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GOLDEN STCR-LIGHTLIKE SUBMANIFOLDS OF GOLDEN SEMI-RIEMANNIAN MANIFOLDS

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Abstract. We introduce golden STCR-lightlike submanifolds of golden semi-Riemannian manifolds. We find new conditions for the induced connection to be a metric connection. Moreover, we find some necessary and sufficient conditions for such submanifolds.

1. Introduction

One can consider lightlike submanifolds of semi-Riemannian manifolds, whose study is important from the viewpoint of applications and challenging in the sense that the intersection of the normal vector bundle and the tangent bundle of these submanifolds is nonempty. This unique feature makes the study of lightlike submanifolds different from that of non-degenerate submanifolds. The theory of lightlike submanifolds is an important research topic in differential geometry due to its applications in mathematical physics, especially in general relativity. The study of this notion was initiated by Duggal and Bejancu [6] and has since been investigated by many authors (see recent results in the two books [8, 12]).

Duggal and Bejancu initiated the study of CR-lightlike submanifolds of indefinite Kähler manifolds [6]. However, this class of submanifolds excludes complex and totally real submanifolds as special cases. To overcome this limitation, Duggal and Şahin introduced screen Cauchy-Riemann (SCR)-lightlike submanifolds of indefinite Kähler manifolds [9], which include complex and totally real submanifolds. Nevertheless, there is no inclusion relation between CR- and SCR-lightlike submanifolds. Consequently, Duggal and Şahin introduced a new class called GCR-lightlike submanifolds of indefinite Kähler manifolds, which acts as an umbrella for real hypersurfaces, invariant, screen real, and CR-lightlike submanifolds [10]. These types of submanifolds have been studied in various manifolds by many authors [11, 16, 17, 19, 20, 22].

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However, CR-lightlike, screen CR-lightlike, and generalized CR-lightlike submanifolds do not contain real lightlike curves. For this reason, Şahin introduced screen transversal lightlike submanifolds of indefinite Kähler manifolds and showed that such submanifolds contain real lightlike curves [26]. These submanifolds have been studied in [15, 28, 29]. As a generalization of CR-lightlike submanifolds and screen transversal lightlike submanifolds, Doğan, Şahin, and Yaşar introduced screen transversal CR-lightlike submanifolds and studied the geometry of these lightlike submanifolds [5].

The number $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$, which is a solution of the equation $x^2 - x - 1 = 0$, is known as the golden ratio. The golden ratio is a fascinating topic that has continually generated new ideas. It has been used in many different areas, including architecture, music, art, and philosophy. Inspired by the golden ratio, Crâşmareanu and Hreţcanu defined golden Riemannian manifolds and studied their submanifolds in [3, 4]. In [27], Şahin and Akyol introduced golden maps between golden Riemannian manifolds and showed that such maps are harmonic. Gök, Keleş, and Kılıç studied several characterizations of submanifolds of golden Riemannian manifolds [14]. Poyraz and Yaşar introduced lightlike submanifolds of golden semi-Riemannian manifolds [24]. Erdoğan studied the geometry of screen transversal lightlike submanifolds, radical screen transversal lightlike submanifolds, and screen transversal anti-invariant lightlike submanifolds of golden semi-Riemannian manifolds [13]. Acet studied screen pseudo-slant lightlike submanifolds of golden semi-Riemannian manifolds [1]. Poyraz introduced golden GCR-lightlike submanifolds of golden semi-Riemannian manifolds [23].

In this paper, we introduce golden STCR-lightlike submanifolds of golden semi-Riemannian manifolds. We derive new conditions under which the induced connection is a metric connection. Moreover, we establish several necessary and sufficient conditions for the existence and characterization of such submanifolds.

2. Preliminaries

Let $\tilde{M}$ be a C^∞−differentiable manifold. If a tensor field $\tilde{P}$ of type $(1, 1)$ satisfies the following equation

$$\tilde{P}^2 = \tilde{P} + I \tag{1}$$

then $\tilde{P}$ is named a golden structure on $\tilde{M}$, where I is the identity transformation [18].

Let $(\tilde{M}, \tilde{g})$ be a semi-Riemannian manifold and $\tilde{P}$ be a golden structure on $\tilde{M}$. If $\tilde{P}$ holds the following equation

$$\tilde{g}(\tilde{P}X, Y) = \tilde{g}(X, \tilde{P}Y) \tag{2}$$

then $(\tilde{M}, \tilde{g}, \tilde{P})$ is named a golden semi-Riemannian manifold [25].

If $\tilde{P}$ be a golden Structure, then the equation (2) is equivalent with

$$\tilde{g}(\tilde{P}X, \tilde{P}Y) = \tilde{g}(\tilde{P}X, Y) + \tilde{g}(X, Y) \tag{3}$$

for any $X, Y \in \Gamma(T\tilde{M})$.

Let $(\tilde{M}, \tilde{g})$ be a real $(m+n)-$dimensional semi-Riemannian manifold of constant index q, such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an $m-$dimensional submanifold of $\tilde{M}$, where g is the induced metric of $\tilde{g}$ on M. If $\tilde{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\tilde{M}$. For a degenerate metric g on M

$$TM^{\perp} = \cup \left\{ u \in T_xM : \tilde{g}(u, v) = 0, \forall v \in T_x\tilde{M}, x \in M \right\}$$

is a degenerate $n-$dimensional subspace of $T_x\tilde{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\mathrm{Rad}(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) space. If the mapping $\mathrm{Rad}(TM) : x \in M \longrightarrow \mathrm{Rad}(T_xM)$, defines a smooth distribution, called radical distribution on M of rank $r > 0$ then the submanifold M of $\tilde{M}$ is called an $r-$lightlike submanifold.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM. This means that

$$TM = S(TM) \perp \mathrm{Rad}(TM) \tag{4}$$

and $S\left(TM^{\perp}\right)$ is a complementary vector subbundle to $\mathrm{Rad}(TM)$ in $TM^{\perp}$. Let $\mathrm{tr}(TM)$ and $\mathrm{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\tilde{M}_{|M}$ and $\mathrm{Rad}(TM)$ in $S\left(TM^{\perp}\right)^{\perp}$, respectively. Then we have

$$\mathrm{tr}(TM) = \mathrm{ltr}(TM) \perp S\left(TM^{\perp}\right), \tag{5}$$

$$T\tilde{M}\mid_M = TM \oplus \mathrm{tr}(TM) = \{\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)\} \perp S(TM) \perp S(TM^{\perp}). \tag{6}$$

Theorem 2.1 ([6]). *Let $(M, g, S(TM), S\left(TM^{\perp}\right))$ be an $r-$lightlike submanifold of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$. Suppose U is a coordinate neighborhood of M and $\{\xi_i\}$, $i \in \{1, \ldots, r\}$ is a basis of $\Gamma(\mathrm{Rad}\left(TM_{|U}\right))$. Then, there exist a complementary vector subbundle $\mathrm{ltr}(TM)$ of $\mathrm{Rad}(TM)$ in $S\left(TM^{\perp}\right)^{\perp}_{|U}$ and a basis $\{N_i\}$, $i \in \{1, \ldots, r\}$ of $\Gamma(\mathrm{ltr}(TM)_{|U})$ such that $\tilde{g}(N_i, \xi_j) = \delta_{ij}$, $\tilde{g}(N_i, N_j) = 0$ for any $i, j \in \{1, \ldots, r\}$.*

We say that a submanifold $(M, g, S(TM), S\left(TM^{\perp}\right))$ of $\tilde{M}$ is

Case 1. *$r-$lightlike* if $r < \min\{m, n\}$;

Case 2. *coisotropic* if $r = n < m$, $S\left(TM^{\perp}\right) = \{0\}$;

Case 3. *isotropic* if $r = m < n$, $S(TM) = \{0\}$;

Case 4. *totally lightlike* if $r = m = n$, $S(TM) = \{0\} = S\left(TM^{\perp}\right)$.

Let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$. Then, using (6), the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{7}$$

$$\tilde{\nabla}_X U = -A_U X + \nabla^t_X U, \tag{8}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(\mathrm{tr}(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(\mathrm{tr}(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $\mathrm{tr}(TM)$, respectively. According to (5), considering the projection morphisms L and S of $\operatorname{tr}(TM)$ on $\operatorname{ltr}(TM)$ and $S\left(TM^{\perp}\right)$, respectively, (7) and (8) become

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + h^l(X,Y) + h^s(X,Y), \\ \tilde{\nabla}_X N &= -A_N X + \nabla^l_X N + D^s(X,N), \\ \tilde{\nabla}_X W &= -A_W X + \nabla^s_X W + D^l(X,W), \end{aligned} \tag{9}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(\operatorname{ltr}(TM))$ and $W \in \Gamma(S\left(TM^{\perp}\right))$, where $h^l(X,Y) = Lh(X,Y)$, $h^s(X,Y) = Sh(X,Y)$, $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$, $\nabla^l_X N, D^l(X,W) \in \Gamma(\operatorname{ltr}(TM))$ and $\nabla^s_X W, D^s(X,N) \in \Gamma(S\left(TM^{\perp}\right))$. Then, by using (9) and taking into account that $\tilde{\nabla}$ is a metric connection we obtain

$$\begin{aligned} \tilde{g}(h^s(X,Y),W) + \tilde{g}(Y, D^l(X,W)) &= \tilde{g}(A_W X, Y), \\ \tilde{g}(D^s(X,N),W) &= \tilde{g}(A_W X, N). \end{aligned} \tag{10}$$

Let Q be a projection of TM on $S(TM)$. Then, using (4) we can write

$$\nabla_X QY = \nabla^*_X QY + h^*(X,QY), \tag{11}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{12}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\operatorname{Rad}(TM))$, where $\{\nabla^*_X QY, A^*_\xi X\}$ and $\{h^*(X,QY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(\operatorname{Rad}(TM))$, respectively.

Using the equations given above, we derive

$$\begin{aligned} &\tilde{g}(h^l(X,QY),\xi) = \tilde{g}(A^*_\xi X, QY), \\ &\tilde{g}(h^*(X,QY),N) = \tilde{g}(A_N X, QY), \quad \tilde{g}(h^l(X,\xi),\xi) = 0, \quad A^*_\xi \xi = 0. \end{aligned} \tag{13}$$

In general, the induced connection ∇ on M is not a metric connection. Since $\tilde{\nabla}$ is a metric connection, from (9) we obtain $(\nabla_X g)(Y,Z) = \tilde{g}(h^l(X,Y),Z) + \tilde{g}(h^l(X,Z),Y)$. However, ∇^* is a metric connection on $S(TM)$.

Definition 2.2. A lightlike submanifold (M,g) of a semi-Riemannian manifold $(\tilde{M},\tilde{g})$ is said to be an irrotational submanifold if $\tilde{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\operatorname{Rad}(TM))$ [21]. Thus M is an irrotational lightlike submanifold if and only if $h^l(X,\xi) = 0$, $h^s(X,\xi) = 0$.

Theorem 2.3 ([6]). *Let M be an r-lightlike submanifold of a semi-Riemannian manifold $\tilde{M}$. Then the induced connection ∇ is a metric connection if and only if* $\operatorname{Rad}(TM)$ *is a parallel distribution with respect to ∇.*

Definition 2.4 ([7]). A lightlike submanifold (M,g) of a semi-Riemannian manifold $(\tilde{M},\tilde{g})$ is said to be totally umbilical in $\tilde{M}$ if there is a smooth transversal vector field $H \in \Gamma(\operatorname{tr}(TM))$ on M, called the transversal curvature vector field of M, such that

$$h(X,Y) = H\tilde{g}(X,Y) \tag{14}$$

for any $X, Y \in \Gamma(TM)$. In case $H = 0$, M is called totally geodesic.

Using (9) and (14) it is easy to see that M is totally umbilical if and only if on each coordinate neighborhood U there exists smooth vector fields $H^l \in \Gamma(\operatorname{ltr}(TM))$ and $H^s \in \Gamma(S\left(TM^{\perp}\right))$ such that $h^l(X,Y) = \tilde{g}(X,Y)H^l$, $h^s(X,Y) = \tilde{g}(X,Y)H^s$ and $D^l(X,W) = 0$, for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S\left(TM^{\perp}\right))$.

3. Golden screen transversal Cauchy-Riemann (STCR)-lightlike submanifolds

Definition 3.1. Let M be a real r-lightlike submanifold of a golden semi-Riemannian manifold $\tilde{M}$. Then we say that M is a *screen transversal Cauchy-Riemann (STCR)-lightlike submanifold* if the following conditions are satisfied:
(A) There exist two subbundles D_1 and D_2 of $\mathrm{Rad}(TM)$ such that

$$\mathrm{Rad}(TM) = D_1 \oplus D_2, \quad \tilde{P}(D_1) \subset S(TM), \quad \tilde{P}(D_2) \subset S(TM^{\perp}).$$

(B) There exist two subbundles D_0 and D' of $S\,(TM)$ such that

$$S(TM) = \{\tilde{P}(D_1) \oplus D'\} \perp D_0, \quad \tilde{P}(D_0) = D_0, \quad \tilde{P}(L_1 \perp S) = D^{'},$$

where D_0 is a non-degenerate distribution on M, L_1 and S are vector subbundles of $\mathrm{ltr}\,(TM)$ and $S\left(TM^{\perp}\right)$, respectively.

With respect to the above definition, the tangent bundle TM of M is decomposed as $TM = D \oplus \bar{D}$, where $D = D_0 \oplus D_1 \oplus \tilde{P}(D_1)$ and $\bar{D} = D_2 \oplus \tilde{P}(L_1) \oplus \tilde{P}(S)$.

It is clear that D is invariant. But $\bar{D}$ is neither invariant nor anti-invariant. Furthermore, we have $\mathrm{ltr}(TM) = L_1 \oplus L_2$, $\tilde{P}(L_1) \subset S(TM)$, $\tilde{P}(L_2) \subset S(TM^{\perp})$ and $S(TM^{\perp}) = \{\tilde{P}(D_2) \oplus \tilde{P}(L_2)\} \perp S$.

If $D_1 \neq \{0\}$, $D_2 \neq \{0\}$, $D_0 \neq \{0\}$ and $S \neq \{0\}$, then M is called a proper golden screen transversal Cauchy-Riemann (STCR)-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$.

Proposition 3.2. *Let M be a golden STCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then M is a semi-invariant lightlike submanifold (respectively, screen transversal lightlike submanifold) if and only if $D_2 = \{0\}$ (respectively, $D_1 = \{0\}$).*

Proof. Let M be a semi-invariant lightlike submanifold of a golden semi-Riemannian manifold $\tilde{M}$. Then $\tilde{P}(\mathrm{Rad}(TM))$ is a distribution on M. Thus, we obtain $D_1 = \mathrm{Rad}(TM)$ and $D_2 = \{0\}$. Then it follows that $\tilde{P}(\mathrm{ltr}(TM)) \subset S(TM)$. Conversely, suppose that M is golden STCR-lightlike submanifold such that $D_2 = \{0\}$. Then, we have $D_1 = \mathrm{Rad}(TM)$. Thus, M is a semi-invariant lightlike submanifold, which completes the proof. Similarly other assertion follows. □

Proposition 3.3. *There exist no coisotropic, isotropic or totally lightlike proper golden STCR-lightlike submanifold M of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Any isotropic golden STCR-lightlike submanifold is a screen transversal lightlike submanifold. Also, a coisotropic golden STCR-lightlike submanifold is a semi-invariant lightlike submanifold.*

Proof. Suppose that M is a proper golden STCR-lightlike submanifold. From definition of proper golden STCR-lightlike submanifold, we know that $D_1 \neq \{0\}$, $D_2 \neq \{0\}$, $D_0 \neq \{0\}$ and $S \neq \{0\}$, that is both $S(TM)$ and $S(TM^{\perp})$ are non-zero. Hence, M can not be a coisotropic, isotropic or totally lightlike golden STCR-lightlike submanifold. On the other hand, if M is an isotropic golden STCR-lightlike submanifold, then $S(TM) = \{0\}$, i.e., $\tilde{P}(D_1) = \{0\}$ and $\mathrm{Rad}(TM) = D_2$. Hence, we obtain $\tilde{P}(\mathrm{Rad}(TM)) = \tilde{P}(D_2) \subset \Gamma(S(TM^\perp))$ and M is a screen transversal lightlike submanifold. Similarly, if M is a coisotropic golden STCR-lightlike submanifold, then $S(TM^\perp) = \{0\}$, i.e., $\tilde{P}(D_2) = \{0\}$ and $\mathrm{Rad}(TM) = D_1$. Since, $\tilde{P}(\mathrm{Rad}(TM)) = \tilde{P}(D_1) \subset \Gamma(S(TM))$ then M is a semi-invariant lightlike submanifold. □

Let M be a golden STCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Thus, for any $X \in \Gamma(TM)$ we derive

$$\tilde{P}X = PX + wX, \tag{15}$$

where PX and wX are tangential and transversal parts of $\tilde{P}X$.

For $V \in \Gamma(\mathrm{tr}(TM))$ we write

$$\tilde{P}V = BV + CV, \tag{16}$$

where BV and CV are tangential and transversal parts of $\tilde{P}V$.

Lemma 3.4. *Let M be a golden STCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, one has*

$$P^2X = PX + X - BwX, \quad wPX = wX - CwX, \tag{17}$$

$$PBV = BV - BCV, \quad C^2V = CV + V - wBV, \tag{18}$$

$$\tilde{g}(PX, Y) - \tilde{g}(X, PY) = \tilde{g}(X, wY) - \tilde{g}(wX, Y), \tag{19}$$

$$\begin{aligned}\tilde{g}(PX, PY) &= \tilde{g}(PX, Y) + \tilde{g}(X, Y) + \tilde{g}(wX, Y) - \tilde{g}(PX, wY) \\ &\quad - \tilde{g}(wX, PY) - \tilde{g}(wX, wY)\end{aligned} \tag{20}$$

for any $X, Y \in \Gamma(TM)$.

Proof. If we apply $\tilde{P}$ to (15), using (1) and taking tangential and transversal parts of the resulting equation, we obtain (17). Similarly, applying $\tilde{P}$ in (16) and using (1) we get (18). From (2), (3) and (15), we derive (19) and (20). □

Theorem 3.5. *Let M be a golden STCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, P is golden structure on D.*

Proof. By the definition of a golden STCR-lightlike submanifold, we have $wX = 0$ for any $X \in \Gamma(D)$. From (17), it follows that $P^2X = PX + X$. Thus, P is a golden structure on D. □

Example 3.6. Let $(\tilde{M} = \mathbb{R}^{12}_4, \tilde{g})$ be a 12-dimensional semi-Euclidean space with signature $(-,-,+,+,-,-,+,+,+,+,+,+)$ and $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12})$ be the standard coordinate system of $\mathbb{R}^{12}_4$. If we define a mapping $\tilde{P}$ by

$$\begin{aligned}&\tilde{P}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) \\ =&(x_1 + x_2, x_1, x_3 + x_4, x_3, x_5 + x_6, x_5, x_7 + x_8, x_7, x_9 + x_{10}, x_9, x_{11} + x_{12}, x_{11})\end{aligned}$$

then $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\mathbb{R}^{12}_4$. Let M be a submanifold of $\tilde{M}$ given by

$$x_1 = u_3 - \frac{1}{2}u_4,\ x_2 = u_1,\ x_3 = 0,\ x_4 = u_2,\ x_5 = u_3 + \frac{1}{2}u_4,\ x_6 = u_1,\ x_7 = 0,$$

$x_8 = u_2,\ x_9 = u_5 + u_6 - u_7,\ x_{10} = u_6,\ x_{11} = u_5 + u_6 + u_7,\ x_{12} = u_6,$

where u_i, $1 \leq i \leq 7$, are real parameters. Thus $TM = \mathrm{Span}\{U_1, U_2, U_3, U_4, U_5, U_6, U_7\}$, where

$$U_1 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6}, \quad U_2 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_8}, \quad U_3 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5},$$
$$U_4 = \frac{1}{2}(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}), \quad U_5 = \frac{\partial}{\partial x_9} + \frac{\partial}{\partial x_{11}},$$
$$U_6 = \frac{\partial}{\partial x_9} + \frac{\partial}{\partial x_{10}} + \frac{\partial}{\partial x_{11}} + \frac{\partial}{\partial x_{12}}, \quad U_7 = -\frac{\partial}{\partial x_9} + \frac{\partial}{\partial x_{11}}.$$

Then M is a $2-$lightlike submanifold of $\mathbb{R}^{12}_4$ with $\mathrm{Rad}(TM) = \mathrm{Span}\{U_1, U_2\}$. It is easy to see $\tilde{P}U_1 = U_3$, thus $D_1 = \mathrm{Span}\{U_1\}$ and $D_2 = \mathrm{Sp}\{U_2\}$. On the other hand, since $\tilde{P}U_5 = U_6 \in \Gamma(S(TM))$, we derive $D_0 = \mathrm{Sp}\{U_5, U_6\}$. We can easily obtain

$$\mathrm{ltr}(TM) = \mathrm{Span}\{N_1 = \frac{1}{2}(-\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6}), N_2 = \frac{1}{2}(-\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_8})\} \quad \text{and}$$

$$S(TM^{\perp}) = \mathrm{Span}\{W_1 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_7}, W_2 = \frac{1}{2}(-\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_7}), W_3 = -\frac{\partial}{\partial x_{10}} + \frac{\partial}{\partial x_{12}}\}.$$

Since $\tilde{P}U_2{=}W_1$, $\tilde{P}N_1{=}U_4$, $\tilde{P}N_2{=}W_2$, $\tilde{P}W_3{=}U_7$, then $L_1{=}\mathrm{Sp}\{N_1\}$, $L_2{=}\mathrm{Sp}\{N_2\}$ and $S{=}\mathrm{Sp}\{W_3\}$. Hence, M is a proper golden STCR lightlike submanifold of $\mathbb{R}^{12}_4$.

Theorem 3.7. *Let M be a golden STCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the induced connection is a metric connection if and only if for any $X \in \Gamma(TM)$, the following hold*

$$P(\nabla^*_X \tilde{P}Y + h^*(X, \tilde{P}Y)) - \nabla^*_X \tilde{P}Y - h^*(X, \tilde{P}Y) \in \Gamma(\mathrm{Rad}(TM))$$
$$Bh(X, \tilde{P}Y) = 0, \quad Y \in \Gamma(D_1),$$
$$-PA_{\tilde{P}Y}X + A_{\tilde{P}Y}X + B\nabla^s_X \tilde{P}Y \in \Gamma(\mathrm{Rad}(TM)),\ BD^l(X, \tilde{P}Y) = 0,\ Y \in \Gamma(D_2).$$

Proof. Since $\tilde{P}$ is the golden structure of $\tilde{M}$, we have$\tilde{\nabla}_X Y = \tilde{P}\tilde{\nabla}_X \tilde{P}Y - \tilde{\nabla}_X \tilde{P}Y$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(\mathrm{Rad}(TM))$. Since $\mathrm{Rad}(TM) = D_1 \oplus D_2$, using (7), (11), (15), (16) and taking the tangential part of the resulting equation, we obtain

$$\nabla_X Y = P(\nabla^*_X \tilde{P}Y + h^*(X, \tilde{P}Y)) - \nabla^*_X \tilde{P}Y - h^*(X, \tilde{P}Y) + Bh(X, \tilde{P}Y) \tag{21}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_1)$. Thus from (21) we obtain $\nabla_X Y \in \Gamma(\mathrm{Rad}(TM))$ if and only if

$$P(\nabla^*_X \tilde{P}Y + h^*(X, \tilde{P}Y)) - \nabla^*_X \tilde{P}Y - h^*(X, \tilde{P}Y) \in \Gamma(\mathrm{Rad}(TM)),\ Bh(X, \tilde{P}Y) = 0 \tag{22}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_1)$. Using (7), (12), (15) and (16) we derive

$$\nabla_X Y = -PA_{\tilde{P}Y}X + A_{\tilde{P}Y}X + B\nabla^s_X \tilde{P}Y + BD^l(X, \tilde{P}Y) \tag{23}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_2)$. From (23) we get $\nabla_X Y \in \Gamma(\mathrm{Rad}(TM))$ if and only if for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_2)$

$$\begin{aligned} &- PA_{\tilde{P}Y}X + A_{\tilde{P}Y}X + B\nabla^s_X \tilde{P}Y \in \Gamma(\mathrm{Rad}(TM)), \\ &BD^l(X, \tilde{P}Y) = 0 \quad \text{and} \quad BD^l(X, \tilde{P}Y) = 0. \end{aligned} \tag{24}$$

Then considering Theorem 2.3, the proof follows from (22) and (24). □

THEOREM 3.8. *Let M be a golden STCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, the distribution D is integrable if and only if $h^s(X, \tilde{P}Y) - h^s(Y, \tilde{P}X) \in \Gamma(\tilde{P}(L_2))$ and $h^l(X, \tilde{P}Y) - h^l(Y, \tilde{P}X) \in \Gamma(L_2)$ for any $X, Y \in \Gamma(D)$.*

Proof. From definition of golden STCR-lightlike submanifolds, D is integrable if and only if$\tilde{g}([X,Y], N_2) = \tilde{g}([X,Y], \tilde{P}\xi_1) = \tilde{g}([X,Y], \tilde{P}W) = 0$ for any $X, Y \in \Gamma(D)$, $\xi_1 \in \Gamma(D_1)$, $N_2 \in \Gamma(L_2)$ and $W \in \Gamma(S)$. Hence from (2), (3) and (9) we derive

$$\begin{aligned}
\tilde{g}([X,Y], N_2) &= \tilde{g}(h^s(X, \tilde{P}Y) - h^s(Y, \tilde{P}X), \tilde{P}N_2),\\
\tilde{g}([X,Y], \tilde{P}\xi_1) &= \tilde{g}(h^l(X, \tilde{P}Y) - h^l(Y, \tilde{P}X), \xi_1), \qquad (25)\\
\tilde{g}([X,Y], \tilde{P}W) &= \tilde{g}(h^s(X, \tilde{P}Y) - h^s(Y, \tilde{P}X), W).
\end{aligned}$$

Equation (25) implies the theorem. □

THEOREM 3.9. *Let M be a golden STCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, the distribution D is integrable if and only if $h(X, \tilde{P}Y) = h(\tilde{P}X, Y)$.*

Proof. Equations (7), (15), and (16) yield $w\nabla_X Y = h(X, \tilde{P}Y) - Ch(X, Y)$ for any $X, Y \in \Gamma(D)$. Consequently, $w[X,Y] = h(X, \tilde{P}Y) - h(Y, \tilde{P}X)$, establishing the assertion. □

THEOREM 3.10. *Let M be a golden STCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, the distribution D defines a totally geodesic foliation in M if and only if $Bh(X, Y) = 0$ for any $X, Y \in \Gamma(D)$.*

Proof. Since D is invariant, if $X \in \Gamma(D)$, then $\tilde{P}X \in \Gamma(D)$. From the definition of golden STCR-lightlike submanifolds, D defines a totally geodesic foliation in M if and only if $g(\nabla_X Y, \tilde{P}\xi_1) = \tilde{g}(\nabla_X \tilde{P}Y, N_2) = g(\nabla_X Y, \tilde{P}W) = 0$ for any $X, Y \in \Gamma(D)$, $\xi_1 \in \Gamma(D_1)$, $N_2 \in \Gamma(L_2)$ and $W \in \Gamma(S)$. Using (2) and (9) we derive

$$\begin{aligned}
g(\nabla_X Y, \tilde{P}\xi_1) &= \tilde{g}(\tilde{\nabla}_X Y, \tilde{P}\xi_1) = \tilde{g}(h^l(X, \tilde{P}Y), \xi_1),\\
g(\nabla_X \tilde{P}Y, N_2) &= \tilde{g}(\tilde{\nabla}_X \tilde{P}Y, N_2) = \tilde{g}(\tilde{\nabla}_X Y, \tilde{P}N_2) = \tilde{g}(h^s(X, Y), \tilde{P}N_2), \qquad (26)\\
g(\nabla_X Y, \tilde{P}W) &= \tilde{g}(\tilde{\nabla}_X Y, W) = \tilde{g}(\tilde{\nabla}_X Y, \tilde{P}W) = \tilde{g}(h^s(X, \tilde{P}Y), W).
\end{aligned}$$

It follows from (26) that D defines a totally geodesic foliation in M if and only if $h^l(X, \tilde{P}Y)$ has no components in L_1 and $h^s(X, Y)$ has no components in L_2 for any $X, Y \in \Gamma(D)$, i.e., from (16), $Bh(X, Y) = 0$ for any $X, Y \in \Gamma(D)$. □

DEFINITION 3.11. A golden STCR-lightlike submanifold of a golden semi-Riemannian manifold is called $D-$geodesic golden STCR-lightlike submanifold if its second fundamental form h satisfies $h(X, Y) = 0$ for any $X, Y \in \Gamma(D)$.

THEOREM 3.12. *Let M be a golden STCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the distribution D defines a totally geodesic foliation in $\tilde{M}$ if and only if M is D-geodesic and D is parallel respect to ∇ on M.*

Proof. Assume that D defines a totally geodesic foliation in $\tilde{M}$, then $\tilde{\nabla}_X Y \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$. Hence from (9), we derive$\tilde{g}(h^l(X,Y),\xi) = \tilde{g}(\tilde{\nabla}_X Y, \xi) = 0$, $\tilde{g}(h^s(X,Y),W) = \tilde{g}(\tilde{\nabla}_X Y, W) = 0$ for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(\mathrm{Rad}(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Hence $h^l(X,Y) = h^s(X,Y) = 0$, which implies that M is $D-$geodesic and D is parallel respect to ∇ on M.

Conversely, we suppose that M is $D-$geodesic and D is parallel respect to ∇ on M. Since $h^l(X,Y) = h^s(X,Y) = 0$ for any $X, Y \in \Gamma(D)$, then $\tilde{\nabla}_X Y \in \Gamma(TM)$. On the other hand, since D is parallel on M, considering (9), we have $\tilde{\nabla}_X Y \in \Gamma(D)$. □

THEOREM 3.13. *Let M be a golden STCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the distribution D defines a totally geodesic foliation in $\tilde{M}$ if and only if for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(RadTM)$, $W \in \Gamma(S(TM^{\perp}))$ $g(A^*_\xi X, \tilde{P}Y) = \tilde{g}(h(X,\xi), \tilde{P}Y)$ and $g(A_W X, \tilde{P}Y) = \tilde{g}(Y, D^l(X,W))$.*

Proof. Using that $\tilde{\nabla}$ is a metric connection and (7), (8), (10) and (13) we obtain

$$\tilde{g}(h(X,\tilde{P}Y),\xi)=\tilde{g}(\tilde{\nabla}_X \tilde{P}Y,\xi)=-\tilde{g}(\tilde{\nabla}_X \xi,\tilde{P}Y)=g(A^*_\xi X,\tilde{P}Y)-\tilde{g}(h(X,\xi),\tilde{P}Y),$$

$$\tilde{g}(h(X,\tilde{P}Y),W)=\tilde{g}(\tilde{\nabla}_X \tilde{P}Y,W)=-\tilde{g}(\tilde{\nabla}_X W,\tilde{P}Y)=g(A_W X,\tilde{P}Y)-\tilde{g}(\tilde{P}Y,D^l(X,W)),$$

for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(\mathrm{Rad}\,TM)$, $W \in \Gamma(S(TM^{\perp}))$. From the preceding equations, we obtain the result. □

DEFINITION 3.14. A golden STCR-lightlike submanifold of a golden semi-Riemannian manifold is called mixed geodesic golden STCR-lightlike submanifold if its second fundamental form h satisfies $h(X,Y) = 0$ for any $X \in \Gamma(D)$, $Y \in \Gamma(\bar{D})$.

THEOREM 3.15. *Let M be a golden STCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the distribution M is mixed geodesic if and only if $A_{\tilde{P}Z} X \in \Gamma(D)$ and $\nabla^t_X \tilde{P}Z = 0$ for any $X \in \Gamma(D)$, $Z \in \Gamma(\tilde{D})$.*

Proof. From (7) and (8) we obtain

$$h(X,Z) = \tilde{P}(-A_{\tilde{P}Z} X + \nabla^t_X \tilde{P}Z) + A_{\tilde{P}Z} X - \nabla^t_X \tilde{P}Z - \nabla_X Z$$

for any $X \in \Gamma(D)$, $Z \in \Gamma(\bar{D})$. Then, using (15), (16) and transversal part we get

$$h(X,Z) = -wA_{\tilde{P}Z} X + C\nabla^t_X \tilde{P}Z - \nabla^t_X \tilde{P}Z = 0,$$

which completes the proof. □

4. Minimal golden STCR-lightlike submanifolds

DEFINITION 4.1 ([2]). We say that a lightlike submanifold $(M, g, S(TM))$ of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ is minimal if:

(i) $h^s = 0$ on $\mathrm{Rad}(TM)$ and

(ii) $\mathrm{trace}\, h = 0$, where trace is written with respect to g restricted to $S(TM)$.

It has been shown in [2] that the above definition is independent of $S(TM)$ and $S(TM^{\perp})$, but it depends on $\mathrm{tr}(TM)$.

Theorem 4.2. *Let M be a totally umbilical golden STCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then M is minimal if and only if M is totally geodesic.*

Proof. Suppose that M is minimal then $h^s(X, Y) = 0$ for any $X, Y \in \Gamma(\mathrm{Rad}(TM))$. Since M is totally umbilical then $h^l(X, Y)=H^l g(X, Y)=0$ for any $X, Y\in\Gamma(\mathrm{Rad}(TM))$. Now, choose an orthonormal basis $\{e_1, e_2, \ldots, e_{m-r}\}$ of $S(TM)$. From (14), we get

$$\operatorname{trace} h(e_i, e_i) = \sum_{i=1}^{m-r} \epsilon_i h^l(e_i, e_i) + \epsilon_i h^s(e_i, e_i) = \epsilon_i(m-r)H^l + \epsilon_i(m-r)H^s.$$

Since M is minimal and $\mathrm{ltr}(TM) \cap S(TM^\perp) = \{0\}$, we obtain $H^l = 0$ and $H^s = 0$. Therefore M is totally geodesic. The converse is clear. □

Theorem 4.3. *Let M be a totally umbilical golden STCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then M is minimal if and only if $\operatorname{trace} A_{W_p} = 0$ and $\operatorname{trace} A^*_{\xi_k} = 0$ on $D_0 \perp \tilde{P}(S)$ for $W_p \in \Gamma(S(TM^\perp))$, where $k \in \{1, 2, \ldots, r\}$ and $p \in \{1, 2, \ldots, n-r\}$.*

Proof. By the definition of a golden STCR-lightlike submanifold, M is minimal if and only if

$$\operatorname{trace} h\,|_{S(TM)}=\sum_{i=1}^{a} h(Z_i, Z_i)+\sum_{j=1}^{b} h(\tilde{P}\xi_j, \tilde{P}\xi_j)+\sum_{j=1}^{b} h(\tilde{P}N_j, \tilde{P}N_j)+\sum_{l=1}^{c} h(\tilde{P}W_l, \tilde{P}W_l), \tag{27}$$

and $h^s = 0$ on $\mathrm{Rad}(TM)$, where $a = dim(D_0)$, $b = dim(D_2)$ and $c = dim(S)$. Since M is totally umbilical then from (14), we derive $h(\tilde{P}\xi_j, \tilde{P}\xi_j) = h(\tilde{P}N_j, \tilde{P}N_j) = 0$. Similarly, $h^s = 0$ on $\mathrm{Rad}(TM)$. Thus from (27) becomes

$$\begin{aligned}
\operatorname{trace} h\,|_{S(TM)} &= \sum_{i=1}^{a} h(Z_i, Z_i) + \sum_{l=1}^{c} h(\tilde{P}W_l, \tilde{P}W_l) \\
&= \sum_{i=1}^{a} \frac{1}{r} \sum_{k=1}^{r} \tilde{g}(h^l(Z_i, Z_i), \xi_k)N_k + \sum_{i=1}^{a} \frac{1}{n-r} \sum_{p=1}^{n-r} \tilde{g}(h^s(Z_i, Z_i), W_p)W_p \\
&\quad + \sum_{l=1}^{c} \frac{1}{r} \sum_{k=1}^{r} \tilde{g}(h^l(\tilde{P}W_l, \tilde{P}W_l), \xi_k)N_k \\
&\quad + \sum_{l=1}^{c} \frac{1}{n-r} \sum_{p=1}^{n-r} \tilde{g}(h^s(\tilde{P}W_l, \tilde{P}W_l), W_p)W_p,
\end{aligned} \tag{28}$$

where $\{W_1, W_2, \ldots, W_{n-r}\}$ is an orthonormal basis of $S(TM^\perp)$. Using (10) and (13) in (28), we obtain

$$\begin{aligned}
\operatorname{trace} h\,|_{S(TM)} &= \sum_{i=1}^{a} \frac{1}{r} \sum_{k=1}^{r} \tilde{g}(A^*_{\xi_k} Z_i, Z_i)N_k + \sum_{i=1}^{a} \frac{1}{n-r} \sum_{p=1}^{n-r} \tilde{g}(A_{W_p} Z_i, Z_i)W_p \\
&\quad + \sum_{l=1}^{c} \frac{1}{r} \sum_{k=1}^{r} \tilde{g}(A^*_{\xi_k} \tilde{P}W_l, \tilde{P}W_l)N_k + \sum_{l=1}^{c} \frac{1}{n-r} \sum_{p=1}^{n-r} \tilde{g}(A_{W_p} \tilde{P}W_l, \tilde{P}W_l)W_p.
\end{aligned}$$

Thus trace $h\mid_{S(TM)}=0$ if and only if trace $A_{W_p}=0$ and trace $A^*_{\xi_k}=0$ on $D_0\perp\tilde{P}(S)$. This completes the proof. □

THEOREM 4.4. *Let M be an irrotational golden STCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then M is minimal if and only if* trace $A^*_{\xi_k}\mid_{S(TM)}=0$ *and* trace $A_{W_p}\mid_{S(TM)}=0$*, where $W_p\in\Gamma(S(TM^\perp))$, $k\in\{1,2,\ldots,r\}$ and $p\in\{1,2,\ldots,n-r\}$.*

Proof. Since M is irrotational, then $h^s(X,\xi)=0$ for any $X\in\Gamma(TM)$ and $\xi\in\Gamma(\mathrm{Rad}(TM))$. Thus $h^s=0$ on $\mathrm{Rad}(TM)$. Moreover, we have

$$\begin{aligned}\operatorname{trace} h\mid_{S(TM)}&=\sum_{i=1}^{m-r}\epsilon_i\{h^l(e_i,e_i)+h^s(e_i,e_i)\}\\&=\sum_{i=1}^{m-r}\epsilon_i\{\frac{1}{r}\sum_{k=1}^{r}\tilde{g}(h^l(e_i,e_i),\xi_k)N_k+\frac{1}{n-r}\sum_{p=1}^{n-r}\tilde{g}(h^s(e_i,e_i),W_p)W_p\},\end{aligned}\quad(29)$$

where $\{W_1,W_2,\ldots,W_{n-r}\}$ is an orthonormal basis of $S(TM^\perp)$. Substituting (10) and (13) into (29), we obtain

$$\operatorname{trace} h\mid_{S(TM)}=\sum_{i=1}^{m-r}\epsilon_i\{\frac{1}{r}\sum_{k=1}^{r}\tilde{g}(A^*_{\xi_k}e_i,e_i)N_k+\frac{1}{n-r}\sum_{p=1}^{n-r}\tilde{g}(A_{W_p}e_i,e_i)W_p\}.$$

Thus the proof is completed. □

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