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RESULTS ON RELATIVE MATLIS REFLEXIVE MODULES WITH RESPECT TO A SEMIDUALIZING MODULE

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Abstract. Let R be a commutative local ring and let C be a semidualizing R-module. In [E. Tavasoli, M. Salimi, *Relative Matlis duality with respect to a semidualizing module*, Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér., **66(114), 4** (2023), 433–444], the notion of relative Matlis duality with respect to C, and C-Matlis reflexive modules are introduced, which generalized the notions of Matlis duality and Matlis reflexive modules. In this paper, we investigate conditions under which the R-modules $\mathrm{Ext}_R^{i\geqslant 0}(M,N)$ and $\mathrm{Tor}^R_{i\geqslant 0}(M,N)$ become C-Matlis reflexive, where M and N are R-modules. In addition, we deal with the isomorphic modules to the relative Matlis duality of R-modules $\mathrm{Ext}_R^{i\geqslant 0}(M,N)$, and $\mathrm{Tor}^R_{i\geqslant 0}(M,N)$ in the case that M and N are Matlis reflexive modules over the complete ring R.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring, and we use the notation $\mathrm{E}_R(M)$ for the injective envelope of an R-module M. Let $(R,\mathfrak{m})$ be a local ring. We define the Matlis dual of M to be $M^\vee = \mathrm{Hom}_R(M, E(R/\mathfrak{m}))$. We say that M is Matlis reflexive if the canonical injection $M \to M^{\vee\vee}$ is an isomorphism. In [5], it is shown that M is Matlis reflexive if and only if M has a finitely generated submodule S such that M/S is Artinian. Using this characterization, Belshoff in [2] investigated the category of Matlis reflexive modules over a complete local ring. In particular, it is shown that if M and N are Matlis reflexive R-modules, then so are the R-modules $\mathrm{Hom}_R(M,N)$, $M \otimes_R N$, $\mathrm{Ext}_R^{i\geqslant 1}(M,N)$, and $\mathrm{Tor}^R_{i\geqslant 1}(M,N)$.

In [11], the notion of relative Matlis duality with respect to a semidualizing R-module was introduced, which gives a generalization of the notion of Matlis duality: for an R-module M, $M^{\vee_C} = \mathrm{Hom}_R(M, C^\vee)$ is the relative Matlis dual of M with respect to a semidualizing R-module C. There is also a natural R-homomorphism $\psi : M \to (M^{\vee_C})^{\vee_C}$ defined by $\psi(x)(f) = f(x)$ for all $x \in M$ and $f \in M^{\vee_C}$. An

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R-module M is called *C-Matlis reflexive* if $M \cong (M^{\vee_C})^{\vee_C}$ under the homomorphism ψ.

In this paper, we investigate the conditions under which $\mathrm{Ext}_R^{i\geqslant 0}(M,N)$ and $\mathrm{Tor}^R_{i\geqslant 0}(M,N)$ become C-Matlis reflexive R-modules. We also provide isomorphisms for R-modules $(\mathrm{Ext}_R^i(M,N))^{\vee_C}$ and $(\mathrm{Tor}_i^R(M,N))^{\vee_C}$ to state the connection between relative Matlis duality with respect to C of these R-modules and Matlis duality, in the case that M and N are Matlis reflexive modules over the complete local ring R. In addition, we investigate the conditions that imply the modules $\mathrm{Ext}_R^{i\geqslant 0}(M,N)$ and $\mathrm{Tor}^R_{i\geqslant 0}(M,N)$ are C-Matlis reflexive R-modules and $\widehat{C}$-Matlis reflexive $\widehat{R}$-modules.

2. Relative Matlis reflexive modules

In [11], the notion of relative Matlis duality with respect to a *semidualizing* R-module was introduced, which gives a generalization of the notion Matlis duality over the local ring R. The notion of semidualizing module due to Foxby [7], generalizing Grothendieck's notion of a dualizing module, and introduced independently by Golod [8] and Vasconcelos [12]: a finitely generated R-module C, wherein R is a commutative Noetherian ring, is called semidualizing if the natural homothety morphism $R \to \mathrm{Hom}_R(C,C)$ is an isomorphism and $\mathrm{Ext}_R^{i\geqslant 1}(C,C)=0$. Now we are ready to recall the notion of relative Matlis duality with respect to a semidualizing module over the local ring $(R,\mathfrak{m})$ from [11]: for an R-module M, $M^{\vee_C} = \mathrm{Hom}_R(M, C^\vee)$ is the relative Matlis dual of M with respect to a semidualizing R-module C. There is also a natural R-homomorphism $\psi : M \to (M^{\vee_C})^{\vee_C}$ defined by $\psi(x)(f) = f(x)$ for all $x \in M$ and $f \in M^{\vee_C}$. We say that an R-module M is *C-Matlis reflexive* if $M \cong (M^{\vee_C})^{\vee_C}$ under the homomorphism ψ. The main goal of this paper is to deal with conditions under which $\mathrm{Ext}_R^{i\geqslant 0}(M,N)$ and $\mathrm{Tor}^R_{i\geqslant 0}(M,N)$ are C-Matlis reflexive R-modules.

In the following, we collect some nice isomorphisms from [11, Proposition 3.2] in order to show the connection between $(-)^{\vee_C}$ and $(-)^\vee$.

Fact 2.1. *Let C be a semidualizing R-module, wherein R is a local ring, and let M be an R-module. Then the following statements hold.*
(i) $M^{\vee_C} \cong (C \otimes_R M)^\vee$. *(ii)* $M^{\vee_C} \cong \mathrm{Hom}_R(C, M^\vee)$.
(iii) $(M^{\vee_C})^{\vee_C} \cong \mathrm{Hom}_R(C, C \otimes_R M^{\vee\vee})$. *(iv)* $(M^{\vee_C})^{\vee_C} \cong (\mathrm{Hom}_R(C, C\otimes_R M))^{\vee\vee}$.

Example 2.2. Let $(R,\mathfrak{m})$ be a complete local ring, and let $C = \mathrm{E}(R/\mathfrak{m})$. By [6, Theorem 3.4.1], $\mathrm{Hom}_R(\mathrm{E}(R/\mathfrak{m}), \mathrm{E}(R/\mathfrak{m})) \cong R$, and $\mathrm{Ext}_R^{i\geqslant 1}(\mathrm{E}(R/\mathfrak{m}), \mathrm{E}(R/\mathfrak{m})) = 0$. Then C is a semidualizing R-module. It is routine to check that R is a C-Matlis reflexive module.

The classes defined next are collectively known as *Foxby classes*. The definitions are due to Foxby [1,4].

Definition 2.3. Let C be a semiaualizing R-module. The *Auslander class* with respect to C is the class $\mathcal{A}_C(R)$ of R-modules M such that:
(i) $\mathrm{Tor}_i^R(C, M) = 0 = \mathrm{Ext}_R^i(C, C \otimes_R M)$ for all $i \geqslant 1$, and

(ii) the natural map $\gamma_C^M : M \to \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism.
The *Bass class* with respect to C is the class $\mathcal{B}_C(R)$ of R-modules M such that:
(i) $\mathrm{Ext}_R^i(C, M) = 0 = \mathrm{Tor}_i^R(C, \mathrm{Hom}_R(C, M))$ for all $i \geqslant 1$, and

(ii) the natural evaluation map $\xi_M^C : C \otimes_R \mathrm{Hom}_R(C, M) \to M$ is an isomorphism.

Proposition 2.4. *Let C be a semidualizing R-module, wherein R is a local ring, and let $M \in \mathcal{A}_C(R)$. Then M is Matlis reflexive if and only if M is C-Matlis reflexive.*

Proof. Let M be a Matlis reflexive R-module. Then M is C-Matlis reflexive by [11, Proposition 3.3 (i)]. For the reverse, we have the following R-isomorphisms.

$$M \cong (M^{\vee_C})^{\vee_C} \cong (\mathrm{Hom}_R(C, C \otimes_R M))^{\vee\vee} \cong M^{\vee\vee}.$$

In the above sequence, second isomorphism follows from Fact 2.1 (iv), and the last one follows from the assumption that $M \in \mathcal{A}_C(R)$. □

Using [2, Proposition 1] and Proposition 2.4, we have the following results.

Proposition 2.5. *Let C be a semidualizing module over local ring R, M be a finitely generated R-module, and let N be a Matlis reflexive R-module. Then the following statements hold.*
(i) If $\mathrm{Hom}_R(M, N) \in \mathcal{A}_C(R)$, then $\mathrm{Hom}_R(M, N)$ is C-Matlis reflexive.

(ii) If $M \otimes_R N \in \mathcal{A}_C(R)$, then $M \otimes_R N$ is C-Matlis reflexive.

(iii) If $\mathrm{Ext}_R^{i\geqslant 1}(M, N) \in \mathcal{A}_C(R)$, then $\mathrm{Ext}_R^{i\geqslant 1}(M, N)$ is C-Matlis reflexive.

(iv) If $\mathrm{Tor}^R_{i\geqslant 1}(M, N) \in \mathcal{A}_C(R)$, then $\mathrm{Tor}^R_{i\geqslant 1}(M, N)$ is C-Matlis reflexive.

Corollary 2.6. *Let C be a semidualizing module over local ring R, M be a finitely generated R-module, and let N be a Matlis reflexive R-module. Then the following statements hold.*
(i) Let $M \in \mathcal{B}_C(R)$. If $\mathrm{id}_R N < \infty$ and $\mathrm{Ext}_R^{i\geqslant 1}(M, N) = 0$, then $\mathrm{Hom}_R(M, N)$ is C-Matlis reflexive.

(ii) Let $M \in \mathcal{A}_C(R)$. If $\mathrm{fd}_R N < \infty$ and $\mathrm{Tor}^R_{i\geqslant 1}(M, N) = 0$, then $M \otimes_R N$ is C-Matlis reflexive.

Proof. (i) By [10, Proposition 3.3.16 (a)], $\mathrm{Hom}_R(M, N) \in \mathcal{A}_C(R)$. Now the assertion follows from Proposition 2.5 (i).

(ii) By [10, Proposition 3.3.14 (a)], $M \otimes_R N \in \mathcal{A}_C(R)$. Now the assertion follows from Proposition 2.5 (ii). □

Proposition 2.7. *Let C be a semidualizing module over local ring R, M be an Artinian R-module and let N be a finitely generated R-module. Then the following statements hold.*
(i) If $\mathrm{Hom}_R(M, N) \in \mathcal{A}_C(R)$, then $\mathrm{Hom}_R(M, N)$ is C-Matlis reflexive.

(ii) Let $M \in \mathcal{B}_C(R)$. If $\operatorname{id}_R N < \infty$ and $\operatorname{Ext}_R^{i\geqslant 1}(M,N) = 0$, then $\operatorname{Hom}_R(M,N)$ is C-Matlis reflexive.

Proof. (i) By [2, Proposition 4], $\operatorname{Hom}_R(M,N)$ is a Matlis reflexive R-module. Now the assertion follows from Proposition 2.4.

(ii) By [2, Proposition 4], $\operatorname{Hom}_R(M,N)$ is a Matlis reflexive R-module, and [10, Proposition 3.3.16 (a)] implies that $\operatorname{Hom}_R(M,N) \in \mathcal{A}_C(R)$. Using Proposition 2.4 we get the assertion. □

PROPOSITION 2.8. *Let R be a complete local ring, C be a semidualizing R-module and let M and N be Matlis reflexive R-modules.*
(i) If $\operatorname{Hom}_R(M,N) \in \mathcal{A}_C(R)$, then $\operatorname{Hom}_R(M,N)$ is C-Matlis reflexive, and $\operatorname{Hom}_R(M,N)^{\vee_C} \cong \operatorname{Hom}_R(C, M \otimes_R N^{\vee})$.

(ii) Let $M \in \mathcal{B}_C(R)$. If $\operatorname{id}_R N < \infty$ and $\operatorname{Ext}_R^{i\geqslant 1}(M,N) = 0$, then $\operatorname{Hom}_R(M,N)$ is C-Matlis reflexive, and $\operatorname{Hom}_R(M,N)^{\vee_C} \cong \operatorname{Hom}_R(C, M \otimes_R N^{\vee})$.

(iii) If $M \otimes_R N \in \mathcal{A}_C(R)$, then $M \otimes_R N$ is C-Matlis reflexive, and $(M \otimes_R N)^{\vee_C} \cong \operatorname{Hom}_R(M, N^{\vee_C})$.

(iv) Let $M \in \mathcal{A}_C(R)$. If $\operatorname{fd}_R N < \infty$ and $\operatorname{Tor}^R_{i\geqslant 1}(M,N) = 0$, then $M \otimes_R N$ is C-Matlis reflexive, and $(M \otimes_R N)^{\vee_C} \cong \operatorname{Hom}_R(M, N^{\vee_C})$.

(v) If $\operatorname{Ext}_R^{i\geqslant 1}(M,N) \in \mathcal{A}_C(R)$, then $\operatorname{Ext}_R^{i\geqslant 1}(M,N)$ is C-Matlis reflexive, and for all $i \geqslant 0$ we have: $\operatorname{Ext}_R^i(M,N)^{\vee_C} \cong \operatorname{Hom}_R(C, \operatorname{Tor}_i^R(M, N^{\vee}))$.

(vi) If $\operatorname{Tor}^R_{i\geqslant 1}(M,N) \in \mathcal{A}_C(R)$, then $\operatorname{Tor}^R_{i\geqslant 1}(M,N)$ is C-Matlis reflexive, and for all $i \geqslant 0$ we have: $\operatorname{Tor}_i^R(M,N)^{\vee_C} \cong \operatorname{Hom}_R(C, \operatorname{Ext}_R^i(M, N^{\vee}))$.

Proof. (i) By [2, Theorem 1], $\operatorname{Hom}_R(M,N)$ is a Matlis reflexive R-module, and Proposition 2.4 implies that $\operatorname{Hom}_R(M,N)$ is C-Matlis reflexive. To see the isomorphism we have the following sequence of R-isomorphisms, where the first isomorphism follows from Fact 2.1 (ii), and the second one follows from [2, Theorem 4(a)]:

$$\operatorname{Hom}_R(M,N)^{\vee_C} \cong \operatorname{Hom}_R(C, \operatorname{Hom}_R(M,N)^{\vee}) \cong \operatorname{Hom}_R(C, M \otimes_R N^{\vee}).$$

(ii) By [10, Proposition 3.3.16 (a)], $\operatorname{Hom}_R(M,N) \in \mathcal{A}_C(R)$. Now the assertion follows from item (i).

(iii) By [2, Theorem 2], $M \otimes_R N$ is a Matlis reflexive R-module and Proposition 2.4 implies that $M \otimes_R N$ is C-Matlis reflexive. To see the isomorphism, we have the following sequence of R-isomorphisms, where the first isomorphism follows from Fact 2.1 (ii), second one follows from [2, Theorem 4(a)], third and forth ones follow from adjointness, and the last one follows from Fact 2.1 (ii).

$$\begin{aligned}(M \otimes_R N)^{\vee_C} &\cong \operatorname{Hom}_R(C, (M \otimes_R N)^{\vee}) \cong \operatorname{Hom}_R(C, \operatorname{Hom}_R(M, N^{\vee})) \\ &\cong \operatorname{Hom}_R(C \otimes_R M, N^{\vee}) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_R(C, N^{\vee})) \cong \operatorname{Hom}_R(M, N^{\vee_C}).\end{aligned}$$

(iv) By [10, Proposition 3.3.14 (a)], $M \otimes_R N \in \mathcal{A}_C(R)$. Now the assertion follows from (iii).

Items (v) and (vi) follow from [2, Theorem 4] and Proposition 2.4. To see the isomorphisms, for each i we have:

$$\operatorname{Ext}^i_R(M,N)^{\vee_C} \cong \operatorname{Hom}_R(C, \operatorname{Ext}^i_R(M,N)^{\vee}) \cong \operatorname{Hom}_R(C, \operatorname{Tor}^R_i(M, N^{\vee})),$$

where the first isomorphism follows from Fact 2.1 (ii) and the second one follows from [2, Theorem 4(c)]. Finally,

$$\operatorname{Tor}^R_i(M,N)^{\vee_C} \cong \operatorname{Hom}_R(C, \operatorname{Tor}^R_i(M,N)^{\vee}) \cong \operatorname{Hom}_R(C, \operatorname{Ext}^i_R(M, N^{\vee})),$$

where the first isomorphism follows from Fact 2.1 (ii) and the second one follows from [2, Theorem 4(d)]. $\square$

Proposition 2.9. *Let C be a semidualizing R-module, wherein R is a local ring, and let M and N be R-modules such that $\operatorname{Ext}^{i\geqslant 1}_R(M,N) \in \mathcal{A}_C(R)$. Then the following statements hold.*

(i) If M is Matlis reflexive and N is Artinian, then $\operatorname{Ext}^{i\geqslant 1}_R(M,N)$ is C-Matlis reflexive.

(ii) If M is Artinian and N is finitely generated, then $\operatorname{Ext}^{i\geqslant 1}_R(M,N)$ is C-Matlis reflexive.

(iii) If M and N are Artinian, then $\operatorname{Ext}^{i\geqslant 1}_R(M,N)$ is C-Matlis reflexive.

(iv) If M is Artinian and N is Matlis reflexive, then $\operatorname{Ext}^{i\geqslant 1}_R(M,N)$ is C-Matlis reflexive.

(v) If M is finitely generated and N is Matlis reflexive, then $\operatorname{Ext}^{i\geqslant 1}_R(M,N)$ is C-Matlis reflexive.

(vi) If M and N are Matlis reflexive, then $\operatorname{Ext}^{i\geqslant 1}_R(M,N)$ is C-Matlis reflexive.

Proof. In the light of Proposition 2.4, items (i)–(vi) follow from [2, Proposition 7], [2, Proposition 8], [2, Proposition 9], [2, Proposition 10], [2, Proposition 11] and [2, Theorem 3], respectively. $\square$

Recall that an R-module M is mini-max if there is a Noetherian submodule N of M such that M/N is artinian. In [3, Theorem 12] it is shown that an R-module M is Matlis reflexive if and only if it is mini-max and $R/\operatorname{Ann}_R(M)$ is complete.

Remark 2.10. Let R be a local ring and let C be a semidualizing R-module. By [10, Proposition 2.2.1], $\widehat{C}$ is a semidualizing $\widehat{R}$-module. Let M be an $\widehat{R}$-module. Since $\widehat{R} \in \mathcal{A}_C(R)$, [10, Proposition 3.4.6] implies that $M \in \mathcal{A}_C(R)$ if and only if $M \in \mathcal{A}_{\widehat{C}}(\widehat{R})$.

Proposition 2.11. *Let C be a semidualizing module over local ring R and let $M \in \mathcal{A}_C(R)$ such that $R/\operatorname{Ann}_R(M)$ is complete. The following conditions are equivalent.*

(i) M is C-Matlis reflexive over R.

(ii) M is mini-max over R.

(iii) M is mini-max over $\widehat{R}$.

(iv) M is $\widehat{C}$-Matlis reflexive over $\widehat{R}$.

Proof. By Remark 2.10, $M \in \mathcal{A}_{\widehat{C}}(\widehat{R})$. Now the assertion follows from [9, Lemma 1.20] and Proposition 2.4. $\square$

Definition 2.12. Let R be a local ring with maximal ideal $\mathfrak{m}$ and let M be an R-module. For each integer i, the i-th Bass number of M is $\mu_R^i(M)=\mathrm{len}_R(\mathrm{Ext}_R^i(R/\mathfrak{m},M))$, where $\mathrm{len}_R(N)$ denotes the length of an R-module N.

Proposition 2.13. *Let C be a semidualizing R-module over local ring R and let M and N be R-modules such that $R/(\mathrm{Ann}_R(M)+\mathrm{Ann}_R(N))$ is complete and M is Artinian. For each $i\geqslant 0$ such that $\mu_R^i(N)<\infty$, the module $\mathrm{Ext}_R^i(M,N)$ is C-Matlis reflexive over R and $\widehat{C}$-Matlis reflexive over $\widehat{R}$, provided that $\mathrm{Ext}_R^i(M,N)\in\mathcal{A}_C(R)$.*

Proof. By Remark 2.10, $\mathrm{Ext}_R^i(M,N)\in\mathcal{A}_{\widehat{C}}(\widehat{R})$ for each $i\geqslant 0$. Now the assertion follows from [9, Corollary 2.4] and Proposition 2.4. □

Proposition 2.14. *Let C be a semidualizing R-module over local ring R and let M and N be R-modules such that N is Artinian and M is mini-max. For each $i\geqslant 0$, the module $\mathrm{Ext}_R^i(M,N)$ is $\widehat{C}$-Matlis reflexive over $\widehat{R}$, provided that $\mathrm{Ext}_R^i(M,N)\in\mathcal{A}_C(R)$.*

Proof. By Remark 2.10, $\mathrm{Ext}_R^i(M,N)\in\mathcal{A}_{\widehat{C}}(\widehat{R})$ for each $i\geqslant 0$. Now the assertion follows from [9, Theorem 2.6] and Proposition 2.4. □

Proposition 2.15. *Let C be a semidualizing R-module over local ring R and let M and N be R-modules. Then the following statements hold.*
(i) If M is mini-max and N is Noetherian such that $R/(\mathrm{Ann}_R(M)+\mathrm{Ann}_R(N))$ is complete, then $\mathrm{Ext}_R^{i\geqslant 0}(M,N)$ is C-Matlis reflexive over R and $\widehat{C}$-Matlis reflexive over $\widehat{R}$, provided that $\mathrm{Ext}_R^{i\geqslant 0}(M,N)\in\mathcal{A}_C(R)$.

(ii) If M and N are mini-max R-modules such that $R/(\mathrm{Ann}_R(M)+\mathrm{Ann}_R(N))$ is complete, then $\mathrm{Ext}_R^{i\geqslant 0}(M,N)$ is C-Matlis reflexive over R and $\widehat{C}$-Matlis reflexive over $\widehat{R}$, provided that $\mathrm{Ext}_R^{i\geqslant 0}(M,N)\in\mathcal{A}_C(R)$.

(iii) Let M and N be R-modules such that M is mini-max and N is Matlis reflexive. The modules $\mathrm{Ext}_R^{i\geqslant 0}(M,N)$ and $\mathrm{Ext}_R^{i\geqslant 0}(N,M)$ are C-Matlis reflexive over R and $\widehat{C}$-Matlis reflexive over $\widehat{R}$, provided that $\mathrm{Ext}_R^{i\geqslant 0}(N,M)$ and $\mathrm{Ext}_R^{i\geqslant 0}(M,N)$ belong to $\mathcal{A}_C(R)$.

Proof. By Remark 2.10, $\mathrm{Ext}_R^{i\geqslant 0}(M,N)\in\mathcal{A}_{\widehat{C}}(\widehat{R})$. In the light of Proposition 2.4, items (i)–(iii) follow from [9, Theorem 2.7], [9, Theorem 2.8], and [9, Corolary 2.9], respectively. □

Proposition 2.16. *Let C be a semidualizng R-module over local ring R and let M and N be R-modules such that M is Artinian and $R/(\mathrm{Ann}_R(M)+\mathrm{Ann}_R(N))$ is complete. Then for each $i\geqslant 0$ such that $\beta_i^R(N)<\infty$, the module $\mathrm{Tor}_i^R(M,N)$ is Artinian, C-Matlis reflexive over R and $\widehat{C}$-Matlis reflexive over $\widehat{R}$, provided that $\mathrm{Tor}_i^R(M,N)\in\mathcal{A}_C(R)$.*

Proof. By [9, Theorem 3.5], $\mathrm{Tor}_i^R(M,N)$ is a Matlis reflexive R-module for each $i\geqslant 0$. On the other hand, Remark 2.10 implies that $\mathrm{Tor}_i^R(M,N)\in\mathcal{A}_{\widehat{C}}(\widehat{R})$ for each $i\geqslant 0$. Now the assertion follows from Proposition 2.4. □

Proposition 2.17. *Let C be a semidualizing R-module over local ring R and let M and N be R-modules such that M and N are mini-max R-modules, and fix $i \geqslant 0$. If $R/(\mathrm{Ann}_R(M) + \mathrm{Ann}_R(N))$ is complete, then $\mathrm{Tor}_i^R(M, N)$ is C-Matlis reflexive over R and $\widehat{C}$-Matlis reflexive over $\widehat{R}$, provided that $\mathrm{Tor}_i^R(M, N) \in \mathcal{A}_C(R)$.*

Proof. By [9, Theorem 3.5], $\mathrm{Tor}_i^R(M, N)$ is a Matlis reflexive R-module for each $i \geqslant 0$. On the other hand, Remark 2.10 implies that $\mathrm{Tor}_i^R(M, N) \in \mathcal{A}_{\widehat{C}}(\widehat{R})$ for each $i \geqslant 0$. Now the assertion follows from Proposition 2.4. □

Proposition 2.18. *Let C be a semidualizing R-module over local ring R and let M and N be R-modules such that M is mini-max, N is Matlis reflexive, and $\mathrm{Tor}_{i \geqslant 0}^R(M, N) \in \mathcal{A}_C(R)$. Then the module $\mathrm{Tor}_{i \geqslant 0}^R(M, N)$ is C-Matlis reflexive over R and $\widehat{C}$-Matlis reflexive over $\widehat{R}$.*

Proof. By Remark 2.10, $\mathrm{Tor}_{i \geqslant 0}^R(M, N) \in \mathcal{A}_{\widehat{C}}(\widehat{R})$. Now the assertion follows from [9, Corollary 3.6] and Proposition 2.4. □

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