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ALMOST n-MULTIPLIERS ON BANACH ALGEBRAS

Abbas Zivari-Kazempour

Abstract. In this paper, we introduce the notion of *almost n-multiplier* on Banach algebras. This new notion generalizes the concept of n-multiplier introduced and studied in [J. Laali, M. Fozouni, *n-multipliers and their relations with n-homomorphisms*, Vietnam J. Math., **45** (2017), 451-457]. We gave some general results and the continuity of such maps with some examples for this new notion on Banach algebras. In particular, we generalize the celebrated theorem of Johnson to (left) n-multipliers on Banach algebras.

1. Introduction and preliminaries

The concept of a multiplier first appears in harmonic analysis. This notion has been employed in several important areas in harmonic analysis and Banach algebras. For instance, in the investigation of homomorphisms of group algebras, the study of Banach modules, general theory of Banach algebras, and so on, see [8].

Let A be a Banach algebra. A mapping $T : A \longrightarrow A$ is called a *left multiplier* $\big($*right multiplier*$\big)$ if for all $a, b \in A$,

$$T(ab) = T(a)b, \quad \big(T(ab) = aT(b)\big),$$

and T is called a *multiplier*, if $aT(b) = T(a)b$, for every $a, b \in A$.

We note that a number of authors use the term "centralizer" instead of "multiplier". This is true, for example, of Johnson [5,6], and Wendel [9]. However multiplier seems to be the older and more common term and for this reason we prefer it to that of centralizer.

The general theory of multipliers (centralizers) on Banach algebras has been developed by Johnson [5]. One may refer to the monograph [8] for the theory of multipliers.

Definition 1.1. Let A be a Banach algebra and $T : A \longrightarrow A$ be a map. Then T is called a *right n-multiplier* $\big($*left n-multiplier*$\big)$ if

$$T(a_1 a_2 \cdots a_n) = a_1 T(a_2 \cdots a_n), \quad \big(T(a_1 a_2 \cdots a_n) = T(a_1 \cdots a_{n-1}) a_n\big),$$

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for all $a_1, a_2, \cdots, a_n \in A$, and it is called *$n$-multiplier* if $a_1 T(a_2 \cdots a_n) = T(a_1 \cdots a_{n-1}) a_n$, for every $a_1, a_2, \cdots, a_n \in A$.

The notion of n-multiplier was introduced and studied by Laali and Fozouni [7], and some interesting results related to these maps were obtained by these authors. Following [7], let $Mul_n(A)$ be the set of all n-multipliers of Banach algebra A. For the case $n = 2$, this concept coincide with the classical definition of multipliers. Note that in [7], n-multipliers is assumed to be linear and bounded, but we consider no assumptions of linearity or continuity. The approximate θ-multipliers on Banach algebras can be found in [11], and the pseudo version of n-multipliers was introduced in [12]. For results concerning multipliers on algebras, we refer the reader to [4, 10–12] and the references therein.

In this paper, we introduce the notion of *almost n-multiplier* on Banach algebras. We gave some general results and investigate the continuity of such maps with some examples. In particular, we generalize the celebrated theorem of Johnson to (left) n-multipliers on Banach algebras.

2. Continuity of almost n-multipliers

We first introduce the concept of *almost n-multiplier* on Banach algebras with some examples to illustrate this new notion.

Definition 2.1. Let A be a Banach algebra. A map $T : A \longrightarrow A$ is called *almost right n-multiplier*, if there exists $\varepsilon \geqslant 0$ such that for every $a_1, a_2, \cdots, a_n \in A$,

$$\|T(a_1 a_2 \cdots a_n) - a_1 T(a_2 \cdots a_n)\| \leqslant \varepsilon \|a_1\| \cdots \|a_n\|.$$

The *almost left n-multiplier* and *almost n-multiplier* can be defined analogously.

The set of all almost n-multipliers on Banach algebra A is denoted by $AMul_n(A)$. Note that $Mul_n(A) \subseteq AMul_n(A)$ and $Mul_n(A) \neq \{0\}$, because $Mul_n(A)$ contains the identity operator. The case $n = 2$ is simply called almost multiplier. If T is both left and right n-multiplier, then T is an n-multiplier, but the converse is not true in general. The next example confirm this fact.

Example 2.2. Let

$$A = \left\{ \begin{bmatrix} 0 & a & x & y \\ 0 & 0 & a & z \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix} : \quad a, x, y, z \in \mathbb{R} \right\},$$

with the usual matrix operations. Then A is a Banach algebra with respect to the l_1-norm, that is, the sum of all absolute values of entries. Define $T : A \longrightarrow A$ via

$$T\left(\begin{bmatrix} 0 & a & x & y \\ 0 & 0 & a & z \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & x & 0 \\ 0 & 0 & a & z \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For all $u, v, w \in A$, $T(uvw) = 0$, and $uT(vw) = T(uv)w$. Therefore T is a 3-multiplier, but it not right (left) 3-multiplier.

We recall that a Banach algebra A is called *without order*, if for all $x \in A$, $xA = \{0\}$ $[Ax = \{0\}]$ implies $x = 0$.

For without order Banach algebra A, every n-multiplier is left and right n-multiplier. Indeed, let $a_1T(a_2 \cdots a_n) = T(a_1 \cdots a_{n-1})a_n$, for all $a_1, a_2, \cdots, a_n \in A$. Then for each $x \in A$, $x\big(T(a_1 \cdots a_n)\big) = T(xa_1 \cdots a_{n-2})a_{n-1}a_n = \big(xT(a_1 \cdots a_{n-1})\big)a_n = x\big(T(a_1 \cdots a_{n-1})\big)a_n$. Since A is without order, we get $T(a_1 \cdots a_n) = T(a_1 \cdots a_{n-1})a_n$, for all $a_1, a_2, \cdots, a_n \in A$. So T is a left n-multiplier. Similarly, T is a right n-multiplier.

The same is true for almost version. That is, each left and right almost n-multiplier is an almost n-multiplier, but the converse may not holds, in general. The following example illustrates this fact.

Example 2.3. Let X be the normed algebra of all polynomials defined on $[0, 1]$, and let $\varphi : X \longrightarrow X$ be a linear unbounded function. Let

$$A = \left\{ \begin{bmatrix} 0 & f & g \\ 0 & 0 & h \\ 0 & 0 & 0 \end{bmatrix} : \quad f, g, h \in X \right\},$$

and define $T : A \longrightarrow A$ by

$$T\left(\begin{bmatrix} 0 & f & g \\ 0 & 0 & h \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & \varphi(f) & 0 \\ 0 & 0 & \varphi(h) \\ 0 & 0 & 0 \end{bmatrix}.$$

Then for all $u, v \in A$, $T(uv) = 0$ and $uT(v) = T(u)v$. Thus, T is a multiplier and hence it is almost multiplier. On the other hand, let

$$u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & h \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $\|T(uv) - uT(v)\| = \|\varphi(h)\|$, which yields that T is not almost left (right) multiplier, because φ is unbounded.

Recall that a bounded approximate identity for A is a bounded net $(e_\alpha)_{\alpha \in I}$ in A such that $ae_\alpha \longrightarrow a$ and $e_\alpha a \longrightarrow a$, for all $a \in A$. For example, it is known that the group algebra $L^1(G)$, for a locally compact group G, and C^*-algebras have a bounded approximate identity bounded by one, see [3].

Clearly, every multiplier is an n-multiplier. As our first general result, with the extra condition, we show that the same is true for an almost n-multiplier.

Theorem 2.4. *Let A be a Banach algebra with a bounded approximate identity, and let $T \in AMul_2(A)$. Then T is an almost n-multiplier.*

Proof. For all $x, a_1, a_2, \cdots, a_n \in A$ we have

$$\|xT(a_1 \cdots a_{n-1})a_n - T(x)a_1 \cdots a_n\| = \|\big(xT(a_1 \cdots a_{n-1}) - T(x)a_1 \cdots a_{n-1}\big)a_n\|$$

$$\begin{aligned}&\leqslant \|xT(a_1\cdots a_{n-1})-T(x)a_1\cdots a_{n-1}\|\|a_n\|\\&\leqslant \varepsilon\|x\|\|a_1\cdots a_{n-1}\|\|a_n\|\leqslant \varepsilon\|x\|\|a_1\|\cdots\|a_n\|.\end{aligned}$$

Therefore

$$\begin{aligned}\|xT(a_1\cdots a_n)-xT(a_1\cdots a_{n-1})a_n\|\leqslant&\|xT(a_1\cdots a_n)-T(x)a_1\cdots a_n\|\\&+\|T(x)a_1\cdots a_n-xT(a_1\cdots a_{n-1})a_n\|\\\leqslant&2\varepsilon\|x\|\|a_1\|\cdots\|a_n\|.\end{aligned}$$

Let $(e_\alpha)_{\alpha\in I}$ be a bounded approximate identity in A with bound c. Replacing x by $(e_\alpha)_{\alpha\in I}$ and letting $\alpha\longrightarrow\infty$, we get $\|T(a_1\cdots a_n)-T(a_1\cdots a_{n-1})a_n\|\leqslant 2\varepsilon c\|a_1\|\cdots\|a_n\|$. Consequently, T is an almost left n-multiplier. Similarly, T is an almost right n-multiplier and hence it is an almost n-multiplier. □

Corollary 2.5. *Let A be a C^*-algebra. Then every almost multiplier $T:A\longrightarrow A$ is an almost n-multiplier.*

Corollary 2.6. *Suppose that A is a unital Banach algebra, and $T\in AMul_n(A)$. Then T is an almost left and right n-multiplier.*

Proof. Let e_A be the unit of A. Then for all $a,b\in A$,

$$\|aT(b)-T(a)b\|=\|aT(e_A\cdots e_Ab)-T(ae_A\cdots e_A)b\|\leqslant\varepsilon\|a\|\|b\|.$$

Thus, T is an almost multiplier, and hence the result follows from Theorem 2.4. □

The following theorem is a well-known result, due to Johnson, concerning the automatic continuity of multiplier.

Theorem 2.7 ([6, Corollary]). *Let A be a Banach algebra with a left approximate identity. Then every left multiplier T on A is linear and continuous.*

Note that every Banach algebra equipped with a left approximate identity is without order. However, we have the next version of Theorem 2.7.

Theorem 2.8 ([8, Theorem 1.1.1]). *Let A be a without order Banach algebra. Then every multiplier T on A is linear and continuous.*

The following famous result is due to Cohen [2, Theorem 1] and its proof is also adapted by Johnson in [6], see also [1, § 11].

Theorem 2.9. *If A is a Banach algebra with a left approximate identity and $\{x_m\}$ is a sequence of elements of A with $x_m\longrightarrow 0$ as $m\longrightarrow\infty$, then there exists $z\in A$ and a sequence y_m in A with $y_m\longrightarrow 0$ such that $x_m=zy_m$.*

We now generalize Theorem 2.7 for n-multipliers on Banach algebras.

Theorem 2.10. *Let A be a Banach algebra with a left approximate identity. Then every left n-multiplier T on A is linear and continuous.*

Proof. Let $x_1, x_2 \in A$ be arbitrary and let $\mu_1, \mu_2 \in \mathbb{C}$. Then by preceding theorem, one can find $y_1, y_2, z \in A$ such that $x_1 = zy_1$ and $x_2 = zy_2$. Since $z \in A$, again we get $z = a_1a_2 \cdots a_{n-1}$, for some $a_1, \cdots, a_{n-1} \in A$. Thus, T is linear:

$$\begin{aligned} T(\mu_1x_1 + \mu_2x_2) =& T(z(\mu_1y_1 + \mu_2y_2)) = T\big(a_1a_2 \cdots a_{n-1}(\mu_1y_1 + \mu_2y_2)\big) \\ =& T(a_1a_2 \cdots a_{n-1})(\mu_1y_1 + \mu_2y_2) \\ =& T(a_1a_2 \cdots a_{n-1})\mu_1y_1 + T(a_1a_2 \cdots a_{n-1})\mu_2y_2 \\ =& \mu_1T(a_1a_2 \cdots a_{n-1}y_1) + \mu_2T(a_1a_2 \cdots a_{n-1}y_2) \\ =& \mu_1T(zy_1) + \mu_2T(zy_2) = \mu_1T(x_1) + \mu_2T(x_2). \end{aligned}$$

Now, let $x_m \in A$ and $x_m \longrightarrow 0$. Then $x_m = zy_m$, where $y_m \longrightarrow 0$ and $z = a_1a_2 \cdots a_{n-1}$, for some $a_1, \cdots, a_{n-1} \in A$. So $T(x_m) = T(zy_m) = T(a_1a_2 \cdots a_{n-1}y_m) = T(a_1a_2 \cdots a_{n-1})y_m \longrightarrow 0$, as $m \longrightarrow \infty$. Consequently, T is continuous. □

From above theorem we obtain the next result.

Corollary 2.11. *Let A be a Banach algebra with a left approximate identity. Then every n-multiplier T on A is linear and continuous.*

The Jacobson radical $\mathfrak{J}(A)$ of A is the intersection of the maximal modular left ideals of A. An algebra A is called *semisimple* whenever $\mathfrak{J}(A) = \{0\}$.

The spectral radius of $a \in A$ is $\rho_A(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$. The element $a \in A$ is quasi-nilpotent if $\rho_A(a) = 0$. The set of quasi-nilpotent elements in A is denoted by $\mathfrak{D}(A)$.

For Banach algebra A, the spectral radius ρ_A is always continuous at zero, but it may be discontinuous at other points. If A is commutative, then ρ_A is continuous at all points of A.

Let X and Y be Banach spaces and let $T : X \longrightarrow Y$ be a linear map. The *separating space* $\mathfrak{S}(T)$ of T is defined by

$$\mathfrak{S}(T) = \{y \in Y : \text{ there exists } (x_n) \in X \text{ s.t. } x_n \longrightarrow 0 \text{ and } T(x_n) \longrightarrow y\}.$$

By [3, Proposition 5.1.2], the separating space $\mathfrak{S}(T)$ is a closed linear subspace of Y, and T is continuous if and only if $\mathfrak{S}(T) = \{0\}$.

Proposition 2.12. *Let A be a unital Banach algebra, and $T : A \longrightarrow A$ be a linear almost right (left) n-multiplier. Then $\mathfrak{S}(T)$ is a closed left (right) ideal of A.*

Proof. It is enough to prove that $\mathfrak{S}(T)$ is an ideal of A. Let $a \in A$ and $b \in \mathfrak{S}(T)$. Then there exists a sequence (a_m) in A such that $a_m \longrightarrow 0$ and $T(a_m) \longrightarrow b$. Hence

$$\begin{aligned} \|T(aa_m) - ab\| =& \|T(aa_me_A \cdots e_A) - ab\| \\ \leqslant& \|T(aa_me_A \cdots e_A) - aT(a_me_A \cdots e_A)\| + \|aT(a_me_A \cdots e_A) - ab\| \\ \leqslant& \varepsilon\|a\|\|a_m\| + \|a\|\|T(a_m) - b\| \longrightarrow 0, \end{aligned}$$

as $m \longrightarrow \infty$. Thus, $T(aa_m) \longrightarrow ab$ and since $aa_m \longrightarrow 0$, we get $ab \in \mathfrak{S}(T)$. Consequently, $\mathfrak{S}(T)$ is a closed left ideal of A. □

THEOREM 2.13. *Let A be a unital Banach algebra, $T : A \longrightarrow A$ be a linear almost right n-multiplier such that $\rho_A(T(a)) \leqslant \rho_A(a)$, for all $a \in A$. If ρ_A is continuous on $\mathfrak{S}(T)$, then $\mathfrak{S}(T) \subseteq \mathfrak{D}(A)$. Suppose, further, that A is semisimple. Then T is automatically continuous.*

Proof. Let b be arbitrary element of $\mathfrak{S}(T)$. Then there exists a sequence (a_m) in A such that $a_m \longrightarrow 0$ and $T(a_m) \longrightarrow b$. It follows from the continuity of ρ_A on $\mathfrak{S}(T)$ that $\rho_A(T(a_m)) \longrightarrow \rho_A(b)$. On the other hand, $\rho_A(T(a_m)) \leqslant \rho_A(a_m) \longrightarrow 0$. Therefore $\rho_A(b) = 0$, and hence $\mathfrak{S}(T) \subseteq \mathfrak{D}(A)$. By Proposition 2.12, $\mathfrak{S}(T)$ is a closed left ideal in A. Hence $\mathfrak{S}(T) \subseteq \mathfrak{J}(A)$ by [3, Proposition 1.5.32]. If A is semisimple, then $\mathfrak{S}(T) = \{0\}$ and hence T is continuous. □

As a consequence of Theorem 2.13 and Corollary 2.6, we get the next result.

COROLLARY 2.14. *Let A be a unital Banach algebra and $T \in AMul_n(A)$ be a linear map such that $\rho_A(T(a)) \leqslant \rho_A(a)$, for all $a \in A$. If A is commutative and semisimple, then T is automatically continuous.*

3. Continuity on uniform Banach algebras

The next example provided that we cannot assert that almost n-multipliers of A are always n-multipliers.

EXAMPLE 3.1. Let

$$A = \left\{ \begin{bmatrix} 0 & a & x & t \\ 0 & 0 & b & y \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{bmatrix} : \quad a, b, c, x, y \in \mathbb{R} \right\},$$

with the usual matrix operations and l_1-norm. Define $T : A \longrightarrow A$ via

$$T\left(\begin{bmatrix} 0 & a & x & t \\ 0 & 0 & b & y \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & y & 0 \\ 0 & 0 & b & x \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, for all $u, v, w \in A$, we have $T(uvw) = 0$, but

$$uT(vw) = \begin{bmatrix} 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad T(uv)w = \begin{bmatrix} 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $s \neq t$. Thus, neither T is a right (left) 3-multiplier nor it is a 3-multiplier. However, $\|T(uvw) - uT(vw)\| = |s| \leqslant \|u\|\|v\|\|w\|$, hence T is an almost right 3-multiplier. Similarly, T is an almost left 3-multiplier and so it is an almost 3-multiplier.

The Banach algebra A is called uniform Banach algebra if the norm $\|.\|$ on A is multiplicative, i.e., $\|xy\| = \|x\|\|y\|$, for all $x, y \in A$.

THEOREM 3.2. *Suppose that A is a uniform Banach algebra. For $T \in AMul_2(A)$, at least one of the following holds:*
(i) $T \in Mul_n(A)$,

(ii) there exist a constant k such that $\|T(x)\| \leqslant k\|x\|$, for all $x \in A$.

Proof. Suppose that T is not n-multiplier, hence there exist $a_1, a_2, \cdots, a_n \in A$ such that $a_1(Ta_2 \cdots a_n) \neq (Ta_1 \cdots a_{n-1})a_n$. Take $u = a_1T(a_2 \cdots a_n)$ and $v = T(a_1a_2 \cdots a_{n-1})a_n$. Thus, $\|u - v\| \neq 0$. Since $T \in AMul_2(A, X)$, for every $a, b \in A$ $\|aT(b) - T(a)b\| \leqslant \varepsilon\|a\|\|b\|$, for some $\varepsilon > 0$. Then for all $x \in A$,

$$\begin{aligned}\|T(x)\|\|u - v\| &= \|(Tx)u - (Tx)v\| \leqslant \|(Tx)u \pm x(Tu) \pm x(Tv) - (Tx)v\| \\ &\leqslant \|(Tx)u - x(Tu)\| + \|x(Tv) - (Tx)v\| + \|x(Tu) - x(Tv)\| \\ &\leqslant \varepsilon\|x\|\big(\|u\| + \|v\| + \|Tu - Tv\|\big).\end{aligned}$$

Therefore,

$$\|T(x)\|\|u - v\| \leqslant \varepsilon\|x\|\big(\|u\| + \|v\| + \|Tu - Tv\|\big).$$

Thus, if we set

$$k = \frac{\varepsilon\big(\|u\| + \|v\| + \|Tu - Tv\|\big)}{\|u - v\|},$$

then $\|T(x)\| \leqslant k\|x\|$, as required. □

From preceding theorem and Corollary 2.11, we have the following result.

COROLLARY 3.3. *Let A be a uniform Banach algebra with left approximate identity. If $T \in AMul_2(A)$ is additive, then it is continuous.*

The following example shows that the inclusion $Mul_n(A) \subseteq AMul_n(A)$ is strict.

EXAMPLE 3.4. Let $A = C([0, 1])$, the Banach algebra of all continuous complex-valued functions on $[0, 1]$ with uniform norm $\|f\| = \sup\{|f(x)| : x \in [0, 1]\}$. Define $T : A \longrightarrow A$ by $T(f) = \overline{f}$, where $\overline{f}(a) = \overline{f(a)}$. Since $\|f\| = \|\overline{f}\|$, for all $f_1, f_2 \in A$, we get

$$\begin{aligned}\|T(f_1)f_2 - f_1T(f_2)\| &= \|\overline{f_1}f_2 - f_1\overline{f_2}\| \leqslant \|\overline{f_1}f_2\| + \|f_1\overline{f_2}\| \\ &\leqslant \|\overline{f_1}\|\|f_2\| + \|f_1\|\|\overline{f_2}\| \leqslant 2\|f_1\|\|f_2\|.\end{aligned}$$

Thus, $T \in AMul_2(A)$ with $\varepsilon = 2$, and hence T is an almost n-multiplier by Theorem 2.4, but T is not n-multiplier. Moreover, T is continuous by Corollary 3.3.

THEOREM 3.5. *Let A be a uniform Banach algebra and $T \in Mul_2(A)$. Then, at least one of the following holds:*
(i) T is additive and $T \in Mul_n(A)$,

(ii) there exist a constant k such that $\|T(x)\| \leqslant c\|x\|$, for all $x \in A$.

Proof. Assume that T is neither n-multiplier nor additive. If $T \notin Mul_n(A)$, then the conclusion follows from Theorem 3.2. If T is not additive, then $T(a+b) \neq T(a)+T(b)$, for some $a, b \in A$. Take $u = T(a+b)$, $v = T(a)$ and $w = T(b)$. Hence $\|u-v-w\| \neq 0$. Since $T \in AMul_2(A)$, $\|aT(b) - T(a)b\| \leqslant \varepsilon\|a\|\|b\|$, for some $\varepsilon \geqslant 0$ and for every $a, b \in A$. Then, for all $x \in A$ we have

$$\begin{aligned}\|T(x)\|\|u-v-w\| &= \|T(x)u - T(x)v - T(x)w\| \\ &\leqslant \|T(x)u \pm xT(u) - T(x)v \pm xT(v) - T(x)w \pm xT(w)\| \\ &\leqslant \|T(x)u - xT(u)\| + \|xT(v) - T(x)v\| \\ &\quad + \|xT(w) - T(x)w\| + \|xT(u) - xT(v) - xT(w)\| \\ &\leqslant \varepsilon\|x\|\big(\|u\| + \|v\| + \|w\| + \|T(u) - T(v) - T(w)\|\big).\end{aligned}$$

By setting

$$k = \frac{\varepsilon\big(\|u\| + \|v\| + \|w\| + \|T(u) - T(v) - T(w)\|\big)}{\|u-v-w\|},$$

we deduce $\|T(x)\| \leqslant k\|x\|$, for all $x \in A$. □

As a consequence of Theorem 3.5, we have the next result.

Corollary 3.6. *With the same hypotheses of Theorem 3.5, if the mapping T is additive, then either $T \in Mul_n(A)$, or it is continuous.*

Next we prove that each mapping which is near to almost n-multiplier T is an almost n-multiplier.

Theorem 3.7. *Let A be a Banach algebra, and $T \in AMul_n(A)$. If $\varphi : A \longrightarrow A$ is a mapping such that $\|\varphi(x) - T(x)\| \leqslant \varepsilon\|x\|$, for every $x \in A$, then $\varphi \in AMul_n(A)$.*

Proof. By assumption $\|a_1T(a_2\cdots a_n) - T(a_1\cdots a_{n-1})a_n\| \leqslant \varepsilon_1\|a_1\|\cdots\|a_n\|$, for some $\varepsilon_1 \geqslant 0$. Hence for every $a_1, a_2, \cdots, a_n \in A$, we have

$$\begin{aligned}&\|a_1\varphi(a_2\cdots a_n) - \varphi(a_1\cdots a_{n-1})a_n\| \\ \leqslant&\|a_1\varphi(a_2\cdots a_n) \pm a_1T(a_2\cdots a_n) \pm T(a_1\cdots a_{n-1})a_n - \varphi(a_1\cdots a_{n-1})a_n\| \\ \leqslant&\|a_1\varphi(a_2\cdots a_n) - a_1T(a_2\cdots a_n)\| + \|a_1T(a_2\cdots a_n) - T(a_1\cdots a_{n-1})a_n\| \\ &+ \|T(a_1\cdots a_{n-1})a_n - \varphi(a_1\cdots a_{n-1})a_n\| \\ \leqslant&\varepsilon\|a_1\|\cdots\|a_n\| + \varepsilon_1\|a_1\|\cdots\|a_n\| + \varepsilon\|a_1\|\cdots\|a_n\| \leqslant (2\varepsilon + \varepsilon_1)\|a_1\|\cdots\|a_n\|.\end{aligned}$$

Thus, $\|a_1\varphi(a_2\cdots a_n) - \varphi(a_1\cdots a_{n-1})a_n\| \leqslant \delta\|a_1\|\cdots\|a_n\|$, where $\delta = 2\varepsilon + \varepsilon_1$. □

Remark 3.8. (i) If A is a unital commutative Banach algebra and $T \in AMul_n(A)$, then there exist exact n-multiplier $h : A \longrightarrow A$ such that

$$\|h(a) - T(a)\| \leqslant \varepsilon\|a\|, \tag{1}$$

for every $a \in A$. Moreover, if T is linear, then it is continuous. To see this, let $\|a_1T(a_2\cdots a_n) - T(a_1\cdots a_{n-1})a_n\| \leqslant \varepsilon\|a_1\|\cdots\|a_n\|$, for some $\varepsilon \geqslant 0$ and for every $a_1, a_2, \cdots, a_n \in A$. Taking $a_2 = \cdots = a_n = e_A$, we get $\|aT(e_A) - T(a)\| \leqslant \varepsilon\|a\|$, $(a \in A)$.

Now the mapping $h : A \longrightarrow A$ defined by $h(a) = aT(e_A)$ is an n-multiplier and it satisfies in (1). The continuity of T is now follows from (1).

(ii) Let A be a unital Banach algebra and $T \in AMul_n(A)$. If T is linear and unital, then T is continuous. In fact, for all $a \in A$, we have

$$\|T(a)\| - \|ae_A\| = \|T(a)\| - \|aT(e_A)\| \leqslant \|T(ae_A \cdots e_A) - aT(e_A \cdots e_A)\| \leqslant \varepsilon\|a\|.$$

Thus $\|T(a)\| \leqslant (1+\varepsilon)\|a\|$, and so T is continuous.

4. Conclusion

In this work, we introduced and studied the notion of *almost n-multiplier* on Banach algebras. Under some conditions, we proved the continuity and automatic continuity of such maps on Banach algebras. Some general theory and useful results on the notion of *almost n-multiplier* are established. We presented some useful and important examples to illustrate our results. In particular, we generalize the celebrated theorem of Johnson to (left) n-multipliers on Banach algebras. The results obtained in this work complement and extend some existing results in the literature.

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Department of Mathematics, Faculty of Basic Sciences, Ayatollah Boroujerdi University, Boroujerd, Iran
E-mail: zivari@abru.ac.ir, zivari6526@gmail.com
ORCID iD: https://orcid.org/0000-0001-8362-8490