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$\mathcal{I}^{\mathcal{K}}$-LIMIT POINTS, $\mathcal{I}^{\mathcal{K}}$-CLUSTER POINTS AND $\mathcal{I}^{\mathcal{K}}$-FRÉCHET COMPACTNESS

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Abstract. Notions of $\mathcal{I}^{\mathcal{K}}$-limit points and $\mathcal{I}^{\mathcal{K}}$-cluster points of functions are studied in topological spaces. In the first countable space, all $\mathcal{I}^{\mathcal{K}}$-cluster points of a function $f : S \to X$ belong to the closure of each member of the filter base $\mathcal{B}_f(\mathcal{I}^{\mathcal{K}})$. Fréchet compactness is studied in the light of ideals $\mathcal{I}$ and $\mathcal{K}$ of subsets of S and showed that in an $\mathcal{I}$-sequential Hausdorff space, Fréchet compactness and $\mathcal{I}$-Fréchet compactness are equivalent. Using the FDS-property introduced by D. Shakmatov, M. Tkachenko, R. Wilson in Houston J. Math., it is seen that $\mathcal{I}^{\mathcal{K}}$-Fréchet compactness, Fréchet compactness and $\mathcal{I}$-Fréchet compactness are equivalent for a particular class of ideals on S.

1. Introduction

During last two decades, statistical convergence [6,17] has proudly achieved its place into the theory of convergence. As an extension of statistical convergence, theory of ideal convergence of real numbers was introduced by P. Kostyrko, T. Šalát, W. Wilczyński [9]. A significant development of the theory of statistical and ideal convergence have been made by a number of mathematicians, few of which are J. A. Fridy [7], G. Di. Maio, L. D. R. Kočinac [14], P. Kostyrko, T. Šalát, W. Wilczyński [9], B. K. Lahiri, P. Das [11], A. K. Banerjee, A. Banerjee [1] etc.. An ideal $\mathcal{I}$ on an arbitrary set S is a family $\mathcal{I} \subset 2^S$ that is closed under finite unions and taking subsets [10]. An ideal $\mathcal{I}$ is called trivial if $\mathcal{I} = \{\emptyset\}$ or $S \in \mathcal{I}$. A non-trivial ideal $\mathcal{I} \subset 2^S$ is called admissible if it contains all the singleton sets [10].

The ideal Fin is the class of all finite subsets of $\mathbb{N}$. Various examples of non-trivial admissible ideals are given in [9]. A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space X is said to be $\mathcal{I}$-convergent to $\alpha \in X$ ($\alpha = \mathcal{I} - \lim_{n\to\infty} x_n$) if for any open set U containing α, $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ [11]. Also in the theory of statistical convergence, the following result of T. Salát [16] and J. A. Fridy [7] is well-known. A sequence of real numbers is statistically convergent to x if and only if there exists a

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set $M \subset \mathbb{N}$ with $\delta(M) = 1$, asymptotic density or natural density of M [15] such that the corresponding subsequence converges to x. Influenced by this result, the concept of $\mathcal{I}^*$-convergence [11] was introduced. A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space X is said to be $\mathcal{I}^*$-convergent to $\xi \in X$ if there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I})$ (i.e. $\mathbb{N} \setminus M \in \mathcal{I}$) such that $\lim_{k\to\infty} x_{m_k} = \xi$. For any admissible ideal on $\mathbb{N}$, $\mathcal{I}^*$-convergence always implies $\mathcal{I}$-convergence but the converse may not be true. A class of ideals (property AP) was defined in [9] and showed that the notions $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence are equivalent if and only if the ideal $\mathcal{I}$-satisfies the property AP.

The notion of $\mathcal{I}$-limit point and $\mathcal{I}$-cluster point in a metric space (X, d) were defined by P. Kostyrko, T. Šalát, W. Wilczyński [9]. An element $\xi \in X$ is said to be an $\mathcal{I}$-limit point of (x_n) provided that there is a set $M = \{m_1 < m_2 < \ldots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim\limits_{k\to\infty} x_{m_k} = \xi$. An element $\xi \in X$ is said to be an $\mathcal{I}$-cluster point of (x_n) if for each $\epsilon > 0$, $\{n \in \mathbb{N} : d(x_n, \xi) < \epsilon\} \notin \mathcal{I}$.

B. K. Lahiri, P. Das [11] generalized these concepts in topological spaces and characterized $\mathcal{I}(C_x)$, the collection of all $\mathcal{I}$-cluster points of a sequence $x = (x_n)$ in a topological space X, as closed subsets of X (see [11, Theorem 10]). Also for any ideal $\mathcal{I}$ on $\mathbb{N}$, the collection $\mathcal{I}(L_x)$ of all $\mathcal{I}$-limit points of x is a subset of $\mathcal{I}(C_x)$ (see [11, Theorem 9]).

In 2011, the $\mathcal{I}^{\mathcal{K}}$-convergence of function in a topological space was introduced by Mačaj and Sleziak [13] as a generalization of $\mathcal{I}^*$-convergence of function. Suppose $\mathcal{I}$ is an ideal on a nonempty set S and X is a topological space. A function $f : S \to X$ is said to be $\mathcal{I}$-convergent to $l \in X$ if $f^{-1}(W) = \{s \in S : f(s) \in W\} \in \mathcal{F}(\mathcal{I})$ holds for every neighbourhood W of the point l [13].

A function f from S into a topological space X is said to be $\mathcal{I}^*$-convergent to some point $a \in X$ if there exists a set $K \in \mathcal{F}(\mathcal{I})$ such that the function $g : S \to X$ defined by $g(s) = f(s)$ if $s \in K$ and $g(s) = a$ if $s \in S \setminus K$ is Fin-convergent to a [13]. Whenever $S = \mathbb{N}$, $\mathcal{I}^*$-convergence of functions coincide with $\mathcal{I}^*$-convergence of sequences as a special case. $\mathcal{I}^{\mathcal{K}}$-convergence is defined by replacing Fin by an arbitrary ideal $\mathcal{K}$ on S. A function $f : S \to X$ is said to be $\mathcal{I}^{\mathcal{K}}$-convergent to some point $a \in X$ if there exists a set $K \in \mathcal{F}(\mathcal{I})$ such that the function $g : S \to X$ defined by $g(s) = f(s)$ if $s \in K$ and $g(s) = a$ if $s \in S \setminus K$ is $\mathcal{K}$-convergent to a [13]. In [4], they investigated the relation between $\mathcal{I}^{\mathcal{K}}$-convergence and $\mathcal{I}$-convergence and also find the condition under which these two convergence are equivalent.

The notion of $\mathcal{I}^{\mathcal{K}}$-limit point and $\mathcal{I}^{\mathcal{K}}$-cluster point of a function in a topological space X were studied in [19]. Let $f : S \to X$ be a function and $\mathcal{I}$, $\mathcal{K}$ be ideals on S. A point $x \in X$ is called an $\mathcal{I}^{\mathcal{K}}$-limit point of f if there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that for the function $g : S \to X$ defined by $g/_M = f/_M$ and $g[S \setminus M] = \{x\}$ has a $\mathcal{K}$-limit point x. A point $x \in X$ is called an $\mathcal{I}^{\mathcal{K}}$-cluster point of f if there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that for the function $g : S \to X$ defined by $g/_M = f/_M$ and $g[S \setminus M] = \{x\}$ has a $\mathcal{K}$-cluster point x, i.e., for any open set U containing x, $\{s \in S : g(s) \in U\} \notin \mathcal{K}$.

The collection of all $\mathcal{I}^{\mathcal{K}}$-limit points and $\mathcal{I}^{\mathcal{K}}$-cluster points of a function f in a topological space X is denoted by $L_f(\mathcal{I}^{\mathcal{K}})$ and $C_f(\mathcal{I}^{\mathcal{K}})$ respectively. For admissible ideals $\mathcal{I}$ and $\mathcal{K}$ on S, $L_f(\mathcal{I}^{\mathcal{K}}) \subset C_f(\mathcal{I}^{\mathcal{K}})$ [19].

2. Properties of $\mathcal{I}^{\mathcal{K}}$-limit points and $\mathcal{I}^{\mathcal{K}}$-cluster points

Let's consider two ideals $\mathcal{I} = \{A : A \cap (\mathbb{N} \setminus 4\mathbb{N}) \text{ is finite}\}$ and $\mathcal{K} = \{A : A \cap (\mathbb{N} \setminus 4\mathbb{N}+1) \text{ is finite}\}$ on $\mathbb{N}$. Consider the sequence $x = (x_n)$ in $\mathbb{R}$ defined by $x_n = n$ if $n \in 4\mathbb{N}$, $x_n = 0$ if $n \in 4\mathbb{N}+1$, $x_n = 2n$ if $n \in 4\mathbb{N}+2$ and $x_n = 3n$ if $n \in 4\mathbb{N}+3$. Then for an open set $(-\frac{1}{2}, \frac{1}{2})$ containing 0, $\{n \in \mathbb{N} : x_n \in (-\frac{1}{2}, \frac{1}{2})\} = 4\mathbb{N}+1 \in \mathcal{K}$. Therefore $0 \notin C_x(\mathcal{K})$. Now consider $\mathbb{N} \setminus 4\mathbb{N} \in \mathcal{F}(\mathcal{I})$ and a sequence (y_n) defined by $y_n = x_n$ if $n \notin 4\mathbb{N}$ and $y_n = 0$ if $n \in 4\mathbb{N}$. For any open set U_0 containing 0, $\{n \in \mathbb{N} : y_n \in U_0\} \supset 4\mathbb{N}$. Since $4\mathbb{N} \notin \mathcal{K}$, $\{n \in \mathbb{N} : y_n \in U_0\} \notin \mathcal{K}$. Therefore $0 \in C_x(\mathcal{I}^{\mathcal{K}})$.

So $C_x(\mathcal{I}^{\mathcal{K}}) \not\subset C_x(\mathcal{K})$, even if $\mathcal{I} \cup \mathcal{K}$ is a proper ideal.

As a consequence of this non inclusiveness, the truth of [19, Theorem 4.6 (ii)] is questionable, though no doubt about [19, Theorem 4.6 (i)] which can be translated in the language of $\mathcal{I}^{\mathcal{K}}$-convergence of functions as follows: *For any function $f : S \to X$ and ideals $\mathcal{I}$, $\mathcal{K}$ on S, $C_f(\mathcal{I}^{\mathcal{K}})$ is closed.*

Let X be a completely separable space and C be any non-empty closed subset of it. Assume that there exists a pairwise disjoint sequence of sets $\{T_p\}$ such that $T_p \subset S$, $T_p \notin \mathcal{K}$ for all $p \in \mathbb{N}$. Being a closed subset of a completely separable space C is separable and let $A = \{c_1, c_2, \ldots\}$ be a countable dense subset of C. Define a function $f : S \to X$ as $f(s) = c_i$ if $s \in T_i$ and b elsewhere, where $b \in C$.

Let $p \in C_f(\mathcal{I}^{\mathcal{K}})$. If $p = b$ or $p = c_i$ for some i, then $p \in C$. Suppose neither $p = b$ nor $p = c_i$ for any i. There exists a set $M \in \mathcal{F}(\mathcal{I})$ such that function $g : S \to X$ given by $g/_M = f/_M$ and $g[S \setminus M] = \{p\}$, has $\mathcal{K}$-cluster point p. Let U be any open set containing p. Therefore $\{s \in M : f(s) \in U\} \cup (S \setminus M) = \{s \in S : g(s) \in U\} \notin \mathcal{K}$. If $\mathcal{I} \subset \mathcal{K}$, $\{s \in M : f(s) \in U\} \notin \mathcal{I}$. Therefore $U \cap C \neq \phi$. Thus p is a limit point of C and so $p \in C$. Hence $C_f(\mathcal{I}^{\mathcal{K}}) \subset C$.

Suppose $p \in C$. Let U be any open set containing p. Then there is $c_i \in A$ such that $c_i \in U$. Thus $T_i \subset \{s \in S : f(s) \in U\}$. Consider a proper subset M of S such that $M \in \mathcal{F}(\mathcal{I})$ and define a function $g : S \to X$ by $g/_M = f/_M$ and $g[S \setminus M] = \{p\}$. Since $\{s \in S : g(s) \in U\} \supset \{s \in S : f(s) \in U\}$ and $\{s \in S : f(s) \in U\} \notin \mathcal{K}$, $\{s \in S : g(s) \in U\} \notin \mathcal{K}$. This implies that $p \in C_f(\mathcal{I}^{\mathcal{K}})$. Hence $C \subset C_f(\mathcal{I}^{\mathcal{K}})$.

The above discussion can be summarized in the form of a theorem given below.

THEOREM 2.1. *(i) For any function $f : S \to X$ and ideals $\mathcal{I}$, $\mathcal{K}$ on S, $C_f(\mathcal{I}^{\mathcal{K}})$ is closed.*

(ii) Suppose X is completely separable and $\mathcal{I}$, $\mathcal{K}$ are ideals on S. If there exists a pairwise disjoint sequence of sets $\{T_p\}$ such that $T_p \subset S$, $T_p \notin \mathcal{K}$ for all $p \in \mathbb{N}$ and $\mathcal{I} \subset \mathcal{K}$ then for any non-empty closed set $C \subset X$, there is a function $f : S \to X$ such that $C = C_f(\mathcal{I}^{\mathcal{K}})$.

THEOREM 2.2. *Let $f : S \to X$ and $g : S \to X$ be functions. Then*
(i) $C_f(\mathcal{I}^{\mathcal{K}}) \subset C_f(Fin)$ and $L_f(\mathcal{I}^{\mathcal{K}}) \subset L_f(Fin)$,

(ii) $C_f(\mathcal{I}^{\mathcal{K}_2}) \subset C_f(\mathcal{I}^{\mathcal{K}_1})$ *and* $L_f(\mathcal{I}^{\mathcal{K}_2}) \subset L_f(\mathcal{I}^{\mathcal{K}_1})$ *with* $\mathcal{K}_1 \subset \mathcal{K}_2$,

(iii) If $\{s \in S : f(s) \neq g(s)\} \in \mathcal{K}$, *then* $C_f(\mathcal{I}^{\mathcal{K}}) = C_g(\mathcal{I}^{\mathcal{K}})$ *and* $L_f(\mathcal{I}^{\mathcal{K}}) = L_g(\mathcal{I}^{\mathcal{K}})$, *where* $\mathcal{I}$, $\mathcal{K}$, $\mathcal{K}_1$,$\mathcal{K}_2$ *are ideals on* S.

Proof. (i) and (ii) follow from the definitions. For (iii), let $p \in C_f(\mathcal{I}^{\mathcal{K}})$. Then there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that the function $h_1 : S \to X$ given by $h_1(s) = f(s)$ if $s \in M$ and p otherwise, has $\mathcal{K}$-cluster point p. So for any open set U containing p, $\{s \in S : h_1(s) \in U\} \notin \mathcal{K}$. Consider a function $h_2 : S \to X$ given by $h_2(s) = g(s)$ if $s \in M$ and p otherwise. Therefore $\{s \in S : h_1(s) \in U\} \subset \{s \in S : h_2(s) \in U\} \cup \{s \in S : h_1(s) \neq h_2(s)\}$. If $\{s \in S : h_2(s) \in U\} \in \mathcal{K}$, $\{s \in S : h_1(s) \in U\} \in \mathcal{K}$ which is a contradiction. Therefore $\{s \in S : h_2(s) \in U\} \notin \mathcal{K}$ and so $p \in C_g(\mathcal{I}^{\mathcal{K}})$. Similarly, $C_g(\mathcal{I}^{\mathcal{K}}) \subset C_f(\mathcal{I}^{\mathcal{K}})$ and consequently $C_f(\mathcal{I}^{\mathcal{K}}) = C_g(\mathcal{I}^{\mathcal{K}})$. The proof of $\mathcal{I}^{\mathcal{K}}$-limit points is same as previous one. □

Let (X, τ) be a topological space. Suppose S is a non-empty set and X^S is the set of all functions from S to X. For ideals $\mathcal{I}$, $\mathcal{K}$ on S, $\{X \setminus C \subset X : C = \bigcup_{f \in C^S} C_f(\mathcal{I}^{\mathcal{K}})\}$ forms a topology on X, denoted by $\tau(\mathcal{I}^{\mathcal{K}})$. In addition, if $\mathcal{I}$, $\mathcal{K}$ are ideals on $\mathbb{N}$ with $\mathcal{I} = \mathcal{K}$, then $\tau(\mathcal{I}^{\mathcal{K}})$ coincides with $\tau(\mathcal{I})$ cf. [12].

Theorem 2.3. *For any topological space* (X, τ) *and ideals* $\mathcal{I}$, $\mathcal{K}$ *on a nonempty set* S, $\tau \subset \tau(\mathcal{I}^{\mathcal{K}})$. *In addition, if* X *is first countable,* $\tau = \tau(\mathcal{I}^{\mathcal{K}})$.

Proof. Let G be a τ-closed set. For each element $\alpha \in G$, consider the constant function $f : S \to G$ at α. Then $G \subset \bigcup_{f \in G^S} C_f(\mathcal{I}^{\mathcal{K}}) \subset \bigcup_{f \in G^S} C_f(Fin) = G$ (as in Theorem 2.2 (i)).

Consider a $\tau(\mathcal{I}^{\mathcal{K}})$-closed set G and $\alpha \in G$. There is a set $M \in \mathcal{F}(\mathcal{I})$ such that the function $g : S \to G$ given by $g/_M = f/_M$ and $g(S \setminus M) = \{\alpha\}$ has $\mathcal{K}$-cluster point α. Let $\{U_p\}$ be a decreasing local base at α. Then $E_i = \{s \in S : g(s) \in U_i\} \notin \mathcal{K}$, for each $i \in \mathbb{N}$. Take $s_1 \in E_1$ and for each $i \in \mathbb{N}$, $s_{i+1} \in E_{i+1} \setminus \{s_1, s_2, \ldots s_i\}$. Suppose $C = \{s_1, s_2, \ldots\}$, then α is a limit point of $\{g(s) : s \in C\}$ that is, α is a limit point of G. Hence G is a τ-closed set. □

Lemma 2.4. *Suppose* A *is a compact subset of a topological space* X *and* $f : S \to X$ *is a function. Then for ideals* $\mathcal{I}$, $\mathcal{K}$ *on* S, $\{s \in S : f(s) \in A\} \notin \mathcal{K}$ *implies that* $A \cap C_f(\mathcal{I}^{\mathcal{K}}) \neq \phi$.

Proof. If possible let, $A \cap C_f(\mathcal{I}^{\mathcal{K}}) = \phi$. Let $a \in A$. Then there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that the function $g : S \to X$ given by $g(s) = f(s)$ if $s \in M$ and a otherwise, has no $\mathcal{K}$-cluster point. So for each $a \in A$ there exists an open set U_a containing a such that $\{s \in S : g(s) \in U_a\} \in \mathcal{K}$. Since $\{s \in S : f(s) \in U_a\} \subset \{s \in S : g(s) \in U_a\}$, so $L_a = \{s \in S : f(s) \in U_a\} \in \mathcal{K}$. Now consider the collection $\{U_a : a \in A\}$ which forms an open cover of the compact set A and so it has a finite subcover $\{U_{a_1}, U_{a_2}, \ldots U_{a_l}\}$ (say). Then $\{s \in S : f(s) \in A\} \subset L_{a_1} \cup L_{a_2} \cup \ldots \cup L_{a_l} \in \mathcal{K}$. This implies $\{s \in S : f(s) \in A\} \in \mathcal{K}$, which is a contradiction. □

For any topological space X, $f : S \to X$ is a function then $\mathcal{I}^{\mathcal{K}}$-filter generated by f is defined by $\mathcal{G}_f(\mathcal{I}^{\mathcal{K}}) = \{Y \subset X :$ there exist $A \in \mathcal{F}(\mathcal{I})$ and $y \in X$ such that

$\{s \in S : g(s) \notin Y\} \in \mathcal{K}\}$, where $g : S \to X$ defined by $g(s) = f(s)$ if $s \in A$ and $g(s) = y$ if $s \in S \setminus A$. Therefore $\mathcal{G}_f(\mathcal{I}^{\mathcal{K}})$ forms a filter on X. The corresponding filter base $\mathcal{B}_f(\mathcal{I}^{\mathcal{K}}) = \{\{g(s) : s \notin K\} : K \in \mathcal{K}$ and there is $A \in \mathcal{F}(\mathcal{I})$ with $g/_A = f/_A$ and $g(S \setminus A) = \{y\}, y \in X\}$. In addition, if $\mathcal{I}$, $\mathcal{K}$ are ideals on $\mathbb{N}$ with $\mathcal{I} = \mathcal{K}$, then $\mathcal{G}_f(\mathcal{I}^{\mathcal{K}})$ coincides with $\mathcal{I}$-filter generated by the function $f : \mathbb{N} \to X$, cf. [12] and in particular if $\mathcal{I} = \mathcal{K} = \text{Fin}$ on $\mathbb{N}$, then $\mathcal{G}_f(\mathcal{I}^{\mathcal{K}})$ coincides with the elementary filter associated with the function $f : \mathbb{N} \to X$, cf. [2, Definition 7, p.64].

THEOREM 2.5. *For any topological space X, suppose $f : S \to X$ is a function and $\mathcal{I}$, $\mathcal{K}$ are ideals on S. Then $\bigcap_{B \in \mathcal{B}_f(\mathcal{I}^{\mathcal{K}})} \bar{B} \subset C_f(\mathcal{I}^{\mathcal{K}})$. In addition if X is first countable, $\bigcap_{B \in \mathcal{B}_f(\mathcal{I}^{\mathcal{K}})} \bar{B} = C_f(\mathcal{I}^{\mathcal{K}})$.*

Proof. Let $p \in \bigcap_{B \in \mathcal{B}_f(\mathcal{I}^{\mathcal{K}})} \bar{B}$. Suppose p is not an $\mathcal{I}^{\mathcal{K}}$-cluster point of f. Then for any set $L \in \mathcal{F}(\mathcal{I})$ such that function $h : S \to X$ defined by $h(s) = f(s)$ if $s \in L$ and $h(s) = p$ if $s \in S \setminus L$ has no $\mathcal{K}$-cluster point. So there exists an open set U containing p such that $L_1 = \{s \in S : h(s) \in U\} \in \mathcal{K}$. This implies that $\{h(s) : s \notin L_1\} \in \mathcal{B}_f(\mathcal{I}^{\mathcal{K}})$. It follows $p \in \bigcap_{B \in \mathcal{B}_f(\mathcal{I}^{\mathcal{K}})} \bar{B} \subset \overline{\{h(s) : s \notin L_1\}} = \overline{\{h(s) : h(s) \notin U\}} \subset X \setminus U$ that is $p \in X \setminus U$, which leads to a contradiction. Conversely suppose $p \in C_f(\mathcal{I}^{\mathcal{K}})$. Then there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that function $g : S \to X$ defined by $g(s) = f(s)$ if $s \in M$ and $g(s) = p$ if $s \in S \setminus M$ has a $\mathcal{K}$-cluster point p. Let $\{W_i\}$ be a decreasing local base at p. Then for each i, $A_i = \{s \in S : g(s) \in W_i\} \notin \mathcal{K}$. Take $K \in \mathcal{K}$ and $B_i = A_i \setminus K \notin \mathcal{K}$, for each i. Put $i_1 \in B_1$ and $i_{n+1} \in B_{n+1} \setminus \{i_1, i_2, \ldots i_n\}$, for all n. Suppose $L = \{i_1, i_2, \ldots\}$, then clearly $i_n \notin K$. Then p is a limit point of the set $\{g(s) : s \notin K\}$ that is $p \in \overline{\{g(s) : s \notin K\}}$. Hence $p \in \bigcap_{K \in \mathcal{K}} \overline{\{g(s) : s \notin K\}}$. □

Consider a Hausdorff uniform space $(Y, \mathbb{U})$. Suppose $K(Y)$ and $CL(Y)$ are the collection of all nonempty compact and closed subsets of Y respectively. Recall that a sequence (y_n) in $(Y, \mathbb{U})$ is said to be bounded if $\{y_n : n \in \mathbb{N}\} \subset V[a] = \{b \in Y : (a, b) \in V\}$ for some $a \in Y$ and $V \in \mathbb{U}$. Y is said to be boundedly compact if every closed bounded subset in Y is compact. Let $bs(Y)$ be the set of all bounded sequences in $(Y, \mathbb{U})$ and by $cs(Y)$ the set of all sequences $y = (y_n)$ in Y with $C_y(\mathcal{I}^{\mathcal{K}}) \neq \phi$. According to Lemma 2.4, for every sequences $y \in bs(Y)$, $C_y(\mathcal{I}^{\mathcal{K}}) \neq \phi$ i.e., $bs(Y) \subset cs(Y)$. If $(Y, \mathbb{U})$ is boundedly compact, then $C_y(\mathcal{I}^{\mathcal{K}})$ is compact (it is closed and bounded). Hence the assignment $y \mapsto C_y(\mathcal{I}^{\mathcal{K}})$ defines a mapping $\Gamma_{\mathcal{I}^{\mathcal{K}}}$ of the set $bs(Y)$ to the set $K(Y)$ of all nonempty compact subsets of $(Y, \mathbb{U})$. As in [3], endow $cs(Y)$ with a uniformity $\tilde{\mathbb{U}}$ defined by $\tilde{\mathbb{U}} = \{\tilde{V} = ((y_n), (z_n)) :$ for all n, $(y_n \in V[z_n]$ and $z_n \in V[y_n]), V \in \mathbb{U}\}$ and on $K(Y)$ consider the Hausdorff-Bourbaki uniformity $\mathbb{U}_H$ (see [5]) inherited from the space $CL(Y)$ defined by $\mathbb{U}_H = \{\underline{V} = (B, C) \in CL(Y) \times CL(Y) : V \in \mathbb{U}, B \subset V[C]$ and $C \subset V[B]\}$.

We recall the definition of Vietoris topology on the set $\mathcal{A}$ of all non empty closed subsets of a Hausdorff topological space Y. For any subset B of Y, take $B^- = \{C \in \mathcal{A} : C \cap B \neq \phi\}$ and $B^+ = \{C \in \mathcal{A} : C \subset B\}$. Then the upper Vietoris topology on $\mathcal{A}$ denoted by τ_V^+, is the topology with subbase $S_B^- = \{B^+ : B$ is open in $Y\}$ and the lower Vietoris topology on $\mathcal{A}$ denoted by τ_V^-, is the topology with subbase

$S_B^- = \{B^- : B$ is open in $Y\}$. The Vietoris topology on $\mathcal{A}$ denoted by τ_V, is the topology with subbase $S_B^- \cup S_B^+$.

THEOREM 2.6. *Suppose $(X, \mathbb{U})$ is a boundedly compact uniform space and $\mathcal{I}$, $\mathcal{K}$ are ideals on $\mathbb{N}$. Then the mapping $\Gamma_{\mathcal{I}^\mathcal{K}} : (bs(X), \tilde{\mathbb{U}}) \to (K(X), \mathbb{U}_H)$ is uniformly continuous.*

Proof. Suppose $U \in \mathbb{U}$. Then there exists $V \in \mathbb{U}$ such that $V^3 \subset U$. Let $((x_n),(y_n)) \in \tilde{V}$ and $a \in C_x(\mathcal{I}^\mathcal{K})$, where $x = (x_n)$. Then there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that sequence (z_n) given by $z_n = x_n$ if $n \in M$ and a otherwise, has $\mathcal{K}$-cluster point a. Therefore $K = \{n \in \mathbb{N} : z_n \in V[a]\} \notin \mathcal{K}$. Consider a sequence (w_n) given by $w_n = y_n$ if $n \in M$ and a otherwise. Since for each n, $(x_n, y_n) \in V$ which implies $(z_n, w_n) \in V$. So for each $n \in K$, $(z_n, a) \in V$ and $(z_n, w_n) \in V$ and then $(w_n, a) \in V^2$ i.e., $w_n \in V^2[a]$. Hence $K \subset \{n \in \mathbb{N} : w_n \in \overline{V^2[a]}\}$. As $K \notin \mathcal{K}$, $\{n \in \mathbb{N} : w_n \in \overline{V^2[a]}\} \notin \mathcal{K}$. Since $\overline{V^2[a]}$ is compact, by Lemma 2.4, $\overline{V^2[a]} \cap C_y(\mathcal{I}^\mathcal{K}) \neq \phi$ (where $y = (y_n)$), which implies $a \in V^3[C_y(\mathcal{I}^\mathcal{K})]$. Therefore $C_x(\mathcal{I}^\mathcal{K}) \subset V^3[C_y(\mathcal{I}^\mathcal{K})] \subset U[C_y(\mathcal{I}^\mathcal{K})]$. Similarly $C_y(\mathcal{I}^\mathcal{K}) \subset U[C_y(\mathcal{I}^\mathcal{K})]$ and it follows that $(C_x(\mathcal{I}^\mathcal{K}), C_y(\mathcal{I}^\mathcal{K})) \in \underline{U} \in \mathbb{U}_H$. □

THEOREM 2.7. *Suppose $(X, \mathbb{U})$ is a locally compact uniform space and $\mathcal{I}$, $\mathcal{K}$ are ideals on $\mathbb{N}$. Then the mapping $\Gamma_{\mathcal{I}^\mathcal{K}} : (cs(X), \tilde{\mathbb{U}}) \to (CL(X), \tau_V^-)$ is continuous.*

Proof. Suppose O^- be a basic open set in $(CL(X), \tau_V^-)$, where O is open in X. Let $x = (x_n) \in \Gamma^{-1}_{\mathcal{I}^\mathcal{K}}(O^-)$. Then $C_x(\mathcal{I}^\mathcal{K}) \cap O \neq \phi$. Let $a \in C_x(\mathcal{I}^\mathcal{K}) \cap O$. Since $(X, \mathbb{U})$ is locally compact, there exists $U \in \mathbb{U}$ such that $\overline{U[a]}$ is compact and $a \in \overline{U[a]} \subset O$. So for $U \in \mathbb{U}$, there exists $V \in \mathbb{U}$ such that $V^2 \subset U$. As $a \in C_x(\mathcal{I}^\mathcal{K})$, there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that sequence (y_n) given by $y_n = x_n$ if $n \in M$ and a otherwise, has $\mathcal{K}$-cluster point a. So $B = \{n \in \mathbb{N} : y_n \in V[a]\} \notin \mathcal{K}$. Let $z = (z_n) \in \tilde{V}[(x_n)]$, then for each n, $(x_n, z_n) \in V$. Consider a sequence (w_n) given by $w_n = z_n$ if $n \in M$ and a otherwise. Again for each $n \in B$, $y_n \in V[a]$ and $w_n \in V[y_n]$ which implies $w_n \in V^2[a] \subset U[a]$. Therefore $B \subset \{n \in \mathbb{N} : w_n \in U[a]\} \subset \{n \in \mathbb{N} : w_n \in \overline{U[a]}\}$. As $B \notin \mathcal{K}$, $\{n \in \mathbb{N} : w_n \in \overline{U[a]}\} \notin \mathcal{K}$. Since $\overline{U[a]}$ is compact, by Lemma 2.4, $\overline{U[a]} \cap C_z(\mathcal{I}^\mathcal{K}) \neq \phi$ and so $C_z(\mathcal{I}^\mathcal{K}) \cap O \neq \phi$. Hence $z \in \Gamma^{-1}_{\mathcal{I}^\mathcal{K}}(O^-)$ and consequently $\tilde{V}[(x_n)] \subset \Gamma^{-1}_{\mathcal{I}^\mathcal{K}}(O^-)$. Hence $\Gamma^{-1}_{\mathcal{I}^\mathcal{K}}(O^-)$ is open in $(cs(X), \tilde{\mathbb{U}})$. □

3. $\mathcal{I}$-Fréchet compactness and $\mathcal{I}^\mathcal{K}$-Fréchet compactness

In [7], J. A. Fridy introduced nonthin subsets of natural numbers in terms of natural density. Being motivated by the concept of nonthin subsets, $\mathcal{I}$-nonthin subset is introduced as follows.

DEFINITION 3.1 ([20, Definition 1]). Suppose A is a subset of a nonempty set S and X is a topological space. A function $f : A \to X$ is said to be $\mathcal{I}$-thin, where $\mathcal{I}$ is an ideal on S if $A \in \mathcal{I}$, otherwise it is called $\mathcal{I}$-nonthin.

If $\mathcal{I}$ is a nontrivial admissible ideal on S, $\mathcal{I}/_M = \{A \cap M; A \in \mathcal{I}\}$ is an ideal on M called trace of $\mathcal{I}$ on M [13]. $\mathcal{I}/_M$ is nontrivial if $M \notin \mathcal{I}$.

Definition 3.2. For a subset C of a topological space X and $x \in X$, the $\mathcal{I}$-closure of C is denoted by $\overline{C}^{\mathcal{I}} = \{x \in X :$ there exists an $\mathcal{I}$-nonthin function $f : A \to C$ that $\mathcal{I}/_A$-converges to $x\}$ and the $\mathcal{I}^{\mathcal{K}}$-closure of C is denoted by $\overline{C}^{\mathcal{I}^{\mathcal{K}}} = \{x \in X :$ there exists an $\mathcal{I}$-nonthin function $f : A \to C$ that $(\mathcal{I}/_A)^{\mathcal{K}}$-converges to $x\}$, where $\mathcal{I}$ and $\mathcal{K}$ are ideals on a nonempty set S.

Theorem 3.3. *Suppose $\mathcal{I}$, $\mathcal{K}$ are ideals on S such that $\mathcal{K} \subset \mathcal{I}$. For any subset C of a topological space X, $C \subset \overline{C}^{\mathcal{K}} \subset \overline{C}^{\mathcal{I}^{\mathcal{K}}} \subset \bar{C}$, where $\bar{C}$ is the closure of C. Furthermore, if X is first countable, $\mathbb{N} \subset S$ and $\mathbb{N} \notin \mathcal{I}$, $\bar{C} = \overline{C}^{\mathcal{K}} = \overline{C}^{\mathcal{I}^{\mathcal{K}}}$*

Proof. Consider $\alpha \in \overline{C}^{\mathcal{I}^{\mathcal{K}}}$, there exists an $\mathcal{I}$-nonthin function $f : A \to C$ that $(\mathcal{I}/_A)^{\mathcal{K}}$-converges to α. Therefore there is $M \in \mathcal{F}(\mathcal{I}/_A)$ such that the function $g : A \to C$ given by $g(s) = f(s)$ if $s \in M$ and $g(s) = \alpha$ if $s \in A \setminus M$ is $\mathcal{K}$-convergent to α. So for any open set U containing α, $\{s \in A : g(s) \in U\} \in \mathcal{F}(\mathcal{K}/_A)$. Since $\mathcal{K} \subset \mathcal{I}$, the set $\{s \in A : g(s) \in U\} \in \mathcal{F}(\mathcal{I}/_A)$ and so $\{s \in A : f(s) \in U\} \in \mathcal{F}(\mathcal{I}/_A)$. Then there exists $p \in A$ such that $p \in \{s \in A : f(s) \in U\}$. Thus $f(p) \in C \cap U$ and hence $\alpha \in \bar{C}$. Now suppose $\alpha \in \bar{C}$. Then there exists a function $f : \mathbb{N} \to C$ such that $f : \mathbb{N} \to C$ is convergent to α. Since $\mathcal{I}$ and $\mathcal{K}$ are admissible ideals on S, $f : \mathbb{N} \to C$ is $\mathcal{I}$-convergent as well as $\mathcal{K}$-convergent to α. Thus $\alpha \in \overline{C}^{\mathcal{I}^{\mathcal{K}}}$. □

Definition 3.4. For non-trivial ideals $\mathcal{I}$, $\mathcal{K}$ on a nonempty set S, a subset C of a topological space X is called $\mathcal{I}$-closed if $\overline{C}^{\mathcal{I}} = C$ and $\mathcal{I}^{\mathcal{K}}$-closed if $\overline{C}^{\mathcal{I}^{\mathcal{K}}} = C$.

Theorem 3.3 follows that if $\mathcal{I}$, $\mathcal{K}$ are ideals on S, closed subsets with $\mathcal{K} \subset \mathcal{I}$ are $\mathcal{I}^{\mathcal{K}}$-closed. For any topological space (X, τ), $\{G \subset X : X \setminus G$ is $\mathcal{I}^{\mathcal{K}}$-closed$\}$ forms a topology on X denoted by $\tau_{\mathcal{I}^{\mathcal{K}}}$ having $\tau \subset \tau_{\mathcal{I}^{\mathcal{K}}}$. Consequently combining Theorem 2.3 and Theorem 3.3, $\tau_{\mathcal{I}^{\mathcal{K}}} = \tau(\mathcal{I}^{\mathcal{K}})$ in first countable spaces. In general $\tau(\mathcal{I}^{\mathcal{K}}) \subset \tau_{\mathcal{I}^{\mathcal{K}}}$, but the reverse inclusion may not hold.

Example 3.5. Consider the space $X = \{0, 1\}^{\mathbb{R}}$ with product topology and two ideals $\mathcal{I}$, $\mathcal{K}$ on $\mathbb{R}$, where $\mathcal{I}$ is the collection of all countable subsets of $\mathbb{R}$ and $\mathcal{K}$ is the collection of all finite subsets of $\mathbb{R}$. Let $S = \mathbb{R}$ and $U = \{f \in X : f(x) = 0$ for all $x \in \mathbb{Q}\}$. Suppose $C = X \setminus U$. Then $C = \bigcup_{t \in \mathbb{Q}} \pi_t^{-1}(\{1\})$. Claim that $\overline{C}^{\mathcal{I}^{\mathcal{K}}} = C$. Clearly $C \subset \overline{C}^{\mathcal{I}^{\mathcal{K}}}$. Let $x \in \overline{C}^{\mathcal{I}^{\mathcal{K}}}$. Then there exists an $\mathcal{I}$-nonthin function $F : A \to C$ such that F is $(\mathcal{I}/_A)^{\mathcal{K}}$-convergent to x. So there exists $M \in \mathcal{F}(\mathcal{I}/_A)$ such that the function $g : A \to X$ defined by $g(s) = F(s)$ if $s \in M$ and $g(s) = x$ if $s \in A \setminus M$ is $\mathcal{K}$-convergent to x. Then for any open set V of x, $\{s \in A : g(s) \notin V\} \in \mathcal{K}/_A$. Therefore $\{s \in A : g(s) \notin V\}$ is a finite set and so $\{s \in M : F(s) \notin V\}$ is a finite set. Again $F(s) \in C$ for all $s \in M$. So, $M = \bigcup_{t \in \mathbb{Q}} \{s \in M : F(s) \in \pi_t^{-1}(\{1\})\}$. Since M is uncountable, there exists $t \in \mathbb{Q}$ such that $\{s \in M : F(s) \in \pi_t^{-1}(\{1\})\}$ is infinite. So $\{s \in M : F(s) \notin \pi_t^{-1}(\{0\})\}$ is an infinite set. Therefore $x \notin \pi_t^{-1}(\{0\})$ and so $x \in \pi_t^{-1}(\{1\})$. Hence $x \in C$.
Now define a function $y : \mathbb{R} \to \{0, 1\}$ by $y(t) = 0$ if $t \in \mathbb{Q}$ and $y(t) = 1$ if $t \in \mathbb{R} \setminus \mathbb{Q}$.

Then $y \in U$, that is $y \notin C$. Define a function $G : S \to C$ by $G(s)(t) = 1$ if $t = [s]$ and $G(s)(t) = y(t)$ if $t \neq [s]$, where $[s]$ is the greatest integer less than or equal to s. We will prove that $y \in C_G(\mathcal{I}^{\mathcal{K}})$. For any basic open set W containing y, where $W = \pi_{t_1}^{-1}(\{y(t_1)\}) \cap \pi_{t_2}^{-1}(\{y(t_2)\}) \dots \pi_{t_n}^{-1}(\{y(t_n)\})$, $t_1, t_2, \dots, t_n \in \mathbb{R}$, $\{s \in S : G(s) \in W\} \supset \{s \in S : [s] \notin \{t_1, t_2, \dots t_n\}\}$. Since $\{s \in S : [s] \notin \{t_1, t_2, \dots t_n\}\}$ is infinite so $\{s \in S : [s] \notin \{t_1, t_2, \dots t_n\}\} \notin \mathcal{K}$. Therefore $y \in C_G(\mathcal{I}^{\mathcal{K}}) \subset \bigcup_{g \in C^S} C_g(\mathcal{I}^{\mathcal{K}})$. Hence $C \neq \bigcup_{g \in C^S} C_g(\mathcal{I}^{\mathcal{K}})$.

Definition 3.6. Suppose $\mathcal{I}$ is an ideal on S. A function $f : S \to X$ is said to be $\mathcal{I}$-eventually constant at α if $\{s \in S : f(s) \neq \alpha\} \in \mathcal{I}$. An $\mathcal{I}$-nonthin function $f : A \to X$ is said to be $\mathcal{I}$-eventually constant at α if $\{s \in A : f(s) \neq \alpha\} \in \mathcal{I}/_A$.

Note 3.7. Suppose $A \subset S$ and a function $f : S \to X$ is $\mathcal{I}/_A$-eventually constant at α. Then $\{s \in S : f(s) \neq \alpha\} \in \mathcal{I}/_A$. As $\{s \in A : f(s) \neq \alpha\} \subset \{s \in S : f(s) \neq \alpha\}$, $\{s \in A : f(s) \neq \alpha\} \in \mathcal{I}/_A$. So $f : A \to X$ is $\mathcal{I}$-eventually constant at α.

Every constant function is $\mathcal{I}$-eventually constant. Particularly any eventually constant sequence is $\mathcal{I}$-eventually constant, but the reverse implication may not hold. For example, suppose $E \subset \mathbb{N}$, $E_n = \{r \in E : r \leq n\}$. The natural density of E is defined by $d(E) = \lim_{n\to\infty} \frac{|E_n|}{n}$, if the limit exists [8, 15]. Consider $\mathcal{I}_d = \{E \subset \mathbb{N} : d(E) = 0\}$ [9]. Suppose A is an infinite subset of $\mathbb{N}$ with $d(A) = 0$. Take $x_n = 0$ if $n \in A$ and $x_n = 1$ if $n \notin A$. Then obviously (x_n) is $\mathcal{I}$-eventually constant but not eventually constant.

Definition 3.8. For any ideal $\mathcal{I}$ on S, a point α in a topological space X is said to be an $\mathcal{I}_{ev}$-limit point of $Y \subset X$ if there exists an $\mathcal{I}$-nonthin non $\mathcal{I}$-eventually constant function $f : A \to Y \setminus \{\alpha\}$ that $\mathcal{I}/_A$-converges to α and $\mathcal{I}^{\mathcal{K}}_{ev}$-limit point of $Y \subset X$ if there exists an $\mathcal{I}$-nonthin non $\mathcal{I}$-eventually constant function $f : A \to Y \setminus \{\alpha\}$ that $(\mathcal{I}/_A)^{\mathcal{K}}$-converges to α.

Definition 3.9. A topological space X is said to be $\mathcal{I}$-Fréchet compact if every infinite subset of X has an $\mathcal{I}_{ev}$-limit point and is called $\mathcal{I}^{\mathcal{K}}$-Fréchet compact if every infinite subset of X has an $\mathcal{I}^{\mathcal{K}}_{ev}$-limit point.

Theorem 3.10. *For ideals $\mathcal{I}$, $\mathcal{K}$ on S, $\mathcal{I}^{\mathcal{K}}$-closed ($\mathcal{I}$-closed) subset of $\mathcal{I}^{\mathcal{K}}$-Fréchet compact space ($\mathcal{I}$-Fréchet compact space) is $\mathcal{I}^{\mathcal{K}}$-Fréchet compact (resp. $\mathcal{I}$-Fréchet compact).*

Proof. Consider an $\mathcal{I}^{\mathcal{K}}$-closed subset Y of $\mathcal{I}^{\mathcal{K}}$-Fréchet compact space X. Let E be an infinite subset of Y. Since X is $\mathcal{I}^{\mathcal{K}}$-Fréchet compact, E has an $\mathcal{I}^{\mathcal{K}}_{ev}$-limit point say, $\alpha \in X$. So there exists an $\mathcal{I}$-nonthin non $\mathcal{I}$-eventually constant function $f : A \to E \setminus \{\alpha\}$ that $(\mathcal{I}/_A)^{\mathcal{K}}$-converges to α. Since $E \subset Y$, $\alpha \in \overline{Y}^{\mathcal{I}^{\mathcal{K}}} = Y$. Therefore E has an $\mathcal{I}^{\mathcal{K}}_{ev}$-limit point in Y and so Y is $\mathcal{I}^{\mathcal{K}}$-Fréchet compact. □

Corollary 3.11. *Closed subspace of $\mathcal{I}^{\mathcal{K}}$-Fréchet compact space ($\mathcal{I}$-Fréchet compact space) is $\mathcal{I}^{\mathcal{K}}$-Fréchet compact (resp. $\mathcal{I}$-Fréchet compact).*

Theorem 3.12. *A topological space X is not $\mathcal{I}^{\mathcal{K}}$-Fréchet compact if and only if there exists an infinite $\mathcal{I}^{\mathcal{K}}$-closed discrete subspace.*

Proof. Let G be an infinite $\mathcal{I}^{\mathcal{K}}$-closed discrete subspace. Then $\overline{G}^{\mathcal{I}^{\mathcal{K}}} = G$. If α is an $\mathcal{I}^{\mathcal{K}}_{ev}$-limit point of G, there exists an $\mathcal{I}$-nonthin non $\mathcal{I}$-eventually constant function $f : A \to G \setminus \{\alpha\}$ that $(\mathcal{I}/_A)^{\mathcal{K}}$-converges to α. There is $B \in \mathcal{F}(\mathcal{I}/_A)$ such that the function $g : B \to G \setminus \{\alpha\}$ given by $g(s) = f(s), s \in B$ is $\mathcal{K}$-convergent to α. Since G is a discrete subspace, there is an open set U containing α, $\{s \in B : g(s) \in U\}$ is an empty set, which is a contradiction. Therefore X is not $\mathcal{I}^{\mathcal{K}}$-Fréchet compact space. Now let $A \subset X$ be an infinite $\mathcal{I}^{\mathcal{K}}$-closed discrete subspace. If possible let, X is $\mathcal{I}^{\mathcal{K}}$-Fréchet compact, then A has an $\mathcal{I}^{\mathcal{K}}_{ev}$-limit point say l. But since A is $\mathcal{I}^{\mathcal{K}}$-closed, $l \in A$. Also A is a discrete subspace, which contradicts the fact that l is an $\mathcal{I}^{\mathcal{K}}_{ev}$-limit point. □

Corollary 3.13. *A topological space X is $\mathcal{I}^{\mathcal{K}}$-Fréchet compact if and only if all $\mathcal{I}^{\mathcal{K}}$-closed discrete subspaces are finite.*

The following theorem is a translation of the proof of the Theorem 3.12 in terms of $\mathcal{I}$-Fréchet compact space.

Theorem 3.14. *A topological space X is not $\mathcal{I}$-Fréchet compact if and only if there exists an infinite $\mathcal{I}$-closed discrete subspace.*

Example 3.15. Consider the space $X = \mathbb{N} \times \{0, 1\}$, where discrete topology on $\mathbb{N}$ and indiscrete topology on $\{0, 1\}$. Clearly X is Fréchet compact. Consider an infinite set $A = \{(n, 0); n \in \mathbb{N}\}$. For any ideal $\mathcal{I}$ on $\mathbb{N}$, A does not have an $\mathcal{I}$-nonthin non $\mathcal{I}$-eventually constant function that $\mathcal{I}$-converges to some point in X. So X is not $\mathcal{I}$-Fréchet compact.

Example 3.16. Consider the space $X = \{\frac{1}{n}; n \in \mathbb{N}\} \cup \{0\}$ where discrete topology on $\{\frac{1}{n}; n \in \mathbb{N}\}$ and cofinite topology containing 0. Suppose $\mathcal{I} = \{A \subset \mathbb{N} : A \cap \Delta_i$ is finite for all but finitely many i$\}$, where $\mathbb{N} = \cup_{j=1}^{\infty} \Delta_j$ is a decomposition of $\mathbb{N}$ such that each Δ_j is infinite and $\Delta_i \cap \Delta_j = \phi$ for $i \neq j$ and $\mathcal{K} = Fin$. So $\mathcal{I}$, $\mathcal{K}$ are non-trivial admissible ideals on $\mathbb{N}$. Then for any infinite subset A of X, 0 is an $\mathcal{I}_{ev}$-limit point of A. Thus X is $\mathcal{I}$-Fréchet compact. But there is no $\mathcal{I}$-nonthin non $\mathcal{I}$-eventually constant sequence that is $\mathcal{K}$-convergent, so X is not $\mathcal{I}^{\mathcal{K}}$-Fréchet compact.

Definition 3.17. A topological space is called $\mathcal{I}$-sequential if every $\mathcal{I}$-closed set is closed.

Theorem 3.18. *For any $\mathcal{I}$-sequential Hausdorff space, Fréchet compactness and $\mathcal{I}$-Fréchet compactness are equivalent, provided $\mathcal{I}$ is an ideal on $\mathbb{N}$.*

Proof. Suppose X is an $\mathcal{I}$-sequential Hausdorff Fréchet compact space. Let A be an infinite subset of X having no $\mathcal{I}_{ev}$-limit point. Thus A is $\mathcal{I}$-closed and so closed. Since X is countable compact and closed subset of a countable compact space is countable compact, A is countable compact. Therefore A is first countable in the relative topology. Suppose B is an infinite subset of A. Then B has a limit point say α. Let $\{V_n\}$ be a countable base at α such that $V_{n+1} \subset V_n$, for all n. So there exists a sequence (x_n) in X such that $x_n \in A \cap (V_n \setminus \{\alpha\})$ and (x_n) converges to α. Thus (x_n) is a non $\mathcal{I}$-eventually constant sequence that $\mathcal{I}$-converges to α. Therefore α is an $\mathcal{I}_{ev}$-limit point of B and so A is $\mathcal{I}$-Fréchet compact. This contradicts the fact that A has no $\mathcal{I}_{ev}$-limit point. Hence X is $\mathcal{I}$-Fréchet compact. □

Definition 3.19. Let $\mathcal{I}$, $\mathcal{K}$ be ideals on S. $\mathcal{I}$ is said to satisfy shrinking condition (A) with respect to $\mathcal{K}$ or shrinking condition $A(\mathcal{I}, \mathcal{K})$ holds, if for any sequence $\{A_i\}$ of sets none in $\mathcal{I}$, there exists a sequence $\{B_i\}$ of sets in $\mathcal{K}$ such that $B_i \subset A_i$ and $\cup_{i=1}^{\infty} B_i \notin \mathcal{I}$.

The following example is an witness of such ideal.

Example 3.20. Let $\mathbb{N} = \cup_{j=1}^{\infty} \Delta_j$ be a decomposition of $\mathbb{N}$ such that each Δ_j is infinite and $\Delta_i \cap \Delta_j = \phi$ for $i \neq j$. Consider $\mathcal{I} = \{A \subset \mathbb{N} : A \cap \Delta_i$ is finite for all but finitely many i$\}$ and $\mathcal{K}$ denote the class of all subset A of $\mathbb{N}$ which intersect at most finite number of Δ_j [9]. Then $\mathcal{I}$ satisfies shrinking condition (A) with respect to $\mathcal{K}$.

The following theorem possessed by finite derived set property or FDS-property, which was introduced in [18] to study the properties of T_1-independent topologies on a set.

Definition 3.21 ([18]). A topological space X has the finite derived set property or FDS-property if every infinite subset of X contains an infinite subset with only finitely many limit points in X.

Theorem 3.22. *Suppose $\mathcal{I}$, $\mathcal{K}$ are ideals on S with $\mathcal{K} \subset \mathcal{I}$. Then $\mathcal{I}$-Fréchet compactness, $\mathcal{I}^{\mathcal{K}}$-Fréchet compactness, Fréchet compactness are equivalent provided shrinking condition $A(\mathcal{I}, \mathcal{K})$ holds and the underlying space is first countable with the FDS-property.*

Proof. Suppose X is a first countable $\mathcal{I}$- Fréchet compact space. Consider an infinite subset $A \subset X$ which has an $\mathcal{I}_{ev}$-limit point say $\alpha \in X$. There is an $\mathcal{I}$-nonthin non $\mathcal{I}$-eventually constant function $f : B \to A \setminus \{\alpha\}$ that $\mathcal{I}/_B$-converges to α. Let $\{U_n\}$ be countable base for X at the point α such that $U_{n+1} \subset U_n$, for all $n \in \mathbb{N}$. Therefore for all $m \in \mathbb{N}$, $A_m = \{s \in B : f/_B(s) \in U_m\} \notin \mathcal{I}$. If shrinking condition $A(\mathcal{I}, \mathcal{K})$ holds, there exists a sequence of sets $\{B_i\}$ in $\mathcal{K}$ such that $B_i \subset A_i$ and $D = \cup_{i \in \mathbb{N}} B_i \notin \mathcal{I}$. Let U be any open set containing α. Then there exists $U_p \in \{U_n\}$ such that $U_n \subset U$, for all $n \geq p$. So $\{s \in D : f(s) \notin U\} \subset B_1 \cup B_2 \cup \ldots \cup B_p \in \mathcal{K}$. Therefore the restriction $f : D \to A \setminus \{\alpha\}$ is $\mathcal{K}$-convergent to α. So X is $\mathcal{I}^{\mathcal{K}}$- Fréchet compact.

Suppose X is $\mathcal{I}^{\mathcal{K}}$- Fréchet compact. From Corollary 3.13, all $\mathcal{I}^{\mathcal{K}}$-closed discrete subspaces are finite. Theorem 3.3 follows that if $\mathcal{K} \subset \mathcal{I}$, closed subsets are $\mathcal{I}^{\mathcal{K}}$-closed. Thus every closed discrete subspace of X is finite and so X is Fréchet compact.

Suppose X is a Fréchet compact space with the FDS-property and Y is an infinite subset of X. Since X has the FDS-property, there exists an infinite subset A of

Y with finite set of limit points say $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Let $f : M_0 \to A$ be an $\mathcal{I}$-nonthin function of distinct elements. If $f : M_0 \to A$ is $\mathcal{I}/_{M_0}$-convergent to α_1, then the proof is done. If not then there exists an open set U_1 containing α_1 such that $M_1 = \{s \in M_0 : f(s) \notin U_1\} \notin \mathcal{I}$. Let $A_1 = \{f(s); s \in M_1\}$. Then A_1 is an infinite set and $f : M_1 \to A_1$ is an $\mathcal{I}$-nonthin function of distinct elements. If $f : M_1 \to A_1$ is $\mathcal{I}/_{M_1}$-convergent to α_2, then the proof is done. If not then there exists an open set U_2 containing α_2 such that $M_2 = \{s \in M_1 : f(s) \notin U_2\} \notin \mathcal{I}$. Therefore $\{f(s); s \in M_2\}$ is an infinite set say A_2. Proceeding in this way we get for some $k \leq n$, the $\mathcal{I}$-nonthin function $f : M_{k-1} \to A_{k-1}$ is $\mathcal{I}/_{M_{k-1}}$-convergent to α_k. Otherwise the infinite set $A \setminus \cup_{i=1}^{n} A_i$ has no limit point, which contradicts our assumption. Hence every infinite subset of X has an $\mathcal{I}_{ev}$-limit point. □

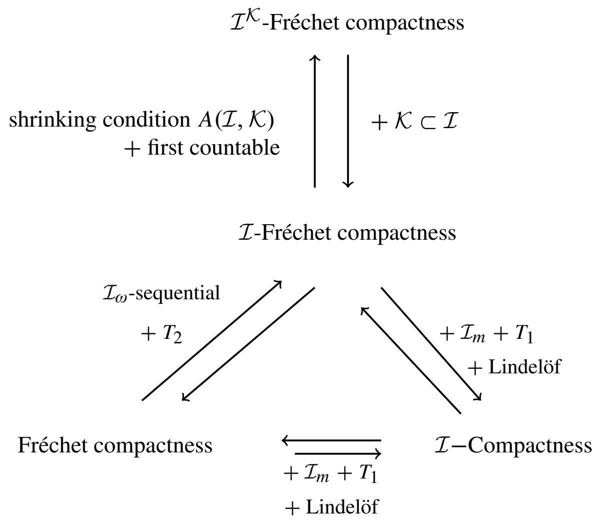

Figure 1: Relation among $\mathcal{I}^{\mathcal{K}}$-Fréchet compactness, $\mathcal{I}$-Fréchet compactness, $\mathcal{I}$-compactness and Fréchet compactness

We conclude the article with diagram (see Figure 1) which shows relations among different types of compactness exhibited in the present work as well as the article [20]. In Figure 1, $\mathcal{I}_\omega$ is an ideal on $\mathbb{N}$ and $\mathcal{I}_m$ is the dual maximal ideal to the free ultrafilter on $\mathbb{N}$.

$\mathcal{I}$-compactness [20] was introduced and showed that even in metric spaces $\mathcal{I}$-compactness and compactness are different. If $\mathcal{I}_m$ is the dual maximal ideal to the free ultrafilter on $\mathbb{N}$, then every compact space is $\mathcal{I}$-compact (see [20, Note 4]). For any nontrivial ideal on $\mathbb{N}$, every $\mathcal{I}$-compact space is $\mathcal{I}$-Fréchet compact. Also, Theorem 3.18 showed that in first countable T_1 space Fréchet compactness and $\mathcal{I}$-Fréchet compactness are equivalent, provided $\mathcal{I}$ is an ideal on $\mathbb{N}$. So neither of Fréchet compactness and $\mathcal{I}$-Fréchet compactness imply $\mathcal{I}$-compactness in general (see [20, Example 3]).

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