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OPEN-LOCATING DOMINATING NUMBER FOR FLOWER SNARKS

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Abstract. The problem of finding an open-locating dominating set is a variant of the domination problem where both domination and the ability to identify a certain vertex are required. The cardinality of such a dominating set is called the open-locating dominating number. The open-locating domination problem has been proven to be NP-hard in the general case. In this paper, the exact value of the old domination number is provided for the class of Flower snark graphs.

1. Introduction

The problem of open neighborhood locating-dominating sets came from certain problems in security and protection especially in computer networks. The problem is to detect a node (for example faulty part, intruder) by placing devices in certain nodes of the network where each device is capable to detect if considered activities appeared in its neighborhood. Each device could not detect such activities in the node where it is placed. The problem is to find the minimal set of nodes which covers entire network, and where each vertex with "undesired behavior" could be uniquely identified.

Let $G = (V, E)$ be a finite, simple graph where V is its set of vertices and E set of edges. For arbitrary $v \in V$ the set $N(v) = \{u \in V | uv \in E\}$ is called *open neighborhood* and the set $N[v] = N(v) \cup \{v\}$ is called *closed neighborhood* of the vertex v. In this paper we consider only those graph where neighoborhood of any vertex is nonempty. We say that a set of vertices $S \subseteq V$ dominates graph G if each vertex from V is either in S or adjacent to a vertex from S. The set S is open neighborhood locating dominating set of graph G if the set S dominates the graph G and for arbitrary two distinct vertices $u, v \in V$ it stands that $N(u) \cap S \neq N(v) \cap S$. Such set is called OLD-set of the graph G. The minimal cardinality of OLD-sets is called open-locating-dominating number of a graph G and is denoted as $\gamma_{old}(G)$,

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and the problem of finding such set is called open neighborhood locating dominating problem.

Suppose that each device placed in a vertex $x \in V$ could detect problem in its neighborhood $N(x)$, but could not detect the problem in the vertex x. Solving OLD problem for a graph we can determine the minimal necessary number of devices that needs to be placed and its locations. The open neighborhood locating-dominating problem is was introduced by Honkala et al. in [9] for Q_k class of graphs and by Seo and Slater in [18, 19] for graphs in general.

Lobstain collected more than 375 references related to the papers on distinguishing sets. Seo and Slater stated in [18] that graph G has an OLD set if and only if G has no isolated vertices and for every pair of vertices x and y from V it hold that $N(x) \neq N(y)$. For finite graphs, several bounds for $\gamma_{old}(G)$ were found. Chelali et al. in [3] proved that for a connected graph G that is C_4 free with minimal degree $\delta(G) \geq 3$ it holds $\gamma_{old}(G) \leq n - \rho(G)$, where $\rho(G)$ is the maximum number of vertices that are at a distance at least 3 pair by pair. Chelali et al. in [3] and Seo and Slater in [19] proved that for a graph G where the maximum degree is Δ it holds that if the graph G has an OLD set then $\gamma_{old}(G) \geq \frac{2|V(G)|}{1+\Delta}$. Henning and Yeo in [8] proved that for cubic graphs of order $|V(G)|$ it holds that $\gamma_{old}(G) \leq \frac{3|V(G)|}{4}$.

Chelali et al. in [3] and Seo and Slater in [19] proved the more specific statement for trees of order $|V(G)| \geq 3$: Trees has an OLD set if and only if doesn't contain vertices that have two or more leaves attached to them, i.e. it doesn't contain vertices that have two or more neighbors with the degree equal to one. Seo and Slater proved in [18] that OLD problem is NP-hard. In [1] authors study change in minimum cardinality under operations such as adding a universal vertex, taking the generalized corona of a graph, and taking a square of graph. They applied these operations to paths and cycles allowing them to find ecact values in resulting cases. Foucaud et el. in [5] showed that this problem is NP-complete even for interval and permutation graphs with diameter equal to 2. Savić at al. in [17] studied OLD number for certain class of symmetric graphs. They determined exact values of OLD number for convex polytopes D_n and R_n, as well as upper bounds for convex polytopes T_n, B_n, C_n and E_n. Raza, in [16] continued this work finding finding exact values of OLD number convex poltopes R_n and H_n, as well as upper bounds for S_n, R'_n, A_n, Q_n, U_n. In the same paper author calculated exact values for cycle and some prism graphs. The open-locating dominating number for generalized Petersen graphs was studied by Maksimović et al. in [14]. In [7] authors proposed the optimal OLD-set size for a particular circulant graph using Halls Theorem. In [4] authors consider type of a fault-tolerant open-locating dominating set called error-detecting open-locating-dominating sets. Their results show its NP-completeness proof, results for extremal graphs, and a characterization of cubic graphs that permit an error-detecting open-locating-dominating set.

Sweigart et al. in [20] and Kincaid et. al in [12] proposed an ILP models for solving OLD problem and upper bound of density for certain classes of infinite grids. In [2] authors studied the three problems from a polyhedral point of view. They provided the according linear relaxations, discussedd their combinatorial structure, and demonstrate how the associated polyhedra can be entirely described or polyhedral arguments can be applied to find minimum such sets for special graphs.

Isaacs in [11] introduced Flower snarks as an example of a class of connected cubic graphs without bridges that have chromatic index equal to 4. Flower snarks denoted by J_n, where n is odd, have $4n$ vertices and $6n$ edges. The set of vertices can be written as $V = \{a_i, b_i, c_i, d_i | i = 0, 1, \ldots, n-1\}$ and the set of edges as

$$E = \{a_i b_i, a_i c_i, a_i d_i, b_i b_{i+1} | i = 0, 1, \ldots, n-1\} \cup \{c_i c_{i+1}, d_i d_{i+1} | i = 0, 1, \ldots, n-2\} \cup \{c_{n-1} d_0, d_n c_0\}$$

where indices are taken modulo n. It was found that Flower snarks have constant metric dimension by Imran et al. in [10]. Ghebleh et al. in [6] studied circular chromatic index for Flower snarks. Maksimović et al. in [15] studied some static Roman domination problems for this class of graphs. The exact values of Roman and restrained Roman domination were determined as well as upper bound of signed total Roman domination problem.

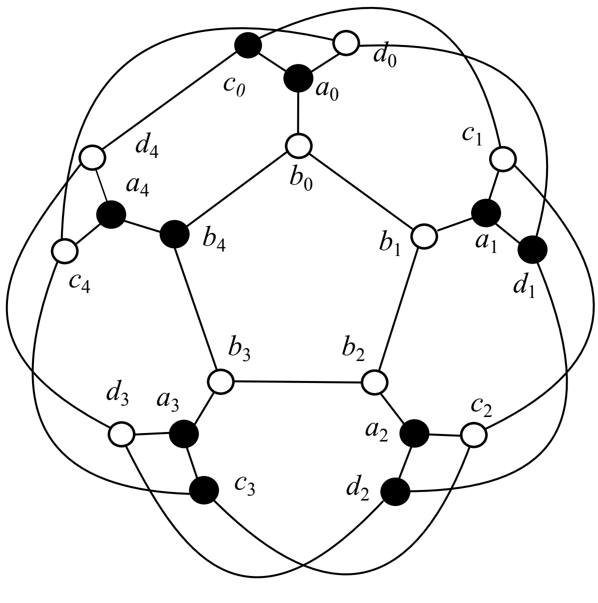

Figure 1: Graph J_5 and it's OLD set

In this paper the open-locating-dominating problem will be studied for Flower snarks class of graphs and it's exact value of OLD-number will be presented and proved. One solution of the OLD-problem for J_5 is presented on Figure 1, where vertices that belongs to the OLD-set are depicted as circles painted in black, while the other vertices are depicted as empty circles.

2. The main result

THEOREM 2.1. $\gamma_{old}(J_n) = 2 \cdot n$.

Proof. Step 1. $\gamma_{old}(J_n) \leq 2 \cdot n$, for $n = 3 \pmod 4$
Let $n = 4t + 3$. In that case the set S could be written as

$$S = \{b_{4t+2}\} \cup \{c_{4t}, c_{4t+1}\} \cup \{a_i | i = 0, \dots, n-1\} \\ \cup \{c_{4i}, c_{4i+1}, d_{4i+2}, d_{4i+3} | i = 0, \dots, t-1\}.$$

Let us present the intersection of open neighborhoods of every vertex with set S. We shell start with a vertices. For $0 \leq i \leq t-1$ it holds that $\mathcal{N}(a_{4i}) \cap S$ is equal to $\{c_{4i}\}$, $\mathcal{N}(a_{4i+1}) \cap S = \{c_{4i+1}\}$, $\mathcal{N}(a_{4i+2}) \cap S = \{d_{4i+2}\}$ and $\mathcal{N}(a_{4i+3}) \cap S = \{d_{4i+3}\}$. Moreover, $\mathcal{N}(a_{4t}) \cap S = \{c_{4t}\}$, $\mathcal{N}(a_{4t+1}) \cap S = \{c_{4t+1}\}$ and $\mathcal{N}(a_{4t+2}) \cap S = \{b_{4t+2}\}$.

Let us consider b vertices. For $1 \leq i \leq n-3$ it holds that $\mathcal{N}(b_i) \cap S = \{a_i\}$. In addition, $\mathcal{N}(b_0) \cap S = \{a_0, b_{n-1}\}$, $\mathcal{N}(b_{n-2}) \cap S = \{a_{n-2}, b_{n-1}\}$ and $\mathcal{N}(b_{n-1}) \cap S = \{a_{n-1}\}$.

In similar fashion we shall now discuss c vertices. For $0 \leq i \leq t-1$ it holds that $\mathcal{N}(c_{4i}) \cap S = \{a_{4i}, c_{4i+1}\}$, $\mathcal{N}(c_{4i+1}) \cap S = \{a_{4i+1}, c_{4i}\}$, $\mathcal{N}(c_{4i+2}) \cap S = \{a_{4i+2}, c_{4i+1}\}$ and $\mathcal{N}(c_{4i+3}) \cap S$ is equal to $\{a_{4i+3}, c_{4i+4}\}$. Moreover, $\mathcal{N}(c_{4t}) \cap S = \{a_{4t}, c_{4t+1}\}$, $\mathcal{N}(c_{4t+1}) \cap S = \{a_{4t+1}, c_{4t}\}$ and $\mathcal{N}(c_{4t+2}) \cap S = \{a_{4t+2}, c_{4t+1}\}$.

Finally, we shall discuss d vertices. For $1 \leq i \leq t-1$ it holds that $\mathcal{N}(d_{4i}) \cap S = \{a_{4i}, d_{4i-1}\}$. For $0 \leq i \leq t-1$ it holds that $\mathcal{N}(d_{4i+1}) \cap S = \{a_{4i+1}, d_{4i+2}\}$, $\mathcal{N}(d_{4i+2}) \cap S = \{a_{4i+2}, d_{4i+3}\}$, $\mathcal{N}(d_{4i+3}) \cap S = \{a_{4i+3}, d_{4i+2}\}$, $\mathcal{N}(d_0) \cap S = \{a_0\}$, $\mathcal{N}(d_{4t}) \cap S = \{a_{4t}, d_{4t-1}\}$, $\mathcal{N}(d_{4t+1}) \cap S = \{a_{4t+1}\}$ and $\mathcal{N}(d_{4t+2}) \cap S = \{a_{4t+2}, c_0\}$.

As it can be seen all neighborhood intersections with the set S are non-empty and mutually different. Therefore, in case $n = 4t + 3$, S is an OLD set for J_n, and its cardinality is equal to $2 \cdot n$, so $\gamma_{old}(J_n) \leq 2 \cdot n$.

Step 2. $\gamma_{old}(J_n) \leq 2 \cdot n$, for $n = 1 \pmod 4$
Let $n = 4t + 1$. In that case the set S could be written as

$$S = \{b_{4t}\} \cup \{a_i | i = 0, \dots, 4t\} \cup \{c_{4i}, d_{4i+1}, d_{4i+2}, c_{4i+3} | i = 0, \dots, t-1\}.$$

In order to distinguish presentation from the previous step, the intersection of open neighborhoods with the set S is given in the Table 1. The first column of the Table contains vertices of the graph J_n. The second column contains conditions for indices, and the last column contains the intersection of open neighborhoods with the set S.

As it can be seen from Table 1, all neighborhood intersections with the set S are non-empty and mutually different. Therefore, in this case $(n = 4t + 1)$, S is an OLD set for J_n, and its cardinality is equal to $2 \cdot n$, so $\gamma_{old}(J_n) \leq 2 \cdot n$.

Step 3. $\gamma_{old}(J_n) \geq 2 \cdot n$
Since flower snarks are 3-regular $(\Delta = 3)$ it holds that $\gamma_{old}(G) \geq \frac{2|V|}{1+\Delta}$, so in the case of J_n $\gamma_{old}(J_n) \geq \frac{2 \cdot 4n}{1+3} = 2n$. □

vertex	condition	$\mathcal{N}(v) \cap S$
a_{4i}	$0 \leq i \leq t-1$	$\{c_{4i}\}$
a_{4i+1}	$0 \leq i \leq t-1$	$\{d_{4i+1}\}$
a_{4i+2}	$0 \leq i \leq t-1$	$\{d_{4i+2}\}$
a_{4i+3}	$0 \leq i \leq t-1$	$\{c_{4i+3}\}$
a_{4t}		$\{b_{4t}\}$
b_0		$\{b_{4t}, a_0\}$
b_i	$1 \leq i \leq 4t-2$	$\{a_i\}$
b_{4t-1}		$\{a_{4t-1}, b_{4t}\}$
b_{4t}		$\{a_{4t}\}$
c_0		$\{a_0\}$
c_{4i}	$1 \leq i \leq t-1$	$\{a_{4i}, c_{4i-1}\}$
c_{4i+1}	$0 \leq i \leq t-1$	$\{a_{4i+1}, c_{4i}\}$
c_{4i+2}	$0 \leq i \leq t-1$	$\{a_{4i+2}, c_{4i+3}\}$
c_{4i+3}	$0 \leq i \leq t-2$	$\{a_{4i+3}, c_{4i+4}\}$
c_{4t-1}		$\{a_{4t-1}\}$
c_{4t}		$\{a_{4t}, c_{4t-1}\}$
d_{4i}	$0 \leq i \leq t-1$	$\{a_{4i}, d_{4i+1}\}$
d_{4i+1}	$0 \leq i \leq t-1$	$\{a_{4i+1}, d_{4i+2}\}$
d_{4i+2}	$0 \leq i \leq t-1$	$\{a_{4i+2}, d_{4i+1}\}$
d_{4i+3}	$0 \leq i \leq t-1$	$\{a_{4i+3}, d_{4i+2}\}$
d_{4t}		$\{a_{4t}, c_0\}$

Table 1: $\mathcal{N}(v) \cap S$ for $n = 4t + 1$

3. Conclusions

In this paper it was proven that the exact value of OLD-number for Flower snark graphs J_n is equal to $2n$. The future work could be directed in several ways. One direction is to determine OLD-number for some other classes of graphs. Another direction could be to determine some other invariants for this class of graphs.

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