MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 07.07.2025

research paper
оригинални научни рад
DOI: 10.57016/MV-4ZBUe111

VALIDATING CONVERGENCE BEHAVIOUR OF C-α NON-EXPANSIVE MAPPINGS IN CAT(0) SPACES WITH AN APPLICATION

J. Prabakaran, Mohammad Owais, Ankush Chanda

Abstract. This paper deals with the validation of convergence results for the newly proposed C-α non-expansive mappings in the CAT(0) space setting. Before analyzing the convergence behavior, we emphasize key results related to C-α non-expansive mappings. The convergence results are obtained using the JF-iteration method. We then illustrate these results with non-trivial examples and compare them to other notable iterations. These comparisons are presented in both tabular and graphical forms. Finally, we discuss a variational inequality problem within the context of these mappings using the same iteration scheme.

1. Introduction

Over the past few decades, fixed point theory has evolved in various ways. These include the development of new iterations, non-expansive mappings, and their applications. Researchers have also explored these elements in different underlying spaces. The theoretical appeal and potential applications have motivated researchers to investigate widely. This has led to several generalizations of non-expansive mappings [5, 14, 15, 23]. Additionally, multi-step iteration algorithms [1, 13, 20, 22] have been developed, which are significantly faster than earlier methods. These concepts have been explored in spaces beyond metric and Banach spaces. One such space that has attached interest is a CAT(0) space. A CAT(0) space refers to a metric space, say $(\mathcal{A}, d)$, where geodesic connectivity is ensured and every triangular geodesic structure is as thin as or thinner than its counterpart in the Euclidean plane. It is well established that any Riemannian manifold that is simply connected, complete, and has a non-positive sectional curvature qualifies as a CAT(0) space. More examples include pre-Hilbert spaces, R-trees and so on. One may go through the texts [2, 3] for a detailed exploration of these spaces. The exploration of CAT(0) spaces in fixed

2020 Mathematics Subject Classification: 47H10, 54H25
Keywords and phrases: C-α non-expansive mappings; CAT(0) spaces; JF-iteration; variational inequality problem; fixed points.

point theory was pioneered by Kirk [10] by proving the existence of fixed point pertaining to a non-expansive mapping defined on a set with suitable conditions within the framework of a complete CAT(0) space. For studying convergence behaviour, the notion of Δ-convergence, which was initially proposed by Lim [12] has been restricted to CAT(0) spaces by Kirk and Panyanak [11].

Now, one may question about the primary causes of developments of fixed point theory in such spaces. The response to this is the development of new non-expansive mappings and fixed point iterations. These developments lead to the exploration of these concepts in different spaces, such as CAT(0) spaces. One may go through [6–8, 10, 11, 21] and references therein to review fixed point theory in the framework of CAT(0) spaces. Nevertheless, a significant direction of development in non-expansive mappings is Suzuki generalized non-expansive mapping or Condition (C) [19]. It is a self-mapping $\mathcal{G} : \mathcal{D} \to \mathcal{D}$, defined on a subset $\mathcal{D} \neq \emptyset$ of an arbitrary Banach space $\mathcal{A}$ such that

$$\frac{1}{2} \left\| p - \mathcal{G}p \right\| \leq \left\| p - q \right\| \Rightarrow \left\| \mathcal{G}p - \mathcal{G}q \right\| \leq \left\| p - q \right\|,$$

for all $p, q \in \mathcal{D}$. Additionally, another distinct class of non-expansive mappings known as α-nonexpansive mappings has been proposed [14]. It is defined as a self-mapping $\mathcal{G} : \mathcal{D} \to \mathcal{D}$, on a non-empty subset $\mathcal{D}$ of an arbitrary Banach space $\mathcal{A}$ such that for all $p, q \in \mathcal{D}$ and $\alpha \in [0, 1)$

$$\left\| \mathcal{G}p - \mathcal{G}q \right\|^2 \leq \alpha \left\| p - \mathcal{G}q \right\|^2 + \alpha \left\| q - \mathcal{G}p \right\|^2 + (1 - 2\alpha) \left\| p - q \right\|^2.$$

It is pertinent to note that Suzuki generalized mappings and α-nonexpansive mappings are independent and generally discontinuous. In contrast, non-expansive mappings are uniformly continuous. Building on this, Pant and Shukla [15] proposed a more generalized class of mappings called a generalized α-nonexpansive mapping. It is a self-mapping $\mathcal{G}$ defined on a subset $\mathcal{D} \neq \emptyset$ of an arbitrary Banach space $\mathcal{A}$ such that for all $p, q \in \mathcal{D}$ and $\alpha \in [0, 1)$

$$\begin{aligned} \frac{1}{2} \left\| p - \mathcal{G}p \right\| \leq \left\| p - q \right\| \text{ implies } & \left\| \mathcal{G}p - \mathcal{G}q \right\| \\ & \leq \alpha \left\| p - \mathcal{G}q \right\| + \alpha \| q - \mathcal{G}p \| + (1 - 2\alpha) \left\| p - q \right\|. \end{aligned}$$

Note that this class of mappings does not properly contain that of α-nonexpansive mappings. To tackle this problem, Pant and Shukla [14] proposed another type of non-expansive mappings known as generalized C-α non-expansive mappings. It is a self-mapping $\mathcal{G}$ defined on a subset $\mathcal{D} \neq \emptyset$ of any arbitrary Banach space $\mathcal{A}$ such that for all $p, q \in \mathcal{D}$ and $\alpha \in [0, 1)$

$$\begin{aligned} \frac{1}{2} \left\| p - \mathcal{G}p \right\| \leq \left\| p - q \right\| \text{ implies } & \left\| \mathcal{G}p - \mathcal{G}q \right\|^2 \\ & \leq \alpha \left\| p - \mathcal{G}q \right\|^2 + \alpha \| q - \mathcal{G}p \|^2 + (1 - 2\alpha) \left\| p - q \right\|^2. \end{aligned}$$

This class of mappings properly contains all the previously defined mappings. Now, since fixed point approximation of non-expansive mappings relies on suitable iterations, introducing novel iterations and modifying existing ones is essential. These iterations help approximate solutions to real world problems formulated as fixed point problems. Some problems for which the solutions can be approximated by this approach are delay differential equations [1], split feasibility problems [16], fractional differential equations [24], image processing [4], boundary value problems [24], variational inequality problems [4,8,23] and so on. In light of this, some notable iterations are Mann [13], Thakur-New [20], JF [1], D [9], K^* [22] and many others. For relevance, we only define the JF-iterative scheme. Assume $\emptyset \neq \mathcal{D} \subseteq \mathcal{A}$, where $\mathcal{D}$ represents a closed and convex set and $\mathcal{A}$ be a Banach space. Then, with an initial point $k_1 \in \mathcal{D}$, the sequence (k_n) is defined by

$$g_n = \mathcal{G}((1-c_n)k_n + c_n\mathcal{G}k_n), \quad b_n = \mathcal{G}g_n, \quad k_{n+1} = \mathcal{G}((1-e_n)b_n + e_n\mathcal{G}b_n), \tag{1}$$

where (c_n) and (e_n) are sequences in $(0,1)$.

In this paper, we first establish auxiliary lemmas related to generalized C-α non-expansive mappings. These lemmas highlight their similarity to the well-known condition (E) and their relevance to our work. Next, we obtain Δ and strong convergence results for these mappings using JF-iteration in a CAT(0) space. We demonstrate that JF-iteration is faster than some recently proposed iterations through tabular and graphical representations, supported by proper examples. One example also illustrates the relationship between this class of mappings and generalized α-nonexpansive mappings. Finally, we use this iteration scheme to discuss the solution of a variational inequality problem.

2. Preliminaries

Through the entirety of this article, standard notations have been used. $\mathbb{N}$ and $F(\mathcal{G})$, respectively, represent the set of natural numbers and the fixed point set corresponding to the mapping $\mathcal{G}$. First, we introduce the definition of C-α non-expansive mappings in the setting of CAT(0) spaces. It is important to note that this class of mappings has originally been introduced within the framework of Banach spaces in [14].

Definition 2.1. Consider a CAT(0) space $\mathcal{A}$ with $\mathcal{D}$ as a non-empty subset. A self-mapping $\mathcal{G} : \mathcal{D} \to \mathcal{D}$ is known to be a C-α non-expansive mapping if

$$\begin{aligned} \frac{1}{2}d(p,\mathcal{G}p) \leq d(p,q) \text{ implies } & d(\mathcal{G}p,\mathcal{G}q)^2 \\ & \leq \alpha d(\mathcal{G}p,q)^2 + \alpha d(p,\mathcal{G}q)^2 + (1-2\alpha)d(p,q)^2 \end{aligned} \tag{2}$$

for all $p, q \in \mathcal{D}$ and $\alpha \in [0,1)$.

Definition 2.2 ([21]). Assuming a CAT(0) space $\mathcal{A}$, let (k_n) be a sequence in $\mathcal{A}$ that is bounded. Then for arbitrary $k \in \mathcal{A}$, we define some relevant concepts as follows

(a) asymptotic center of (k_n) at k as $r(k,(k_n)) = \limsup_{n\to\infty} d(k_n, k)$;

(b) asymptotic radius of (k_n) relative to $\mathcal{D}$ as $r(\mathcal{D},(k_n)) = \inf\{r(k,k_n) : k \in \mathcal{D}\}$;

(c) asymptotic center of (k_n) relative to $\mathcal{D}$ as $A(\mathcal{D},(k_n)) = \{k \in \mathcal{D} : r(k,(k_n)) = r(\mathcal{D},(k_n))\}$.

It is important to note that for an arbitrary uniformly convex, closed CAT(0) space, exactly one point is contained by the set $A(\mathcal{D}, (k_n))$.

Definition 2.3 ([21]). Let (k_n) be any sequence in $\mathcal{A}$. Then (k_n) is said to be Δ-convergent to $k \in \mathcal{A}$ if for every subsequence (p_n) of (k_n), k is the asymptotic center of (p_n), which is unique. We denote it as $\Delta - \lim_{n\to\infty} k_n = k$ and read it as the Δ-limit of (k_n) is k.

Observe that for a given $(k_n) \subseteq \mathcal{A}$ such that (k_n) is Δ-convergent to k and a given $p \in \mathcal{A}$ so that $k \neq p$, by the property of asymptotic center being unique, we have

$$\limsup_{n\to\infty} d(k_n, k) < \limsup_{n\to\infty} d(k_n, p).$$

Therefore, the Opial property is satisfied by every CAT(0) space. We now recollect some properties of CAT(0) spaces that are relevant to this paper.

Definition 2.4 ([18]). For a non-empty subset $\mathcal{D}$ of a normed space $\mathcal{A}$, let $\mathcal{G} : \mathcal{D} \to \mathcal{D}$ be a mapping. Then $\mathcal{G}$ satisfies Condition (I) if there exists a non-decreasing function $g : [0, \infty) \to [0, \infty)$ satisfying $g(0) = 0$ and $g(\gamma) > 0$ for all $\gamma \in (0, \infty)$ so that $d(p, \mathcal{G}p) \geq g(d(p, F(\mathcal{G})))$ for all $p \in \mathcal{D}$, where $d(p, F(\mathcal{G})) = \inf\{d(p, k) : k \in F(\mathcal{G})\}$.

Lemma 2.5 ([11]). *For every sequence that is bounded in a complete $CAT(0)$ space, there exists a subsequence which is Δ-convergent.*

Lemma 2.6 ([7]). *Assume (k_n) to be a bounded sequence in $\mathcal{D}$, where $\mathcal{D}$ is a convex and closed subset of a complete $CAT(0)$ space $\mathcal{A}$. Then the asymptotic center of (k_n) is in $\mathcal{D}$.*

Lemma 2.7 ([6]). *For a $CAT(0)$ space $\mathcal{A}$ with $p, q \in \mathcal{A}$ and $\alpha \in [0, 1]$, there exists a unique $r \in [p, q]$ such that $d(p, r) = \alpha d(p, q)$ and $d(q, r) = (1 - \alpha)d(p, q)$.*

The notation used for the unique point r in the above lemma is $(1 - \alpha)p \oplus \alpha q$.

Lemma 2.8 ([21]). *For $p, q, r \in \mathcal{A}$ and $\alpha \in [0, 1]$, we have*

$$d((1 - \alpha)p \oplus \alpha q, r) \leq (1 - \alpha)d(p, r) + \alpha d(q, r).$$

Lemma 2.9 ([17]). *Assume $\mathcal{A}$ to be a complete $CAT(0)$ space and $p \in \mathcal{A}$. Consider (s_n) to be a sequence in $[\alpha, \beta]$, for some $\alpha, \beta \in (0, 1)$. Further, let (p_n) and (q_n) be sequences in $\mathcal{A}$ so that $\limsup_{n\to\infty} d(p_n, p) \leq m$, $\limsup_{n\to\infty} d(q_n, p) \leq m$, and $\lim_{n\to\infty} d((1 - s_n)p_n \oplus s_n q_n, p) = m$ for some $m \geq 0$. Then $\lim_{n\to\infty} d(p_n, q_n) = 0$.*

3. Auxiliary results

We obtain two auxiliary results, which are not addressed in the work of Pant and Shukla [14]. The following result, although used indirectly in the original paper, is provided separately to emphasize its semblance with Condition (E).

LEMMA 3.1. *The following inequality is admitted by every C-α non-expansive mapping*

$$d(p,\mathcal{G}q)^2 \leq \left(\frac{1+\alpha}{1-\alpha}\right) d(p,\mathcal{G}p)^2+d(p,q)^2+\left(\frac{2}{1-\alpha}\right)(\alpha d(p,q)+d(\mathcal{G}p,\mathcal{G}q))d(p,\mathcal{G}p). \quad (3)$$

Proof. Using the triangle inequality and (2), we have

$$\begin{aligned}
d(p,\mathcal{G}q)^2 &\leq (d(p,\mathcal{G}p)+d(\mathcal{G}p,\mathcal{G}q))^2\\
&= d(p,\mathcal{G}p)^2+d(\mathcal{G}p,\mathcal{G}q)^2+2d(p,\mathcal{G}p)d(\mathcal{G}p,\mathcal{G}q)\\
&\leq d(p,\mathcal{G}p)^2+\alpha d(p,\mathcal{G}q)^2+\alpha d(q,\mathcal{G}p)^2+(1-2\alpha)d(p,q)^2\\
&+2d(p,\mathcal{G}p)d(\mathcal{G}p,\mathcal{G}q)\\
(1-\alpha)d(p,\mathcal{G}q)^2 &\leq d(p,\mathcal{G}p)^2+\alpha(d(q,p)+d(p,\mathcal{G}p))^2+(1-2\alpha)d(p,q)^2\\
&+2d(p,\mathcal{G}p)d(\mathcal{G}p,\mathcal{G}q)\\
&\leq (1+\alpha)d(p,\mathcal{G}p)^2+(1-\alpha)d(p,q)^2+2(d(\mathcal{G}p,\mathcal{G}q)+\alpha d(p,q))d(p,\mathcal{G}p)\\
\Rightarrow d(p,\mathcal{G}q)^2 &\leq \left(\frac{1+\alpha}{1-\alpha}\right)d(p,\mathcal{G}p)^2+d(p,q)^2\\
&+\left(\frac{2}{1-\alpha}\right)(\alpha d(p,q)+d(\mathcal{G}p,\mathcal{G}q))d(p,\mathcal{G}p). \qquad \square
\end{aligned}$$

LEMMA 3.2. *Assume $\mathcal{D}$ to be a non-empty subset of a $CAT(0)$ space $\mathcal{A}$ and $\mathcal{G}:\mathcal{D}\to\mathcal{D}$ to be a C-α non-expansive mapping that has a fixed point. Then $\mathcal{G}$ is quasi-nonexpansive.*

Proof. Let $q\in F(\mathcal{G})$ and $p\in\mathcal{D}$. Since $\frac{1}{2}d(q,\mathcal{G}q)=0\leq d(p,q)$, we utilize (2) to have

$$\begin{aligned}
d(\mathcal{G}p,q)^2 = d(\mathcal{G}p,\mathcal{G}q)^2 &\leq \alpha d(\mathcal{G}p,q)^2+\alpha d(p,\mathcal{G}q)^2+(1-2\alpha)d(p,q)^2\\
&= \alpha d(\mathcal{G}p,q)^2+\alpha d(p,q)^2+(1-2\alpha)d(p,q)^2 = \alpha d(\mathcal{G}p,q)^2+(1-\alpha)d(p,q)^2
\end{aligned}$$

$$\Rightarrow (1-\alpha)d(\mathcal{G}p,q)^2\leq(1-\alpha)d(p,q)^2.$$

From this, we have $d(\mathcal{G}p,q)\leq d(p,q)$. Hence $\mathcal{G}$ is quasi-nonexpansive. $\square$

LEMMA 3.3. *Assume $\mathcal{D}$ be a non-empty subset of a $CAT(0)$ space $\mathcal{A}$ and $\mathcal{G}:\mathcal{D}\to\mathcal{D}$ be a C-α non-expansive mapping that has a fixed point. Then $F(\mathcal{G})$ is closed. Also, $F(\mathcal{G})$ is convex, assuming strict convexity of $\mathcal{A}$ together with convexity of $\mathcal{D}$.*

Proof. This proof can be done by following the steps from [14, Lemma 3.9]. $\square$

4. Convergence behaviour of C-α non-expansive mappings

In this section, we procure the convergence results for a C-α non-expansive mapping by utilizing a three-step iterative algorithm namely JF-iteration, in the setting of an arbitrary $CAT(0)$ space. Assume $\mathcal{G}:\mathcal{D}\to\mathcal{D}$ to be a self-mapping satisfying (2), where $\mathcal{D}$ is a non-empty subset of a $CAT(0)$ space $\mathcal{A}$. Then JF-iteration in this setting is defined as

$$k_1\in\mathcal{D},\ g_n=\mathcal{G}((1-c_n)k_n\oplus c_n\mathcal{G}k_n),\ b_n=\mathcal{G}g_n,\ k_{n+1}=\mathcal{G}((1-e_n)b_n\oplus e_n\mathcal{G}b_n) \quad (4)$$

for all $n \in \mathbb{N}$, where (c_n) and (e_n) are control sequences in $(0,1)$. Foremost, we procure the lemma below about the iterative algorithm defined above in context of C-α non-expansive mappings.

Lemma 4.1. *For a non-empty closed and convex subset $\mathcal{D}$ of a complete $CAT(0)$ space $\mathcal{A}$, assume $\mathcal{G} : \mathcal{D} \to \mathcal{D}$ be a C-α non-expansive mapping with $F(\mathcal{G}) \neq \emptyset$. If the sequence (k_n) is defined by (4), then $\lim_{n\to\infty} d(k_n, k)$ exists for all $k \in F(\mathcal{G})$.*

Proof. Since $\mathcal{G}$ satisfies (2), then it is quasi-nonexpansive due to Lemma 3.2. Therefore from (4) and Lemma 2.8, we have

$$\begin{aligned} d(g_n, k) &= d(\mathcal{G}((1-c_n)k_n \oplus c_n\mathcal{G}k_n), k) \leq d((1-c_n)k_n \oplus c_n\mathcal{G}k_n, k) \\ &\leq (1-c_n)d(k_n, k) + c_n d(\mathcal{G}k_n, k) \leq (1-c_n)d(k_n, k) + c_n d(k_n, k) \\ \Rightarrow d(g_n, k) &\leq d(k_n, k). \end{aligned} \tag{5}$$

Similarly, we have

$$d(b_n, k) = d(\mathcal{G}g_n, k) \leq d(g_n, k) \leq d(k_n, k), \tag{6}$$

and

$$\begin{aligned} d(k_{n+1}, k) &= d(\mathcal{G}((1-e_n)b_n \oplus e_n\mathcal{G}b_n), k) \leq d((1-e_n)b_n \oplus e_n\mathcal{G}b_n, k) \\ &\leq (1-e_n)d(b_n, k) + e_n d(\mathcal{G}b_n, k) \leq (1-e_n)d(b_n, k) + e_n d(b_n, k) \\ d(k_{n+1}, k) &\leq d(b_n, k). \end{aligned} \tag{7}$$

From (5), (6) and (7), we have

$$d(k_{n+1}, k) \leq d(b_n, k) \leq d(g_n, k) \leq d(k_n, k).$$

This implies that $(d(k_n, k))$ is a non-increasing sequence which is bounded below for all $k \in F(\mathcal{G})$. Therefore, $\lim\limits_{n\to\infty} d(k_n, k)$ exists. □

Lemma 4.2. *Consider $\mathcal{A}$, $\mathcal{D}$, $\mathcal{G}$ and (k_n) to be the same as in Lemma 4.1. Then $F(\mathcal{G}) \neq \emptyset$ iff the sequence (k_n) is bounded and $\lim\limits_{n\to\infty} d(\mathcal{G}k_n, k_n) = 0$.*

Proof. Consider $F(\mathcal{G}) \neq \emptyset$ and $k \in F(\mathcal{G})$. By Lemma 4.1, $\lim\limits_{n\to\infty} d(k_n, k)$ exists and (k_n) is bounded. Let $\lim\limits_{n\to\infty} d(k_n, k) = m$. From (5) and (6), we have

$$\limsup_{n\to\infty} d(g_n, k) \leq \limsup_{n\to\infty} d(k_n, k) \leq m, \tag{8}$$

and similarly,

$$\limsup_{n\to\infty} d(b_n, k) \leq \limsup_{n\to\infty} d(k_n, k) \leq m. \tag{9}$$

From (4) we have,

$$\begin{aligned} d(k_{n+1}, k) &= d(\mathcal{G}((1-e_n)b_n \oplus e_n\mathcal{G}b_n), k) \leq d((1-e_n)b_n \oplus e_n\mathcal{G}b_n, k) \\ &\leq (1-e_n)d(b_n, k) + e_n d(\mathcal{G}b_n, k) \leq (1-e_n)d(b_n, k) + e_n d(b_n, k) \\ &\leq (1-e_n)d(k_n, k) + e_n d(g_n, k) \\ \Rightarrow d(k_{n+1}, k) &\leq d(k_n, k) - e_n d(k_n, k) + e_n d(g_n, k) \end{aligned}$$

which implies,

$$d(k_{n+1},k)-d(k_n,k)\leq e_n\left(d(g_n,k)-d(k_n,k)\right)$$
$$\frac{d(k_{n+1},k)-d(k_n,k)}{e_n}\leq d(g_n,k)-d(k_n,k).$$

Since $e_n\in(0,1)$, we have

$$\begin{aligned}d(k_{n+1},k)-d(k_n,k)&\leq\frac{d(k_{n+1},k)-d(k_n,k)}{e_n}\leq d(g_n,k)-d(k_n,k)\\&\Rightarrow d(k_{n+1},k)\leq d(g_n,k).\end{aligned}\tag{10}$$

By applying lim inf on both sides, we have $m\leq\liminf_{n\to\infty}d(g_n,k)$. By using (8) and (10), we have

$$m=\lim_{n\to\infty}d(g_n,k)=\lim_{n\to\infty}d(\mathcal{G}((1-c_n)k_n\oplus c_n\mathcal{G}k_n),k).$$

Also, due to quasi-nonexpansiveness of $\mathcal{G}$ we have $d(\mathcal{G}k_n,k)\leq d(k_n,k)$. By taking lim sup on both sides, we have

$$\limsup_{n\to\infty}d(\mathcal{G}k_n,k)\leq\limsup_{n\to\infty}d(k_n,k)\leq m.$$

Thus all the requirements of Lemma 2.9 are fulfilled and we have $\lim_{n\to\infty}d(\mathcal{G}k_n,k_n)=0$.

Conversely, let (k_n) be bounded and $\lim_{n\to\infty}d(\mathcal{G}k_n,k_n)=0$ and let $k\in A(\mathcal{D},(k_n))$. Then by utilizing Lemma 3.1, we have

$$\begin{aligned}r(k_n,\mathcal{G}k)^2&=\limsup_{n\to\infty}d(k_n,\mathcal{G}k)^2\\&\leq\limsup_{n\to\infty}\left(\frac{1+\alpha}{1-\alpha}\right)d(k_n,\mathcal{G}k_n)^2+\limsup_{n\to\infty}d(k_n,k)^2\\&+\limsup_{n\to\infty}\left(\frac{2}{1-\alpha}\right)(\alpha d(k_n,k)+d(\mathcal{G}k_n,\mathcal{G}k))d(k_n,\mathcal{G}k_n)\\&=\limsup_{n\to\infty}d(k_n,k)^2=r((k_n),k)^2.\end{aligned}$$

Thus we have $\mathcal{G}k\in A(\mathcal{D},(k_n))$. Since $\mathcal{D}$ is a closed and convex subset of the $CAT(0)$ space $\mathcal{A}$, $A(\mathcal{D},(k_n))$ is singleton set. Hence we have $\mathcal{G}k=k$, that is, $F(\mathcal{G})\neq\emptyset$. □

THEOREM 4.3. *Consider $\mathcal{A}$, $\mathcal{D}$, $\mathcal{G}$ and (k_n) to be the same as in Lemma 4.1 with $F(\mathcal{G})\neq\emptyset$. Then the sequence (k_n) is Δ-convergent to a point of $F(\mathcal{G})$.*

Proof. Due to Lemma 4.2, the sequence (k_n) is bounded and $\lim_{n\to\infty}d(k_n,\mathcal{G}k_n)=0$.

Let $\mathcal{K}_j((k_n)):=\cup A(\mathcal{D},(a_n))$, where we take union over all subsequences (a_n) of (k_n). To claim that (k_n) is Δ-convergent to a point of $F(\mathcal{G})$, we show that $\mathcal{K}_j((k_n))\subseteq F(\mathcal{G})$ and $\mathcal{K}_j((k_n))$ is a singleton set. Now, we prove that $\mathcal{K}_j((k_n))\subseteq F(\mathcal{G})$. Let $t\in\mathcal{K}_j((k_n))$. Then there is a subsequence (t_n) of (k_n) such that $A(\mathcal{D},(t_n))=t$. By Lemma 2.5, there exists a subsequence (s_n) of (t_n) such that $\Delta-\lim_{n\to\infty}s_n=s$ and $s\in\mathcal{D}$. As $\lim_{n\to\infty}d(s_n,\mathcal{G}s_n)=0$ and $\mathcal{G}$ satisfies (3), we have

$$d(s_n,\mathcal{G}s)^2\leq\left(\frac{1+\alpha}{1-\alpha}\right)d(s_n,\mathcal{G}s_n)^2+d(s_n,s)^2$$

$$+\left(\frac{2}{1-\alpha}\right)(\alpha d(s_n,s)+d(\mathcal{G}s_n,\mathcal{G}s))d(s_n,\mathcal{G}s_n).$$

By taking lim sup on both sides, we get

$$\limsup_{n\to\infty} d(s_n,\mathcal{G}s)^2 \leq \limsup_{n\to\infty}\left(\frac{1+\alpha}{1-\alpha}\right)d(s_n,\mathcal{G}s_n)^2+\limsup_{n\to\infty} d(s_n,s)^2$$
$$+\limsup_{n\to\infty}\left(\frac{2}{1-\alpha}\right)(\alpha d(s_n,s)+d(\mathcal{G}s_n,\mathcal{G}s))d(s_n,\mathcal{G}s_n).$$

This implies that $\limsup\limits_{n\to\infty} d(s_n,\mathcal{G}s)^2 \leq \limsup\limits_{n\to\infty} d(s_n,s)^2$ which leads to

$$\limsup_{n\to\infty} d(s_n,\mathcal{G}s) \leq \limsup_{n\to\infty} d(s_n,s). \tag{11}$$

As $\Delta - \lim\limits_{n\to\infty} s_n = s$, using Opial property, we have

$$\limsup_{n\to\infty} d(s_n,s) \leq \limsup_{n\to\infty} d(s_n,\mathcal{G}s). \tag{12}$$

From (11) and (12), we have $\mathcal{G}s = s$, that is, $s \in F(\mathcal{G})$. Now, we claim that $s = t$. Assuming not, say $t > s$, then by Lemma 4.1, $\lim\limits_{n\to\infty} d(k_n,s)$ exists and from the uniqueness of asymptotic centers

$$\limsup_{n\to\infty} d(s_n,s) < \limsup_{n\to\infty} d(s_n,t) \leq \limsup_{n\to\infty} d(t_n,t) < \limsup_{n\to\infty} d(t_n,s)$$
$$= \limsup_{n\to\infty} d(k_n,s) = \limsup_{n\to\infty} d(s_n,s).$$

This is a clear contradiction. Hence $s = t$.

We claim that $\mathcal{K}_j((k_n))$ is a singleton. To show this, let (t_n) be a subsequence of (k_n). Due to Lemma 2.5 and Lemma 2.6, a subsequence (s_n) of (t_n) exists such that $\Delta - \lim_{n\to\infty} s_n = s$. Let $A(\mathcal{D},(t_n)) = t = s$ and $A(\mathcal{D},(k_n)) = k$. We now claim that $s = k$. On contrary, assume $s \neq k$, then in view of Lemma 4.1, $d(k_n,s)$ converges. Due to uniqueness of asymptotic centers, we have

$$\limsup_{n\to\infty} d(s_n,s) < \limsup_{n\to\infty} d(s_n,k) \leq \limsup_{n\to\infty} d(k_n,k)$$
$$< \limsup_{n\to\infty} d(k_n,s) = \limsup_{n\to\infty} d(s_n,s),$$

which is a contradiction. Hence our claim follows. □

THEOREM 4.4. *Consider $\mathcal{A}$, $\mathcal{D}$, $\mathcal{G}$ and (k_n) to be the same as in Lemma 4.1 with $F(\mathcal{G}) \neq \emptyset$ such that $\mathcal{D}$ is also a compact subset of $\mathcal{A}$. Then (k_n) strongly converges to a fixed point of $\mathcal{G}$.*

Proof. Given that $F(\mathcal{G}) \neq \emptyset$, we have $\lim_{n\to\infty} d(k_n,\mathcal{G}k_n) = 0$. Also due to compactness of $\mathcal{D}$, a subsequence (k_{n_i}) of (k_n) exists so that $k_{n_i} \to k$ for a $k \in \mathcal{D}$. Thence, by Lemma 3.1, we obtain

$$d(k_{n_i},\mathcal{G}k)^2 \leq \left(\frac{1+\alpha}{1-\alpha}\right)d(k_{n_i},\mathcal{G}k_{n_i})^2+d(k_{n_i},k)^2$$
$$+\left(\frac{2}{1-\alpha}\right)(\alpha d(k_{n_i},k)+d(\mathcal{G}k_{n_i},\mathcal{G}k))d(k_{n_i},\mathcal{G}k_{n_i}).$$

Now for $i \to \infty$, we get $\mathcal{G}k = k$, that is, $k \in F(\mathcal{G})$. Using Lemma 4.1, $\lim_{n\to\infty} d(k_n, k)$ exists for every $k \in F(\mathcal{G})$ and so (k_n) is strongly convergent to a fixed point of $\mathcal{G}$. □

THEOREM 4.5. *Consider $\mathcal{A}$, $\mathcal{D}$, $\mathcal{G}$ and (k_n) to be the same as in Lemma 4.1 with $F(\mathcal{G}) \neq \emptyset$. If $\mathcal{G}$ satisfies the Condition (I) stated as Definition 2.4, then the sequence (k_n) strongly converges to a fixed point of $\mathcal{G}$.*

Proof. From the hypothesis and Condition (I), we have $g(d(k_n, F(\mathcal{G}))) \leq d(k_n, \mathcal{G}k_n)$. As $F(\mathcal{G}) \neq \emptyset$, then in light of Lemma 4.2, we get

$$\lim_{n\to\infty} d(k_n, \mathcal{G}k_n) = 0 \ \Rightarrow \lim_{n\to\infty} g(d(k_n, F(\mathcal{G}))) = 0.$$

We have $\lim\limits_{n\to\infty} d(k_n, F(\mathcal{G})) = 0$, since g is non-decreasing. Then, we have (k_{n_i}), a subsequence of (k_n) and a sequence $(t_n) \subseteq F(\mathcal{G})$ so that, for all $i \in \mathbb{N}$

$$d(k_{n_i}, t_i) < \frac{1}{2^i} \tag{13}$$

and

$$d(k_{n_{i+1}}, t_i) \leq d(k_{n_i}, t_i) < \frac{1}{2^i}.$$

Therefore

$$d(t_{i+1}, t_i) \leq d(t_{i+1}, k_{n_{i+1}}) + d(k_{n_{i+1}}, t_i) \leq \frac{1}{2^{i-1}} \to 0 \text{ as } i \to \infty.$$

This asserts that (t_n) is Cauchy in $F(\mathcal{G})$ and as $F(\mathcal{G})$ is closed due to Lemma 3.3, it converges to some $k \in F(\mathcal{G})$. Again from (13), we derive (k_{ni}) is also convergent to k. Finally as $\lim_{n\to\infty} d(k_n, k)$ exists, we have $k_n \to k$. □

5. Numerical examples

This section exemplifies C-α non-expansive mappings solely, as well as the relationship between C-α and generalized α-nonexpansive mappings. Furthermore, we use these examples to procure the comparative convergence of JF-iteration with other notable iterations which has been demonstrated through tabular and graphical representations.

EXAMPLE 5.1. Consider a function $\mathcal{G} : [0, 2] \to [0, 2]$ defined as

$$\mathcal{G}(s) = \begin{cases} \sin(\frac{s}{3}), & \text{if } s \neq 2, \\ \frac{3}{2}, & \text{if } s = 2. \end{cases}$$

We show that $\mathcal{G}$ is a C-α non-expansive mapping but not a generalized α-nonexpansive mapping for $\alpha \geq \frac{1}{3}$.

Case I: *If $s, t \neq 2$, then*

$$d(\mathcal{G}s, \mathcal{G}t)^2 = d\left(\sin\left(\frac{s}{3}\right), \sin\left(\frac{t}{3}\right)\right)^2 \leq \frac{1}{9} d(s,t)^2 \leq \frac{1}{3} d(s,t)^2 = \left(1 - 2 \times \frac{1}{3}\right) d(s,t)^2$$
$$\leq \frac{1}{3} d(s, \mathcal{G}t)^2 + \frac{1}{3} d(t, \mathcal{G}s)^2 + \left(1 - 2 \times \frac{1}{3}\right) d(s,t)^2.$$

Case II: *If* $s = 2$ *and* $t \neq 2$, *then*

$$d(\mathcal{G}s, \mathcal{G}t)^2 = d\left(\frac{3}{2}, \sin\left(\frac{t}{3}\right)\right)^2 \tag{14}$$

and $\alpha d(s, \mathcal{G}t)^2 + \alpha d(t, \mathcal{G}s)^2 + (1-2\alpha) d(s,t)^2$

$$= \frac{1}{3} d\left(2, \sin\left(\frac{t}{3}\right)\right)^2 + \frac{1}{3} d\left(t, \frac{3}{2}\right)^2 + \frac{1}{3} d\left(\frac{3}{2}, \sin\left(\frac{t}{3}\right)\right)^2. \tag{15}$$

Simplifying (14) and (15), we have

$$d\left(\frac{3}{2}, \sin\left(\frac{t}{3}\right)\right) = \left(\frac{3}{2} - \sin\left(\frac{t}{3}\right)\right)^2 = \frac{9}{4} - 3\sin\left(\frac{t}{3}\right) + \sin^2\left(\frac{t}{3}\right) \tag{16}$$

and

$$\frac{1}{3}\left(\left|2 - \sin\left(\frac{t}{3}\right)\right|^2 + \left|t - \frac{3}{2}\right|^2 + \left|\frac{3}{2} - \sin\left(\frac{t}{3}\right)\right|^2\right)$$

$$= \frac{1}{3}\left(4 - 4\sin\left(\frac{t}{3}\right) + \sin^2\left(\frac{t}{3}\right)\right) + \frac{1}{3}\left(t - \frac{3}{2}\right)^2 + \frac{1}{3}\left(\frac{9}{4} - 3\sin\left(\frac{t}{3}\right) + \sin^2\left(\frac{t}{3}\right)\right)$$

$$= \frac{25}{12} - \frac{7}{3}\sin\left(\frac{t}{3}\right) + \frac{2}{3}\sin^2\left(\frac{t}{3}\right) + \frac{1}{3}\left(t - \frac{3}{2}\right)^2. \tag{17}$$

Now, we compare (16) and (17) by method of contradiction. We claim that (16) is less than or equal to (17). Assume on contrary that for a particular t, (16) is greater than (17), that is,

$$\frac{9}{4} - 3\sin\left(\frac{t}{3}\right) + \sin^2\left(\frac{t}{3}\right) > \frac{25}{12} - \frac{7}{3}\sin\left(\frac{t}{3}\right) + \frac{2}{3}\sin^2\left(\frac{t}{3}\right) + \frac{1}{3}\left(t - \frac{3}{2}\right)^2$$

$$\left(\frac{9}{4} - \frac{25}{12}\right) + \left(-3 + \frac{7}{3}\right)\sin\left(\frac{t}{3}\right) + \left(1 - \frac{2}{3}\right)\sin^2\left(\frac{t}{3}\right) > \frac{1}{3}\left(t - \frac{3}{2}\right)^2$$

$$\frac{1}{6} - \frac{2}{3}\sin\left(\frac{t}{3}\right) + \frac{1}{3}\sin^2\left(\frac{t}{3}\right) > \frac{1}{3}\left(t - \frac{3}{2}\right)^2$$

$$\Rightarrow \sin^2\left(\frac{t}{3}\right) - 2\sin\left(\frac{t}{3}\right) + \frac{1}{2} > \left(t - \frac{3}{2}\right)^2. \tag{18}$$

It can be easily checked that (18) does not hold for any $t \in \mathbb{R}$, which is a contradiction. Thus we have that the expression (14) is less than or equal to that of (15). Therefore the mapping $\mathcal{G}$ is a C-α non-expansive mapping possessing a fixed point 0, which is unique. Now we provide the following case to show its relationship with generalized α-nonexpansive mappings.

If we take $s = 2$ and $t = 1$, then

$$\frac{1}{2} d(s, \mathcal{G}s) = \frac{1}{2} d\left(2, \frac{3}{2}\right) = \frac{1}{4} \leq d(s,t) = d(2,1) = 1.$$

and $$d(\mathcal{G}s, \mathcal{G}t) = d\left(\frac{3}{2}, \sin\left(\frac{1}{3}\right)\right) = |1.5, 0.3271| = 1.1720,$$

$$\frac{1}{3}d(s,\mathcal{G}t) = \frac{1}{3}|2-0.3271| = 0.5576, \quad \frac{1}{3}d(t,\mathcal{G}s) = \frac{1}{3}\,|1.5-1| = 0.1667,$$
$$\Rightarrow d(\mathcal{G}s,\mathcal{G}t) = 1.7720 > 1.0576 = \frac{1}{3}d(s,\mathcal{G}t) + \frac{1}{3}(t,\mathcal{G}s) + \frac{1}{3}d(s,t).$$

This implies that, $\mathcal{G}$ does not qualify as a generalized α-nonexpansive mapping. We now use this $\mathcal{G}$ as the mapping in iteration (4) to showcase the comparative convergence behaviour of JF-iteration with respect to other notable iterations. We take the initial guess as $k_1 = 1.9$ and the two control sequences as $c_n = 0.65$ and $e_n = 0.25$.

n	JF-iteration	D-iteration	K^*-iteration	Picard	Thakur
1	0.0487919093	0.0585032955	0.0954878653	0.5918349050	0.1768256475
2	0.0013377018	0.0019319246	0.0050096056	0.1960011527	0.0175092633
3	0.0000366767	0.0000638012	0.0002628496	0.0652872480	0.0017347048
4	0.0000010056	0.0000021070	0.0000137915	0.0217606983	0.0001718643
5	0.0000000276	0.0000000696	0.0000007236	0.0072535025	0.0000170273
6	0.0000000008	0.0000000023	0.0000000380	0.0024178318	0.0000016870
7	0.0000000000	0.0000000001	0.0000000020	0.0008059438	0.0000001671

Table 1: Comparison of JF-iteration with some notable iterations wrt. Example 5.1

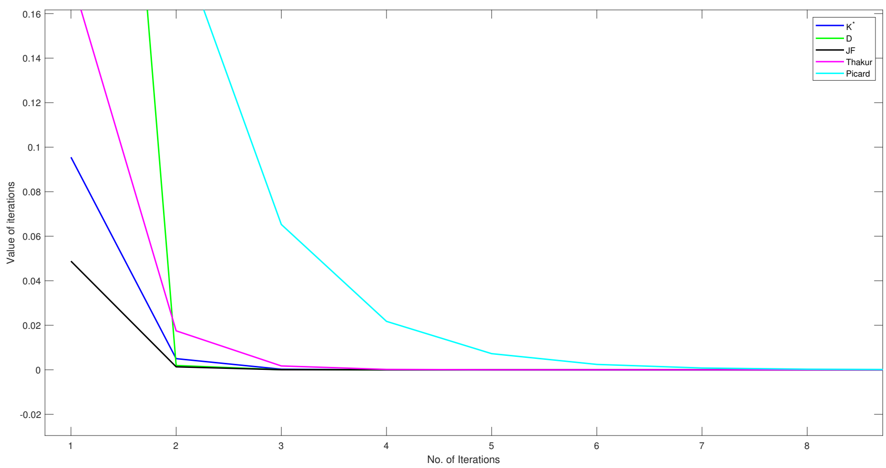

Figure 1: Graphical representation of Table 1

Example 5.2. Consider a function $\mathcal{G} : [0,n] \to [0,n]$ for $n \in \mathbb{N}$, defined as
$$\mathcal{G}(s) = \frac{s}{7} + \frac{1}{7}\log(s+1).$$
We show that $\mathcal{G}$ is a C-α non-expansive mapping. We first have
$$d(\mathcal{G}s,\mathcal{G}t)^2 = d\left(\frac{s}{7} + \frac{1}{7}\log(s+1), \frac{t}{7} + \frac{1}{7}\log(t+1)\right)^2$$

$$= \frac{1}{7}|s - t + \log(s+1) - \log(t+1)|^2. \tag{19}$$

We assert that $|\log(s+1) - \log(t+1)| \leq |s-t|$. Since $\log(s+1)$ is continuous and differentiable in the interval $[0,n], n \in \mathbb{N}$, then by mean value theorem there exists a point, say $c \in (0,n)$ such that $|\log(s+1) - \log(t+1)| \leq \left(\frac{1}{s+1}\right)_c |s-t|$. As $\left(\frac{1}{s+1}\right)_c \leq 1$ for all $s \in [0,n]$, we have $|\log(s+1) - \log(t+1)| \leq |s-t|$. Hence, from (19), we have

$$d(\mathcal{G}s, \mathcal{G}t)^2 \leq \frac{1}{7}(|s-t| + |\log(s+1) - \log(t+1)|)^2 \leq \frac{1}{7}(|s-t| + |s-t|)^2 = \frac{4}{7}|s-t|^2$$
$$\leq \frac{3}{14}d(s,\mathcal{G}t)^2 + \frac{3}{14}d(t,\mathcal{G}s)^2 + \left(1 - 2 \times \frac{3}{14}\right) d(s,t)^2.$$

Therefore, $\mathcal{G}$ is a C-α non-expansive mapping with 0 as its unique fixed point. We now use this $\mathcal{G}$ as the mapping in iteration (4) to draw a comparison of the behaviour of convergence of JF-iteration with other notable iterations. We take the same initial guess as previously, that is, $k_1 = 1.9$ and the two control sequences as $c_n = 0.65$ and $e_n = 0.25$.

n	JF-iteration	D-iteration	K^*-iteration	Picard	Thakur
1	0.0203604686	0.0276397808	0.0519631026	0.4235301053	0.0995380037
2	0.0003105731	0.0005647786	0.0018440403	0.1109528402	0.0069821825
3	0.0000047689	0.0000116416	0.0000662133	0.0308815574	0.0005027752
4	0.0000000732	0.0000002400	0.0000023785	0.0087565535	0.0000362736
5	0.0000000011	0.0000000049	0.0000000854	0.0024964273	0.0000026174
6	0.0000000000	0.0000000001	0.0000000031	0.0007128205	0.0000001889
7	0.0000000000	0.0000000000	0.0000000001	0.0002036267	0.0000000136

Table 2: Comparison of JF-iteration with some notable iterations wrt. Example 5.2

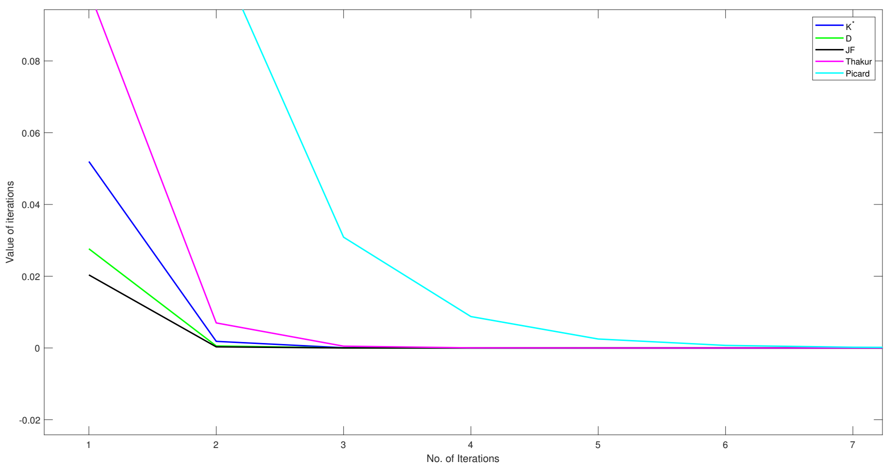

Figure 2: Graphical representation of Table 2

6. Application to a variational inequality problem

In this section, an application to variational inequality problem is discussed. Assume a Hilbert space $\mathcal{A}$ with $\mathcal{D}$ as its non-empty, closed, and convex subset. Then, the mapping $\mathcal{G} : \mathcal{A} \to \mathcal{A}$ is monotone iff $\langle \mathcal{G}s - \mathcal{G}t, s - t \rangle \geq 0$, for all $s, t \in \mathcal{A}$. Now, we define a variational inequality problem $V(\mathcal{G}, \mathcal{D})$ characterized by $\mathcal{G}$ and $\mathcal{D}$ as to find $t^* \in \mathcal{D}$ such that for each $t \in \mathcal{A}$ $\langle \mathcal{G}t^*, t - t^* \rangle \geq 0$.

Let $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ be the identity self-mapping and $\mathcal{P}_{\mathcal{D}}$ denote the closest point projection onto $\mathcal{D}$. Then for $\sigma > 0$, t^* is the solution of $V(\mathcal{G}, \mathcal{D})$ iff $\mathcal{P}_{\mathcal{D}}(1 - \sigma\mathcal{G})(t^*) = t^*$ (see [4]). Further, let the solution set of $V(\mathcal{G}, \mathcal{D})$ be denoted by $\mathcal{S}_{V(\mathcal{G},\mathcal{D})}$. By [4, Theorem 2.3], if $\mathcal{S}_{V(\mathcal{G},\mathcal{D})} \neq \emptyset$ together with $1 - \sigma\mathcal{G}$ and $\mathcal{P}_{\mathcal{D}}(1 - \sigma\mathcal{G})$ as averaged non-expansive mapping, the sequence (k_n) obtained by the iterative method $k_{n+1} = \mathcal{P}_{\mathcal{D}}(1 - \sigma\mathcal{G})k_n$ weakly converges to a point in $\mathcal{S}_{V(\mathcal{G},\mathcal{D})}$.

We now proceed to approximate solution to $V(\mathcal{G}, \mathcal{D})$ in the context of generalized C-α non-expansive mappings. It is important to note here that these mappings are not necessarily continuous. This inference can be directly drawn from the Example 5.1 and relationships provided in [14]. We utilize iteration (4), which has already been shown to be relatively faster previously. Also, since the Opial condition is satisfied by every Hilbert space and every Hilbert space qualifies as a complete CAT(0) space, we state the upcoming convergence results.

Theorem 6.1. *Assume $\mathcal{S}_{V(\mathcal{G},\mathcal{D})} \neq \emptyset$ and $\mathcal{L} := \mathcal{P}_{\mathcal{D}}(1 - \sigma\mathcal{G})$ with $\sigma > 0$ to be a generalized C-α non-expansive mapping. Let (k_n) be a sequence generated by iteration (4). Then (k_n) is Δ-convergent to a point, say $t^* \in \mathcal{S}_{V(\mathcal{G},\mathcal{D})}$.*

Proof. Since the prerequisites of Theorem 4.3 are fulfilled, the conclusion follows from it. □

Theorem 6.2. *Consider the same set of assumptions as in Theorem 6.1. Further let $\mathcal{D}$ be compact. Then (k_n) stongly converges to a point, say $t^* \in \mathcal{S}_{V(\mathcal{G},\mathcal{D})}$.*

Proof. Since the prerequisites of Theorem 4.4 are fulfilled, the conclusion follows from it. □

7. Conclusion

In this article, foremost, Condition (E) analog has been highlighted as a potential generalization of Condition (E) for further investigation in the future. Afterwards, strong and Δ-convergence results have been obtained in the context of the newly proposed C-α non-expansive mappings by using JF-iteration scheme in the setting of CAT(0) spaces. Furthermore, JF-iteration scheme has been shown to be faster than many notable iterations by tabular and graphical representations. This comparison has emphasized the speed of convergence and thus the importance of JF-iteration in the context of the aforementioned kind of mappings. Finally, we have approximated solution to a variational inequality problem by utilizing the procured convergence results. By doing so, we have demonstrated that the CAT(0) spatial approach to approximating solutions to a variational inequality problem is just as feasible as a Banach spatial approach. The future directions from this work would include investigation of other applications within the framework of these concepts such as split feasibility in [16]. A multivalued analog of C-α non-expansive mappings could be defined and a variational inequality problem may be investigated within CAT(0) space setting as in [8]. Also, the exploration of other applications such as image recovery and integral equations are possible. Additionally, these problems can be investigated with a change in the setting, that is, in CAT(k) or hyperbolic spaces.

References

[1] J. Ali, F. Ali, J.J. Nieto, *Some observations on generalized non-expansive mappings with an application*, Comput. Appl. Math., **39(2)** (2020), 74.

[2] M. Bridson, A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer, Berlin, 1999.

[3] D. Burago, Y. Burago, S. Ivanov, *In: A Course in Metric Geometry. Graduate Studies in Math.*, Am. Math. Soc., Providence, 2001.

[4] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Probl., **20(1)** (2003), 103–120.

[5] C. Ciobanescu, *Remarks on some operators of nonexpansive type*, J. Math. Anal., **13(4)** (2022), 42–52.

[6] S. Dhompongsa, B. Panyanak, *On Δ-convergence theorems in CAT(0) spaces*, Comput. Math. Appl., **56(10)** (2008), 2572–2579.

[7] S. Dhompongsa, W. A. Kirk, B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear Convex Anal., **8(1)** (2007), 35–45.

[8] F. Gürsoy, E. Haciöglu, V. Karakaya, G.V. Milovanović, I. Uddin, *Variational inequality problem involving multivalued nonexpansive mapping in CAT(0) spaces*, Results Math., **77** (2022), 131.

[9] A. Hussain, D. Ali, E. Karapınar, *Stability data dependency and errors estimation for a general iteration method*, Alex. Eng. J., **60(1)** (2021), 703–710.

[10] W. A. Kirk, *Geodesic geometry and fixed point theory*, Seminar of Mathematical Analysis (Malaga/Seville), **64** (2003), 195–225.

[11] W. A. Kirk, B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal., **68(12)** (2008), 3689-3696.

[12] T. C. Lim, *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc., **60(1)** (1976), 179–182.

[13] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506–510.

[14] R. Pant, R. Shukla, *Fixed point theorems for a new class of nonexpansive mappings*, Appl. Gen. Topol., **23(2)** (2022), 377–390.

[15] R. Pant, R. Shukla, *Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces*, Numer. Funct. Anal. Optim., **38(2)** (2017), 248–266.

[16] L. U. Rahman, M. Arshad, H. A. Hammad, M. De la Sen, *Fixed point iterations for functional equations and split feasibility problems in CAT(0) spaces*, Contemp. Math., **5(1)** (2024), 365-79.

[17] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Aust. Math. Soc., **43(1)** (1991), 153–159.

[18] H. F. Senter, W.G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Am. Math. Soc., **44(2)** (1974), 375–380.

[19] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl., **340(2)** (2008), 1088–1095.

[20] B. S. Thakur, D. Thakur, M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings*, Appl. Math. Comput., **275** (2016), 147–155.

[21] I. Uddin, S. Khatoon, V. Colao, *Approximating fixed points for generalized α-Reich-Suzuki nonexpansive mapping in $CAT(0)$ spaces*, J. Nonlinear Convex Anal., **21(9)** (2020), 2139–2150.

[22] K. Ullah, M. Arshad, *New three-step iteration process and fixed point approximation in Banach spaces*, J. Linear Topol. Algeb., **7(2)** (2018), 87–100.

[23] K. Ullah, J. Ahmad, H. A. Hammad, R. George, *Iterative schemes for numerical reckoning of fixed points of new nonexpansive mappings with an application*, AIMS Math., **8(5)** (2023), 10711–10727.

[24] S. Zaheer, A. Chanda, H. K. Nashine, *Reckoning applications of $\mathcal{Z}$ -iteration: Data dependence and solution to a delay Caputo fractional differential equations*, Nonlinear Anal. Model. Control, **29(5)** (2024), 833–857.

(received 14.07.2024; in revised form 07.09.2024; available online 07.07.2025)

Department of Mathematics, Vellore Institute of Technology, Vellore, India
E-mail: prabakaran.j2022@vitstudent.ac.in
ORCID iD: https://orcid.org/0009-0004-8144-2421

Department of Mathematics, Vellore Institute of Technology, Vellore, India
E-mail: eowais8@gmail.com
ORCID iD: https://orcid.org/0009-0007-5046-6158

Department of Mathematics, Vellore Institute of Technology, Vellore, India
E-mail: ankushchanda8@gmail.com
ORCID iD: https://orcid.org/0000-0001-6826-2048