) \le \alpha(u,v)$, which simplifies to $cd(u,v) \le 0$ since $f_r u = u$ and $f_r v = v$. Therefore, $d(u,v) = 0$, implying that $u = v$. Hence, each f_r has a unique fixed point.

Now consider the second part of the theorem. If every pair of mappings f_r and f_s satisfies condition (3), then for the fixed points u_r of f_r and u_s of f_s, we have:

$$\alpha(f_r u_r, f_s u_s) + cd(u_r, u_s) \le \alpha(u_r, u_s).$$

Since $f_r u_r = u_r$ and $f_s u_s = u_s$, this reduces to $cd(u_r, u_s) \le 0$, which implies $u_r = u_s$ for all $r, s \in [0,1]$. Thus, the fixed point is common to both f_r and f_s, and this common fixed point is unique for all the mappings $\{f_r : 0 \le r \le 1\}$. □

The following proposition is crucial for the proof of the next theorem.

Proposition 2.2. Let $\{f_r : 0 \le r \le 1\}$ be a family of self-mappings on a metric space (X,d). Suppose $\varphi : X \to [0,\infty)$ is a function such that for each $x, y \in X$, the inequality

$$d(f_r x, f_r y) \le \varphi(x) - \varphi(f_r x) + \varphi(y) - \varphi(f_r y), \tag{4}$$

holds. Then, for each $x \in X$, the sequence of iterates $\{f_r^n x\}$ is a Cauchy sequence.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ by $x_1 = f_r x_0$, $x_2 = f_r x_1$, $\ldots$, $x_n = f_r x_{n-1}$, i.e., $x_n = f_r^n x_0$. Then,

$$\begin{aligned} d(x_1, x_2) &= d(f_r x_0, f_r x_1) \le \varphi(x_0) - \varphi(f_r x_0) + \varphi(x_1) - \varphi(f_r x_1) \\ &= \varphi(x_0) - \varphi(x_1) + \varphi(x_1) - \varphi(x_2) = \varphi(x_0) - \varphi(x_2). \end{aligned}$$

Similarly,

$$\begin{aligned} d(x_2, x_3) &\le \varphi(x_1) - \varphi(x_3), \\ d(x_3, x_4) &\le \varphi(x_2) - \varphi(x_4), \\ &\vdots \\ d(x_{n-1}, x_n) &\le \varphi(x_{n-2}) - \varphi(x_n), \\ d(x_n, x_{n+1}) &\le \varphi(x_{n-1}) - \varphi(x_{n+1}). \end{aligned}$$

Adding these inequalities, we get

$$\begin{aligned} &d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \\ &\le \varphi(x_0) + \varphi(x_1) - \varphi(x_n) - \varphi(x_{n+1}) \le \varphi(x_0) + \varphi(x_1). \end{aligned}$$

Taking the limit as $n \to \infty$, we get $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \le \varphi(x_0) + \varphi(x_1)$. This implies that $\{x_n\}$ is a Cauchy sequence. □

Theorem 2.3. *Let $\{f_r : 0 \le r \le 1\}$ be a family of self-mappings on a complete metric space (X,d). Suppose $\varphi : X \to [0,\infty)$ is a function satisfying (4). If f_r is k-asymptotically continuous for some integer $k \ge 1$ or f_r is weakly orbitally continuous, then f_r has a unique fixed point. Moreover, if every pair of mappings (f_r, f_s) satisfies* *the condition*

$$d(f_r x, f_s y) \leq \varphi(x) - \varphi(f_r x) + \varphi(y) - \varphi(f_s y), \tag{5}$$

then the mappings have a unique common fixed point, which is also the unique fixed point of each f_r.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}$ recursively by $x_n = f_r x_{n-1}$, i.e., $x_n = f_r^n x_0$ for $n = 1, 2, 3, \ldots$. By Proposition 2.2, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists some $z_r \in X$ such that $\lim_{n\to\infty} x_n = z_r$. Additionally, $\lim_{n\to\infty} f_r^k x_n = z_r$ for each integer $k \geq 1$, and $\lim_{k,n\to\infty} f_r^k x_n = z_r$.

Suppose f_r is k-asymptotically continuous. Then, since $\lim_{k,n\to\infty} f_r^k x_n = z_r$, k-asymptotic continuity implies $\lim_{k,n\to\infty} f_r(f_r^k x_n) = f_r(z_r)$. This implies $z_r = f_r(z_r)$, since $\lim_{k,n\to\infty} f_r^{k+1} x_n = z_r$. Therefore, z_r is a fixed point of f_r.

Next, suppose f_r is weakly orbitally continuous. Since the sequence $f_r^n x_0$ converges for each $x_0 \in X$, weak orbital continuity of f_r implies that there exists $y_0 \in X$ such that $f_r^n y_0 \to z_r$ and $f_r(f_r^n y_0) \to f_r z$ for some $z_r \in X$. This implies that $z_r = f_r z_r$.

If u and v are fixed points of f_r, then using (4), we get

$$d(u, v) = d(f_r u, f_r v) \leq \varphi(u) - \varphi(f_r u) + \varphi(v) - \varphi(f_r v) = 0.$$

Therefore, $u = v$, implying that f_r has a unique fixed point.

Moreover, if u_r and u_s are the fixed points of f_r and f_s respectively, then by (5), we get

$$d(u_r, u_s) = d(f_r u_r, f_s u_s) \leq \varphi(u_r) - \varphi(f_r u_r) + \varphi(u_s) - \varphi(f_s u_s) = 0.$$

Hence, $u_r = u_s$ and each mapping f_r has a unique fixed point, which is the unique common fixed point for the entire family of mappings. □

We now give an example to illustrate the above theorem.

Example 2.4. Let $X = [0, 2]$ and let d be the Euclidean metric. Define $f_r : X \to X$, $0 \leq r \leq 1$, by

$$f_r(x) = \begin{cases} 1 & \text{if } x \leq 1, \\ r[2 - x] & \text{if } 1 < x \leq 2, \end{cases}$$

where $[a]$ denotes the greatest integer less than or equal to the non-negative real number a.

Also, let $\varphi : X \to [0, \infty)$ be defined by

$$\varphi(x) = \begin{cases} 1 - x & \text{if } x \leq 1, \\ 1 + x & \text{if } x > 1. \end{cases}$$

Then the mappings f_r satisfy all the conditions of the above theorem and have a unique common fixed point $x = 1$, which is also the unique fixed point of each mapping. The mapping f_r is discontinuous at the fixed point for $0 \leq r < 1$. Moreover, f_r^2 is continuous for each r, and f_r is 2-continuous for each r; hence, f_r is k-asymptotically continuous.

The following theorems demonstrate that the well-known fixed point theorems for families of mappings by Banach, Kannan [10], Chatterjea [5], and Ćirić [6] are special cases of Theorem 2.3 (see [12]).

THEOREM 2.5. *Suppose $\{f_r : 0 \leq r \leq 1\}$ is a family of self-mappings of a complete metric space (X, d) that satisfies the Kannan contraction condition, i.e.,*

$$d(f_r x, f_r y) \leq \frac{k}{2}[d(x, f_r x) + d(y, f_r y)], \quad 0 < k < 1,$$

for all $x, y \in X$. Then, for each mapping f_r in the family, the conditions of Theorem 2.3 are satisfied and f_r has a unique fixed point.

Proof. For any $x \in X$, we have $d(f_r x, f_r^2 x) \leq \frac{k}{2}\left[d(x, f_r x) + d(f_r x, f_r^2 x)\right]$. This implies $(2-k)d(f_r x, f_r^2 x) \leq kd(x, f_r x)$, that is,

$$\left(\frac{2-k}{k}\right) d(f_r x, f_r^2 x) \leq d(x, f_r x). \tag{6}$$

Now, for any $x, y \in X$, we have

$$\begin{aligned} d(f_r x, f_r y) &\leq \frac{k}{2}[d(x, f_r x) + d(y, f_r y)] \\ &= \frac{k(2-k)}{4(1-k)}[d(x, f_r x) + d(y, f_r y)] - \frac{k^2}{4(1-k)}[d(x, f_r x) + d(y, f_r y)]. \end{aligned}$$

Using (6), the above inequality yields

$$\begin{aligned} d(f_r x, f_r y) &\leq \frac{k(2-k)}{4(1-k)}[d(x, f_r x) + d(y, f_r y)] - \frac{k(2-k)}{4(1-k)}[d(f_r x, f_r^2 x) + d(f_r y, f_r^2 y)] \\ &\leq \frac{k(2-k)}{4(1-k)}[d(x, f_r x) - d(f_r x, f_r^2 x) + d(y, f_r y) - d(f_r y, f_r^2 y)]. \end{aligned}$$

Let us define a function $\varphi : X \to [0, \infty)$ by $\varphi(x) = \frac{k(2-k)}{4(1-k)} d(x, f_r x)$ then last inequality gives $d(f_r x, f_r y) \leq \varphi(x) - \varphi(f_r x) + \varphi(y) - \varphi(f_r y)$.

Since $d(f_r x, f_r^2 x) \leq d(x, f_r x)$, it follows that $\varphi(f_r x) \leq \varphi(x)$. Therefore, f_r satisfies the conditions of Theorem 2.3 and possesses a unique fixed point. □

The following theorem shows that Chatterjea's theorem can be regarded as a specific case of Theorem 2.3.

THEOREM 2.6. *Suppose $\{f_r : 0 \leq r \leq 1\}$ is a family of self-mappings of a complete metric space (X, d) that satisfies the Chatterjea contraction condition, i.e.,*

$$d(f_r x, f_r y) \leq \frac{k}{2}[d(x, f_r y) + d(y, f_r x)], \quad 0 < k < 1, \tag{7}$$

for all $x, y \in X$. Then, for each mapping f_r in the family, the conditions of Theorem 2.3 are satisfied and f_r has a unique fixed point.

Proof. For any $x \in X$, we have

$$d(f_r x, f_r^2 x) \leq \frac{k}{2}\left[d(x, f_r^2 x) + d(f_r x, f_r x)\right] \leq \frac{k}{2}\left[d(x, f_r x) + d(f_r x, f_r^2 x)\right].$$

This implies

$$\left(\frac{2-k}{k}\right) d(f_r x, f_r^2 x) \leq d(x, f_r x). \tag{8}$$

Now, for any $x, y \in X$, we have

$$d(f_r x, f_r y) \leq \frac{k}{2}[d(x, f_r y) + d(y, f_r x)]$$
$$\leq \frac{k}{2}[d(x, f_r x) + d(f_r x, f_r y) + d(y, f_r y) + d(f_r y, f_r x)].$$

That is,

$$d(f_r x, f_r y) \leq \frac{k}{2(1-k)}[d(x, f_r x) + d(y, f_r y)]$$
$$= \frac{k(2-k)}{4(1-k)^2}[d(x, f_r x) + d(y, f_r y)] - \frac{k^2}{4(1-k)^2}[d(x, f_r x) + d(y, f_r y)].$$

Using (8), the above inequality holds:

$$d(f_r x, f_r y) \leq \frac{k(2-k)}{4(1-k)^2}[d(x, f_r x) + d(y, f_r y)] - \frac{k^2(2-k)}{4k(1-k)^2}[d(f_r x, f_r^2 x) + d(f_r y, f_r^2 y)]$$
$$= \frac{k(2-k)}{4(1-k)^2}[d(x, f_r x) - d(f_r x, f_r^2 x) + d(y, f_r y) - d(f_r y, f_r^2 y)].$$

Let us define a function $\varphi : X \to [0, \infty)$ by $\varphi(x) = \frac{k(2-k)}{4(1-k)^2} d(x, f_r x)$ then last inequality gives

$$d(f_r x, f_r y) \leq \varphi(x) - \varphi(f_r x) + \varphi(y) - \varphi(f_r y).$$

Since $d(f_r x, f_r^2 x) \leq d(x, f_r x)$, it follows that $\varphi(f_r x) \leq \varphi(x)$. Therefore, f_r satisfies the conditions of Theorem 2.3 and possesses a unique fixed point. □

The following theorem establishes that Ćirić fixed point theorem is a particular instance of Theorem 2.3.

Theorem 2.7. *Suppose $\{f_r : 0 \leq r \leq 1\}$ is a family of self-mappings of a complete metric space (X, d) that satisfies the Ćirić contraction condition, i.e.,*

$$d(f_r x, f_r y) \leq k \max\left\{d(x, y), d(x, f_r x), d(y, f_r y), \frac{d(x, f_r y) + d(y, f_r x)}{2}\right\}, \; 0 < k < 1, \quad (9)$$

for all $x, y \in X$. Then, for each mapping f_r in the family, the conditions of Theorem 2.3 are satisfied and f_r has a unique fixed point.

Proof. It follows from condition (9) that

$$\frac{1}{k} d(f_r x, f_r^2 x) \leq d(x, f_r x) \quad (10)$$

and

$$d(f_r x, f_r y) \leq k[d(x, f_r x) + d(y, f_r y) + d(f_r x, f_r y)]. \quad (11)$$

Inequality (11) gives

$$d(f_r x, f_r y) \leq \frac{k}{(1-k)}[d(x, f_r x) + d(y, f_r y)]$$
$$= \frac{k}{(1-k)^2}[d(x, f_r x) + d(y, f_r y)] - \frac{k^2}{(1-k)^2}[d(x, f_r x) + d(y, f_r y)].$$

Utilizing (10) in the above inequality, this yields (4) for $\varphi(x) = \frac{k}{(1-k)^2} d(x, f_r x)$. Therefore, f_r satisfies the conditions of Theorem 2.3 and possesses a unique fixed point. It demonstrates that Theorem 2.3 includes Ćirić's theorem as a specific case. □

3. Conclusion

We have established new fixed point results on the existence, uniqueness, and convergence for a one-parameter family of mappings, each with a unique fixed point that is the common fixed point of the family. Our results encompass both contractive-type mappings and non-contractive conditions, and include, as special cases, the well-known fixed-point theorems of Banach, Kannan [10], Chatterjea [5] and Ćirić [6] as well as the results of Gel'man [9] and Pant et al. [12].

Acknowledgement. The authors thank the editor and reviewer for their valuable suggestions, which have greatly improved the quality of this paper.

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(received 11.09.2024; in revised form 19.01.2025; available online 01.07.2025)

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