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ON THE EXPONENTIAL CHANGE OF m-th ROOT FINSLER METRICS WITH SPECIAL CURVATURE PROPERTIES

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Abstract. In this paper, we begin by introducing the exponential change of m-th root Finsler metrics, referred to as exponentially transformed m-th root Finsler metric. For this metric, we derive the fundamental metric tensors along with their inverses. Additionally, we determine the spray coefficients and establish the conditions under which the transformed metric is projectively related to an m-th root metric. Furthermore, we investigate the conditions for the transformed Finsler space to exhibit locally duality flatness and projective flatness. We also identify the conditions under which the transformed metrics to be the Berwald metric, weakly Berwald metric, Landsberg metric, and weakly Landsberg metric. Lastly, we show that every exponential change of m-th root Finsler metrics with almost vanishing H-curvature has vanishing H-curvature.

1. Introduction

Exponential Finsler metrics are a specific type of Finsler metric that demonstrate exponential growth along geodesics. They find extensive applications in various fields, including General Relativity, Optimal Transportation, Robotics and Motion Planning, Image Processing and Computer Vision, Medical Imaging, Neuroscience, and Brain Connectivity. Exponential Finsler metrics offer a more flexible approach to describing the geometry of space-time, allowing for the consideration of more general metric structures.

The m-th root Finsler metric was originally proposed by Simada in 1979 [13]. This metric exhibits applications in ecology, as explored by Antonelli [2], as well as in fields like physics, space-time, general relativity, and unified gauge field theory [3, 4]. It generalizes the Riemannian metric, for $m = 2, 3$, and 4, it corresponds to the Riemannian metric, cubic metric, and quartic metric, respectively. Remarkably, in four dimensions, the particular fourth root metric $F = \sqrt[4]{y^1y^2y^3y^4}$, referred to as the Berwald Móor metric [8], is seen by physicists as a promising model for space-time.

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On an n-dimensional Finsler manifold $(M, F) = F^n$, where F represents a Finsler metric, a Finsler change can be defined for a 1-form $\beta(x, y) = b_i(x)y^i$. The Finsler change transforms the Finsler metric $F(x, y)$ into $\bar{F}(x, y) = f(F, \beta)$, where $f(F, \beta)$ is a positively homogeneous function of degree one with respect to both F and β. One specific type of Finsler change is known as the exponential change, characterized by the equation:

$$\bar{F} = Fe^{\beta/F}. \tag{1}$$

If the Finsler metric F is equivalent to a Riemannian metric α, then the resulting transformed metric $\bar{F}$ is equivalent to an exponential metric $\bar{F} = \alpha e^{\beta/\alpha}$.

The work of Tayebi and Najafi [14] describe the characteristics of locally dually flat and Antonelli m-th root metrics. The authors establish that any m-th root metric with isotropic mean Berwald curvature (respectively, isotropic Landsberg curvature) is equivalent to a weakly Berwald metric (respectively, Landsberg metric). Furthermore, authors demonstrate that an m-th root metric with nearly vanishing H-curvature will have exactly vanishing H-curvature [16]. Additionally, the study by Tayebi, Peyghan, and Shahbazi [15] identifies a condition under which a generalized m-th root metric is projectively related to an m-th root metric. Moreover, in the work of Brinzei [5], necessary and sufficient conditions are provided for a Finsler space with an m-th root metric to be projectively flat with respect to a Berwald space. Many geometers have studied Finsler space with m-th root metric [7, 10, 17–20].

This paper focuses on several key aspects of the transformed Finsler space. We begin by deriving the fundamental metric tensors and their inverses for this transformed space. Additionally, the spray coefficients are determined, and the conditions for the transformed metric to be projectively related to an m-th root metric are established. Furthermore, we investigate the properties of the transformed Finsler space, including local duality flatness and projective flatness. It is shown that under certain conditions, the transformed Finsler space exhibits these properties. Moreover, we provide a crucial proof demonstrating that every exponential change of m-th root Finsler metrics with isotropic Berwald curvature, isotropic mean Berwald curvature, isotropic Landsberg curvature, and isotropic mean Landsberg curvature results in a reduction to the Berwald metric, weakly Berwald metric, Landsberg metric, and weakly Landsberg metric, respectively. Additionally, we prove that every exponential change of m-th root Finsler metrics with nearly vanishing H-curvature results in vanishing H-curvature in the transformed metrics.

2. Preliminaries

Let M^n be an n-dimensional C^∞-manifold, T_xM denotes the tangent space of M^n at x. The tangent bundle TM is the union of tangent spaces, i.e., $TM := \bigcup_{x\in M} T_xM$. We denote the elements of TM by (x, y), where $x = (x^i)$ is a point of M^n and $y \in T_xM$ called supporting element. We denote $TM_0 = TM \setminus \{0\}$.

Definition 2.1. A Finsler metric on M^n is a function $F : TM \to [0, \infty)$ with the following properties:
(i) F is C^∞ on TM_0,

(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM and

(iii) The Hessian of $\dfrac{1}{2}F^2$ with element $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.
The pair $(M^n, F) = F^n$ is called a Finsler space. The function F is called the fundamental function and g_{ij} is called the fundamental metric tensor of the Finsler space F^n.

The normalized supporting element l_i, angular metric tensor h_{ij}, and metric tensor g_{ij} of F^n are defined respectively as:

$$l_i = \frac{\partial F}{\partial y^i}, \quad h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j}, \quad \text{and} \quad g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}.$$

Consider a Finsler metric F defined as $F = \sqrt[m]{A}$, where $A = a_{i_1 i_2 \cdots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}$. The symmetric nature of $a_{i_1 \ldots i_m}$ ensures the symmetry of A with respect to all its indices. In this context, F is referred to as an m-th root Finsler metric. It is evident that A exhibits homogeneity of degree m in y. Let

$$\begin{aligned} A_i &= a_{i i_2 \ldots i_m}(x) y^{i_2} \ldots y^{i_m} = \frac{1}{m}\frac{\partial A}{\partial y^i}, \\ A_{ij} &= a_{ij i_3 \ldots i_m}(x) y^{i_3} \ldots y^{i_m} = \frac{1}{m(m-1)}\frac{\partial^2 A}{\partial y^i \partial y^j}, \\ A_{ijk} &= a_{ijk i_4 \ldots i_m}(x) y^{i_4} \ldots y^{i_m} = \frac{1}{m(m-1)(m-2)}\frac{\partial^3 A}{\partial y^i \partial y^j \partial y^k}. \end{aligned} \tag{2}$$

The normalized supporting element of F^n is given by

$$l_i := F_{y^i} = \frac{\partial F}{\partial y^i} = \frac{\partial \sqrt[m]{A}}{\partial y^i} = \frac{1}{m}\frac{\frac{\partial A}{\partial y^i}}{A^{\frac{m-1}{m}}} = \frac{A_i}{F^{m-1}}. \tag{3}$$

Throughout the paper, we refer to the exponentially transformed m-th root Finsler metric as $\bar{F}$, and the exponentially transformed Finsler space is denoted as $(M^n, \bar{F}) = \bar{F}^n$. We focus on the case where $n > 2$ throughout the study and any quantities related to the transformed Finsler space $\bar{F}^n$ are indicated with a bar notation.

3. Fundamental metric tensor of exponential transformed m-th root metric

Let us consider the Finsler metric given in equation (1), where $F = \sqrt[m]{A}$, $\beta = b_i(x) y^i$ is a differential one form. The differentiation of (1) with respect to y^i yields the normalized supporting element $\bar{l}_i$ given by

$$\bar{l}_i = \frac{(F-\beta)}{F^m} A_i e^{\beta/F} + e^{\beta/F} b_i = \bar{F}\left\{\frac{(F-\beta)}{F^{m+1}} A_i + \frac{1}{F} b_i\right\}. \tag{4}$$

Again differentiation of (4) with respect to y^j yields

$$\bar{h}_{ij} = \bar{F}^2\Bigg[\frac{(m-1)(F-\beta)}{F^{m+1}}A_{ij} + \left\{\frac{(m-1)(\beta-F)}{F^{2m+1}} + \frac{\beta^2}{F^{2m+2}}\right\}A_iA_j - \frac{\beta}{F^{m+2}}(A_ib_j + A_jb_i) + \frac{1}{F^2}b_ib_j\Bigg]. \tag{5}$$

From (4) and (5), the fundamental metric tensor $\bar{g}_{ij}$ of Finsler space $\bar{F}^n$ is given by $\bar{g}_{ij} = \bar{h}_{ij} + \bar{l}_i\bar{l}_j$, after simplification, we get

$$\bar{g}_{ij} = \rho A_{ij} + \rho_0 b_ib_j + \rho_1(A_ib_j + A_jb_i) + \rho_2 A_iA_j, \tag{6}$$

where
$$\rho = \frac{(m-1)\bar{F}^2(F-\beta)}{F^{m+1}}, \quad \rho_0 = \frac{2\bar{F}^2}{F^2}, \quad \rho_1 = \frac{\bar{F}^2(F-2\beta)}{F^{m+2}},$$
$$\rho_2 = \frac{\bar{F}^2}{F^{2m+2}}\{(2-m)F^2 + 2\beta^2 + F\beta(m-3)\}.$$

In view of equation (6), the metric tensor $\bar{g}_{ij}$ of $\bar{F}^n$ can be rewritten as

$$\bar{g}_{ij} = \tau_1 g_{ij} + \rho_0 b_ib_j + \rho_1(A_ib_j + A_jb_i) + \tau_2 A_iA_j, \tag{7}$$

where
$$\begin{aligned} g_{ij} &= (m-1)\frac{A_{ij}}{F^{m-2}} - (m-2)\frac{A_iA_j}{F^{2(m-1)}}, \\ \tau_1 &= \frac{(F-\beta)\bar{F}^2}{F^3}, \\ \tau_2 &= \rho_2 + \frac{(m-2)(F-\beta)\bar{F}^2}{F^{2m+1}}. \end{aligned} \tag{8}$$

The contravariant metric tensor $\bar{g}^{ij}$ of Finsler space $\bar{F}^n$ is given by

$$\bar{g}^{ij} = \frac{A^{ij}}{\rho} - t_1 b^ib^j - t_2(b^iy^j + b^jy^i) - t_3 y^iy^j, \tag{9}$$

where
$$t_1 = \rho_4 + \rho_7\rho_5^2, \quad t_2 = \rho_7\rho_6\rho_5, \quad t_3 = \rho_7\rho_6^2, \quad \rho_3 = \frac{\rho_1^2}{\rho_2}, \tag{10}$$
$$\rho_4 = \frac{1}{\rho\left\{b^2 + \dfrac{\rho\rho_2}{\rho_0\rho_2 - \rho_1^2}\right\}}, \quad \rho_5 = \frac{1}{\rho} - \rho_4 b^2 - \frac{\rho_1\rho_4}{\rho_2},$$
$$\rho_6 = \frac{\rho_1}{\rho\rho_2}, \quad \rho_7 = \frac{\rho_3}{1 + d^2\rho_3}, \quad d^2 = \rho_5\left(b^2 + \beta\frac{\rho_2}{\rho_1}\right) + \rho_6\left(\beta + F^m\frac{\rho_2}{\rho_1}\right).$$

Further, in view of equation (9), the contravariant metric tensor $\bar{g}^{ij}$ can be rewritten as

$$\bar{g}^{ij} = \frac{1}{\tau_1}g^{ij} - t_0 b^ib^j - t_1(b^iy^j + b^jy^i) - \tau_3 y^iy^j, \tag{11}$$

where
$$g^{ij} = \frac{F^{m-2}}{(m-1)}A^{ij} + \frac{(m-2)}{(m-1)}\frac{y^iy^j}{F^2}, \tag{12}$$
$$\tau_3 = t_3 + \frac{(m-2)}{\rho F^m}.$$

PROPOSITION 3.1. *The Fundamental metric tensor $\bar{g}_{ij}$ and its inverse tensor $\bar{g}^{ij}$ of exponentially transformed m-th root Finsler metric $\bar{F}$ are given by equations* (7) *and* (11) *respectively.*

REMARK 3.2. It is remarkable to note that the metric tensors $\bar{g}_{ij}$ and $\bar{g}^{ij}$ of $\bar{F}^n$ are not necessarily rational functions in y.

4. Spray coefficients of exponentially transformed m-th root metric

The Geodesics of a Finsler space F^n are given by the following system of equations
$$\frac{d^2x^i}{dt^2} + G^i\left(x, \frac{dx}{dt}\right) = 0, \qquad \text{where} \quad G^i = \frac{1}{4}g^{il}\left\{\left[F^2\right]_{x^k y^l} y^k - \left[F^2\right]_{x^l}\right\},$$
are called the spray coefficients of F^n. Two Finsler metrics F and $\bar{F}$ on a manifold M^n are called projectively related if there is a scalar function $P(x,y)$ defined on TM_0 such that $\bar{G}^i = G^i + Py^i$, where $\bar{G}^i$ and G^i are the geodesic spray coefficients of $\bar{F}^n$ and F^n respectively [9]. In other words, two metrics F and $\bar{F}$ are called projectively related if any geodesic of the first is also geodesic for the second and vice-versa.

The Spray coefficients of exponentially transformed Finsler space $\bar{F}^n$ are given by
$$\bar{G}^i = \frac{1}{4}\bar{g}^{il}\left\{\left[\bar{F}^2\right]_{x^k y^l} y^k - \left[\bar{F}^2\right]_{x^l}\right\}.$$
It can also be written as
$$\bar{G}^i = \frac{1}{4}\bar{g}^{il}\left\{\left(2\frac{\partial \bar{g}_{jl}}{\partial x^k} - \frac{\partial \bar{g}_{jk}}{\partial x^l}\right) y^j y^k\right\}. \tag{13}$$
From (7) and (13), we get
$$\begin{aligned}\bar{G}^i = \frac{\bar{g}^{il}}{4}\Bigg[&2\frac{\partial}{\partial x^k}\Big\{\tau_1 g_{jl} + \rho_0 b_j b_l + \rho_1(A_j b_l + A_l b_j) + \tau_2 A_j A_l\Big\} y^j y^k \\ &- \frac{\partial}{\partial x^l}\Big\{\tau_1 g_{jk} + \rho_0 b_j b_k + \rho_1(A_j b_k + A_k b_j) + \tau_2 A_j A_k\Big\} y^j y^k\Bigg],\end{aligned}$$
which implies that
$$\bar{G}^i = \frac{\bar{g}^{il}}{4}\left[2\left\{\tau_1\frac{\partial}{\partial x^k}g_{jl} + g_{jl}w_k + \frac{\partial}{\partial x^k}X_{jl}\right\} y^j y^k - \left\{\tau_1\frac{\partial}{\partial x^l}g_{jk} + g_{jk}w_l + \frac{\partial}{\partial x^l}X_{jk}\right\} y^j y^k\right],$$
where $X_{jl} = \rho_0 b_j b_l + \rho_1(A_j b_l + A_l b_j) + \tau_2 A_j A_l$ and $w_k = \frac{\partial}{\partial x^k}(\tau_1)$.

Now, using equation (11), we have
$$\begin{aligned}\bar{G}^i = \frac{1}{4}&\left\{\frac{1}{\tau_1}g^{il} - t_0 b^i b^l - t_1(b^i y^l + b^l y^i) - \tau_3 y^i y^l\right\}\times \\ &\left\{\left(2\frac{\partial}{\partial x^k}g_{jl} - \frac{\partial}{\partial x^l}g_{jk}\right)\tau_1 + 2g_{jl}w_k - g_{jk}w_l + 2\frac{\partial}{\partial x^k}X_{jl} - \frac{\partial}{\partial x^l}X_{jk}\right\} y^j y^k,\end{aligned}$$
this equation can be written as
$$\begin{aligned}\bar{G}^i = \frac{1}{4}&\left\{\frac{1}{\tau_1}g^{il} - y^i(t_1 b^l + \tau_3 y^l) - b^i(t_0 b^l - t_1 y^l)\right\}\times \\ &\left\{\left(2\frac{\partial}{\partial x^k}g_{jl} - \frac{\partial}{\partial x^l}g_{jk}\right)\tau_1 + 2g_{jl}w_k - g_{jk}w_l + 2\frac{\partial}{\partial x^k}X_{jl} - \frac{\partial}{\partial x^l}X_{jk}\right\} y^j y^k,\end{aligned}$$

which implies that

$$\bar{G}^i = \frac{g^{il}}{4}\left(2\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right)y^jy^k+\frac{1}{\tau_1}g^{il}R_{jkl}y^jy^k-y^i(t_1b^l+\tau_3y^l)S_{jkl}y^jy^k-b^i(t_0b^l-t_1y^l)S_{jkl}y^jy^k,$$

where

$$R_{jkl} = \frac{1}{4}\left\{2g_{jl}w_k - g_{jk}w_l + 2\frac{\partial}{\partial x^k}X_{jl} - \frac{\partial}{\partial x^l}X_{jk}\right\},$$

$$S_{jkl} = \frac{1}{4}\left\{\left(2\frac{\partial}{\partial x^k}g_{jl} - \frac{\partial}{\partial x^l}g_{jk}\right)\tau_1 + 2g_{jl}w_k - g_{jk}w_l + 2\frac{\partial}{\partial x^k}X_{jl} - \frac{\partial}{\partial x^l}X_{jk}\right\}.$$

Now, using the equations (8) and (12), we get

$$\bar{G}^i = \frac{g^{il}}{4}\left(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)y^jy^k + \frac{F^3R_{jkl}}{(F-\beta)\bar{F}^2}\left\{\frac{F^{m-2}}{(m-1)}A^{il} + \frac{(m-2)}{(m-1)}\frac{y^iy^l}{F^2}\right\}y^jy^k$$
$$- y^i(t_1b^l + \tau_3y^l)S_{jkl}y^jy^k - b^i(t_0b^l - t_1y^l)S_{jkl}y^jy^k.$$

The above equation may be rewritten as $\bar{G}^i = G^i + Py^i + Q^i$, where

$$G^i = \frac{1}{4}g^{il}\left\{\left(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)y^jy^k\right\},$$

$$P = \left\{\frac{(m-2)FR_{jkl}}{(m-1)(F-\beta)\bar{F}^2}y^l - (t_1b^l + \tau_3y^l)S_{jkl}\right\}y^jy^k,$$

$$Q^i = \left\{\frac{F^{m+1}R_{jkl}}{(m-1)(F-\beta)\bar{F}^2}A^{il} - b^i(t_0b^l - t_1y^l)S_{jkl}\right\}y^jy^k.$$

Now, $\bar{F}$ and F will be projectively related if $Q^i = 0$, which implies

$$R_{jkl}A^{il} = \frac{(m-1)(F-\beta)\bar{F}^2}{F^{m+1}}b^i(t_0b^l - t_1y^l)S_{jkl}. \tag{14}$$

Thus, we have the following theorem.

THEOREM 4.1. *The exponentially transformed m-th root metric $\bar{F}$ and m-th root metric F, on an open subset $U \subset R^n$, are projectively related if equation* (14) *is satisfied.*

REMARK 4.2. It is remarkable to note that the spray coefficients $\bar{G}^i$ of $\bar{F}^n$ are rational functions in y.

5. Locally dually flatness of exponentially transformed m-th root metric

In Finsler geometry, Z. Shen [11] extended the notion of locally dually flatness for Finsler metrics that every locally Minkowskian metric is locally dually flat.

DEFINITION 5.1 ([11]). A transformed Finsler Space with Finsler metric $\bar{F} = \bar{F}(x, y)$ on a manifold M^n is said to be locally dually flat if at any point there is a standard coordinate system (x^i, y^i) in TM such that $\left[\bar{F}^2\right]_{x^ky^l}y^k = 2\left[\bar{F}^2\right]_{x^l}$. The coordinate (x^i) is called an adapted local coordinate system.

Consider the exponential transformation $\bar{F} = Fe^{\beta/F}$, where F is an m-th root metric. For the exponentially transformed m-th root Finsler metric $\bar{F}$, we have

$$\left[\bar{F}^2\right]_{x^k} = 2e^{2\beta/F}\left[\frac{F-\beta}{mF^{m-1}}A_{x^k} + F\beta_k\right]. \tag{15}$$

From previous equation, we get

$$\left[\bar{F}^2\right]_{x^k y^l} = 2e^{2\beta/F}\Bigg[\left\{\frac{(\beta-F)}{mF^{2m-1}} + \frac{(2F^2+2\beta^2-3F\beta)}{m^2F^{2m}}\right\}A_{x^k}A_{y^l} + \frac{(F-\beta)}{mF^{m-1}}A_{x^k y^l}$$
$$+\frac{(F-2\beta)}{mF^m}A_{y^l}\beta_k + \frac{(F-2\beta)}{mF^m}A_{x^l}b_l + 2b_l\beta_k + Fb_{lk}\Bigg],$$

and
$$\left[\bar{F}^2\right]_{x^k y^l} y^k = 2e^{2\beta/F}\Bigg[\left\{\frac{(\beta-F)}{mF^{2m-1}} + \frac{(2F^2+2\beta^2-3F\beta)}{m^2F^{2m}}\right\}A_0A_{y^l} + \frac{(F-\beta)}{mF^{m-1}}A_{0l}$$
$$+\frac{(F-2\beta)}{mF^m}A_{y^l}\beta_k y^k + \frac{(F-2\beta)}{mF^m}A_0b_l + 2b_l\beta_k y^k + F\beta_l\Bigg]. \tag{16}$$

For the transformed m-th root Finsler metric $\bar{F}$ to be locally dually flat, we must have

$$\left[\bar{F}^2\right]_{x^k y^l} y^k - 2\left[\bar{F}^2\right]_{x^l} = 0. \tag{17}$$

In view of equations (15), (16), and (17), we have

$$\left[\bar{F}^2\right]_{x^k y^l} y^k - 2\left[\bar{F}^2\right]_{x^l} = 2e^{2\beta/F}\Bigg[\left\{\frac{(\beta-F)}{mF^{2m-1}} + \frac{(2F^2+2\beta^2-3F\beta)}{m^2F^{2m}}\right\}A_0A_{y^l}$$
$$+\frac{(F-\beta)}{mF^{m-1}}A_{0l} + \frac{(F-2\beta)}{mF^m}A_{y^l}\beta_k y^k + \frac{(F-2\beta)}{mF^m}A_0b_l$$
$$+2b_l\beta_k y^k + F\beta_l\Bigg] - 4e^{2\beta/F}\left[\frac{F-\beta}{mF^{m-1}}A_{x^l} + F\beta_l\right] = 0.$$

The above equation can be rewritten as

$$4e^{2\beta/F}\frac{F-\beta}{mF^{m-1}}A_{x^l} = 2e^{2\beta/F}\Bigg[\left\{\frac{(\beta-F)}{mF^{2m-1}} + \frac{(2F^2+2\beta^2-3F\beta)}{m^2F^{2m}}\right\}A_0A_{y^l}$$
$$+\frac{(F-\beta)}{mF^{m-1}}A_{0l} + \frac{(F-2\beta)}{mF^m}A_{y^l}\beta_k y^k + \frac{(F-2\beta)}{mF^m}A_0b_l$$
$$+2b_l\beta_k y^k + F\beta_l\Bigg] - 4e^{2\beta/F}F\beta_l.$$

Therefore, $\bar{F}$ is locally dually flat metric if and only if

$$A_{x^l} = \left\{\frac{-1}{2F^m} + \frac{(2F^2+2\beta^2-3F\beta)}{2m(F-\beta)F^{m+1}}\right\}A_0A_{y^l} + \frac{1}{2}A_{0l} + \frac{(F-2\beta)}{2F(F-\beta)}A_{y^l}\beta_k y^k$$
$$+\frac{(F-2\beta)}{2F(F-\beta)}A_0b_l + \frac{mF^{m-1}}{(F-\beta)}b_l\beta_k y^k - \frac{mF^m}{2(F-\beta)}\beta_l. \tag{18}$$

Thus, we have the following theorem.

THEOREM 5.2. *The exponentially transformed m-th root Finsler metric $\bar{F}$ on a Finsler manifold M^n is locally dually flat metric if and only if the equation (18) holds.*

6. Projective flatness of exponentially transformed m-th root metric

DEFINITION 6.1 ([11]). A transformed Finsler Space with metric $\bar{F} = \bar{F}(x, y)$ on an open subset U of manifold M^n is projectively flat if and only if it satisfies the following equations:

$$\left[\bar{F}^2\right]_{x^k y^l} y^k = \left[\bar{F}^2\right]_{x^l}. \tag{19}$$

Consider the exponential transformation $\bar{F} = Fe^{\beta/F}$, where F is an m-th root metric. Using the equations (15) and (16) in the equation (19), we obtain

$$\begin{aligned}\left[\bar{F}^2\right]_{x^k y^l} y^k - \left[\bar{F}^2\right]_{x^l} = 2e^{2\beta/F}\Bigg[&\left\{\frac{(\beta - F)}{mF^{2m-1}} + \frac{(2F^2 + 2\beta^2 - 3F\beta)}{m^2F^{2m}}\right\}A_0A_{y^l}\\ &+ \frac{(F-\beta)}{mF^{m-1}}A_{0l} + \frac{(F-2\beta)}{mF^m}A_{y^l}\beta_k y^k + \frac{(F-2\beta)}{mF^m}A_0 b_l\\ &+ 2b_l\beta_k y^k + F\beta_l\Bigg] - 2e^{2\beta/F}\left[\frac{(F-\beta)}{mF^{m-1}}A_{x^l} + F\beta_l\right] = 0.\end{aligned}$$

The above equation can be rewritten as

$$\begin{aligned}2e^{2\beta/F}\frac{(F-\beta)}{mF^{m-1}}A_{x^l} = 2e^{2\beta/F}\Bigg[&\left\{\frac{(\beta - F)}{mF^{2m-1}} + \frac{(2F^2 + 2\beta^2 - 3F\beta)}{m^2F^{2m}}\right\}A_0A_{y^l}\\ &+ \frac{(F-\beta)}{mF^{m-1}}A_{0l} + \frac{(F-2\beta)}{mF^m}A_{y^l}\beta_k y^k + \frac{(F-2\beta)}{mF^m}A_0 b_l\\ &+ 2b_l\beta_k y^k + F\beta_l\Bigg] - 2e^{2\beta/F}F\beta_l.\end{aligned}$$

On simplification, we obtain

$$\begin{aligned}A_{x^l} = &\left\{\frac{-1}{F^m} + \frac{(2F^2 + 2\beta^2 - 3F\beta)}{m(F-\beta)F^{m+1}}\right\}A_0A_{y^l} + A_{0l} + \frac{(F-2\beta)}{F(F-\beta)}A_{y^l}\beta_k y^k\\ &+ \frac{(F-2\beta)}{F(F-\beta)}A_0 b_l + \frac{2mF^{m-1}}{(F-\beta)}b_l\beta_k y^k.\end{aligned} \tag{20}$$

Thus, we have the following theorem.

THEOREM 6.2. *The exponentially transformed m-th root Finsler metric $\bar{F}$ on a Finsler manifold M^n is projectively flat if and only if the equation* (20) *holds.*

7. Berwald curvature

For a non-zero tangent vector $y \in T_xM$, in a local coordinate system, define $B_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $E_y : T_xM \otimes T_xM \to \mathbb{R}$ by $B_y(u, v, w) = B^i_{jkl}(x, y)u^j v^k w^l \dfrac{\partial}{\partial x^i}$ and $E_y(u, v) = E_{jk}(x, y)u^j v^k$ respectively, where $B^i_{jkl} = [G^i]_{y^j y^k y^l}$, $E_{jk} = \dfrac{1}{2}B^m_{mjk} = \dfrac{1}{2}\dfrac{\partial^2}{\partial y^j \partial y^k}\left[\dfrac{\partial G^m}{\partial y^m}\right]$, $u = u^i\dfrac{\partial}{\partial x^i}, v = v^i\dfrac{\partial}{\partial x^i}$, and $w = w^i\dfrac{\partial}{\partial x^i}$. Then, $B = \{B_y \mid y \in$

$TM_0\}$ and $E = \{E_y \mid y \in TM_0\}$ are called the Berwald curvature and mean Berwald curvature respectively. A Finsler metric is called a Berwald metric and a mean Berwald metric if $B = 0$ and $E = 0$ respectively.

A Finsler metric F is said to be an isotropic Berwald metric if its Berwald curvature is in the following form:

$$B^i_{jkl} = c(F_{y^jy^k}\delta^i_l + F_{y^jy^l}\delta^i_k + F_{y^ky^l}\delta^i_j + F_{y^jy^ky^l}y^i), \tag{21}$$

where $c = c(x)$ is a scalar function on M (see [6]). A Finsler metric F is said to be an isotropic mean Berwald metric if $E_{ij} = \frac{n+1}{2}c(x)h_{ij}$, where $c = c(x)$ is a scalar function on M and h_{ij} is the angular metric tensor. A Finsler metric F is said to be weakly Berwald metric if its mean curvature $E_{ij} = 0$.

THEOREM 7.1. *Suppose $\bar{F}$ is an exponentially transformed m-th root Finsler metric on a manifold, and $\bar{F}$ has isotropic Berwald curvature. Then, $\bar{F}$ reduces to a Berwald metric.*

Proof. Let $\bar{F} = Fe^{\beta/F}$ be an exponentially transformed m-th root Finsler metric. Suppose that $\bar{F}$ has isotropic Berwald curvature given by equation (21). By Remark 3.2, the left-hand side of equation (21) is a rational function in y, while the right-hand side is an irrational function. This implies that $c = 0$ and hence $\bar{F}$ reduces to a Berwald metric. □

THEOREM 7.2. *Suppose $\bar{F}$ is an exponentially transformed m-th root Finsler metric on a manifold, and $\bar{F}$ has isotropic mean Berwald curvature. Then, $\bar{F}$ reduces to weakly Berwald metric.*

Proof. Let $\bar{F} = Fe^{\beta/F}$ be a metric with isotropic mean Berwald curvature, i.e.,

$$\bar{E}_{ij} = \frac{n+1}{2\bar{F}}c\bar{h}_{ij}, \tag{22}$$

where $c = c(x)$ is a scalar function on M and $\bar{h}_{ij}$ is the angular metric tensor. By putting the angular metric $\bar{h}_{ij}$ given by equation (5) in (22), we have

$$\begin{aligned}\bar{E}_{ij} = \frac{(n+1)c\bar{F}}{2}\Bigg[&\frac{(m-1)(F-\beta)}{F^{m+1}}A_{ij} + \left\{\frac{(m-1)(\beta-F)}{F^{2m+1}} + \frac{\beta^2}{F^{2m+2}}\right\}A_iA_j \\ &- \frac{\beta}{F^{m+2}}(A_ib_j + A_jb_i) + \frac{1}{F^2}b_ib_j\Bigg].\end{aligned} \tag{23}$$

The left side of equation (23) is a rational function in y while its right side is an irrational function in y. Thus, equation (23) implies either $c = 0$ or

$$\begin{aligned}\Bigg[\frac{(m-1)(F-\beta)}{F^{m+1}}A_{ij} + &\left\{\frac{(m-1)(\beta-F)}{F^{2m+1}} + \frac{\beta^2}{F^{2m+2}}\right\}A_iA_j \\ &- \frac{\beta}{F^{m+2}}(A_ib_j + A_jb_i) + \frac{1}{F^2}b_ib_j\Bigg] = 0.\end{aligned} \tag{24}$$

If equation (24) holds, then $\bar{h}_{ij} = 0$, which is not possible. Hence, $c = 0$ and $\bar{E}_{ij} = 0$. Hence, the proof is complete. □

8. Landsberg curvature

For a non-zero tangent vector $y \in T_xM$, in a local coordinate system, define $L_y(u,v,w) := L_{ijk}(x,y)u^iv^jw^k$, where $L_{ijk} := -\frac{1}{2}FF_{y^s}[G^s]_{y^iy^jy^k}$, $u = u^i\frac{\partial}{\partial x^i}, v = v^i\frac{\partial}{\partial x^i}$ and $w = w^i\frac{\partial}{\partial x^i}$. Then, $L = \{L_y \mid y \in TM_0\}$ is called the Landsberg curvature and a Finsler metric is called a Landsberg metric if $L = 0$. For a non-zero tangent vector $y \in T_pM$, the Cartan torsion $C_y = C_{ijk}(x,y)dx^i \otimes dx^j \otimes dx^k : T_pM \otimes T_pM \otimes T_pM \to \mathbb{R}$ is defined by $C_{ijk} = -\frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$.

THEOREM 8.1. *Suppose $\bar{F}$ is an exponentially transformed m-th root Finsler metric on a manifold. If $\bar{F}$ has isotropic Landsberg curvature, then $\bar{F}$ reduces to a Landsberg metric.*

Proof. Let $\bar{F} = Fe^{\beta/F}$ be an isotropic Landsberg metric, that is

$$\bar{L}_{ijk} = c\bar{F}\bar{C}_{ijk}, \tag{25}$$

where $c = c(x)$ is a scalar function on M. The Cartan torsion $\bar{C}_{ijk}$ of $\bar{F}$ is given by the following equation

$$\begin{aligned}\bar{C}_{ijk} = \frac{\bar{F}^2}{2}\Bigg[&\frac{(m-1)(F-\beta)}{F^{m+1}}A_{ijk} + \frac{1}{F^{2m+2}}\{(2-m)F^2 + 2\beta^2 + F\beta(m-3)\}\sum_{(i,j,k)}A_{ij}A_k \\ &- \frac{1}{F^{3m+3}}\{4\beta^3 - (m-1)F(2F^2m - 2mF\beta + 7F\beta - 4F^2 - 6\beta^2)\}A_iA_jA_k \\ &+ \frac{(F-2\beta)}{F^{m+2}}\sum_{(i,j,k)}A_{ij}b_k + \frac{1}{F^{2m+3}}\{4\beta^2 + (m-1)F(2\beta - F)\}\sum_{(i,j,k)}A_iA_jb_k \\ &- \frac{4\beta}{F^{m+3}}\sum_{(i,j,k)}A_ib_jb_k - \frac{4}{F^3}b_ib_jb_k\Bigg],\end{aligned} \tag{26}$$

where $\sum_{(i,j,k)}$ denotes the cyclic interchange of suffices i,j,k and summation, for instance $\sum_{(i,j,k)}A_{ij}A_k = A_{ij}A_k + A_{jk}A_i + A_{ik}A_j$. Also, we know that $\bar{L}_{ijk} = -\frac{1}{2}\bar{F}\bar{F}_{y^s}\bar{G}^s_{y^iy^jy^k}$. In our case,

$$\bar{L}_{ijk} = -\frac{\bar{F}^2}{2}\left\{\frac{(F-\beta)}{F^{m+1}}A_s + \frac{1}{F}b_s\right\}\bar{G}^s_{y^iy^jy^k}. \tag{27}$$

Using the equations (25), (26), and (27), we obtain

$$\begin{aligned}\left\{\frac{(F-\beta)}{F^{m+1}}A_s + \frac{1}{F}b_s\right\}\bar{G}^s_{y^iy^jy^k} = -c\bar{F}\Bigg[&\frac{(m-1)(F-\beta)}{F^{m+1}}A_{ijk} \\ &+ \frac{1}{F^{2m+2}}\{(2-m)F^2 + 2\beta^2 + F\beta(m-3)\}\sum_{(i,j,k)}A_{ij}A_k \\ &- \frac{1}{F^{3m+3}}\{4\beta^3 - (m-1)F(2F^2m - 2mF\beta + 7F\beta - 4F^2 - 6\beta^2)\}A_iA_jA_k\end{aligned}$$

$$+\frac{(F-2\beta)}{F^{m+2}}\sum_{(i,j,k)}A_{ij}b_k+\frac{1}{F^{2m+3}}\{4\beta^2+(m-1)F(2\beta-F)\}\sum_{(i,j,k)}A_iA_jb_k$$
$$-\frac{4\beta}{F^{m+3}}\sum_{(i,j,k)}A_ib_jb_k-\frac{4}{F^3}b_ib_jb_k\Bigg]. \tag{28}$$

The left side of equation (28) is a rational function in y while its right side is an irrational function in y. Therefore, either $c=0$ or A satisfies the following PDEs:

$$\Bigg[\frac{(m-1)(F-\beta)}{F^{m+1}}A_{ijk}+\frac{1}{F^{2m+2}}\{(2-m)F^2+2\beta^2+F\beta(m-3)\}\sum_{(i,j,k)}A_{ij}A_k$$
$$-\frac{1}{F^{3m+3}}\{4\beta^3-(m-1)F(2F^2m-2mF\beta+7F\beta-4F^2-6\beta^2)\}A_iA_jA_k$$
$$+\frac{(F-2\beta)}{F^{m+2}}\sum_{(i,j,k)}A_{ij}b_k+\frac{1}{F^{2m+3}}\{4\beta^2+(m-1)F(2\beta-F)\}\sum_{(i,j,k)}A_iA_jb_k$$
$$-\frac{4\beta}{F^{m+3}}\sum_{(i,j,k)}A_ib_jb_k-\frac{4}{F^3}b_ib_jb_k\Bigg]=0.$$

If the above equation holds, then $\bar{C}_{ijk}=0$. Hence, $\bar{F}$ is a Riemannian metric, which is contraction our assumption. Therefore, $c=0$. □

The mean Cartan torsion $I_y=I_i(x,y)dx^i:T_pM\to\mathbb{R}$ is defined by

$$I_i=g^{jk}C_{ijk}. \tag{29}$$

Thus, the mean Cartan of $\bar{F}$ is given by

$$\bar{I}_i=\frac{\bar{F}^2}{2}\left\{\frac{A^{jk}}{\rho}-t_1b^jb^k-t_2(b^jy^k+b^ky^j)-t_3y^jy^k\right\}$$
$$\Bigg[\frac{(m-1)(F-\beta)}{F^{m+1}}A_{ijk}+\frac{1}{F^{2m+2}}\{(2-m)F^2+2\beta^2+F\beta(m-3)\}\sum_{(i,j,k)}A_{ij}A_k$$
$$-\frac{1}{F^{3m+3}}\{4\beta^3-(m-1)F(2F^2m-2mF\beta+7F\beta-4F^2-6\beta^2)\}A_iA_jA_k$$
$$+\frac{(F-2\beta)}{F^{m+2}}\sum_{(i,j,k)}A_{ij}b_k+\frac{1}{F^{2m+3}}\{4\beta^2+(m-1)F(2\beta-F)\}\sum_{(i,j,k)}A_iA_jb_k$$
$$-\frac{4\beta}{F^{m+3}}\sum_{(i,j,k)}A_ib_jb_k-\frac{4}{F^3}b_ib_jb_k\Bigg].$$

LEMMA 8.2. *Suppose $\bar{F}$ is an exponentially transformed m-th root Finsler metric on a manifold. Then, mean Cartan torsions $\bar{I}_i$ of $\bar{F}^n$ are rational functions in y.*

The horizontal covariant derivative of I along a vector $u\in T_xM$ gives rise to the mean Landsberg curvature $J_y(u):=J_i(y)u^i$, where $J_i=I_{i|s}y^s$. A Finsler metric with $J=0$ is called a weakly Landsberg metric.

The mean Landsberg curvature of $\bar{F}$ is defined by

$$\bar{J}_i=\bar{g}^{jl}\bar{L}_{ijl}. \tag{30}$$

In view of equations (9), (27) and (30), we have

$$\bar{J}_i = -\frac{\bar{F}^2}{2}\left\{\frac{A^{jl}}{\rho} - t_1 b^j b^l - t_2(b^j y^l + b^l y^j) - t_3 y^j y^l\right\}\left\{\frac{(F-\beta)}{F^{m+1}}A_s + \frac{1}{F}b_s\right\}\bar{G}^s_{y^i y^j y^l},$$

where t_1, t_2 and t_3 are given in equation (10).

LEMMA 8.3. *Suppose $\bar{F}$ is an exponentially transformed m-th root Finsler metric on a manifold. Then, mean Landsberg curvatures $\bar{J}_i$ of $\bar{F}^n$ are rational functions in y.*

In view of equations (25), (29) and (30), we have

$$\bar{J}_i = c\bar{F}\bar{I}_i. \tag{31}$$

THEOREM 8.4. *Suppose $\bar{F}$ is an exponentially transformed m-th root Finsler metric on a manifold with isotropic mean Landsberg curvature. Then, $\bar{F}$ is a weakly Landsberg metric.*

Proof. Using Lemma 8.2 and Lemma 8.3, the left side of equation (31) is a rational function in y while its right side is an irrational function in y. Thus, if equation (31) holds, then either $c = 0$ or $\bar{I}_i = 0$. If $\bar{I}_i = 0$, $\bar{F}$ is a Riemannian metric, which contradicts our assumption. Therefore, $c = 0$ and the proof is complete. □

9. *H*-curvature

In [1], Akbar-Zadeh introduced the non-Riemannian quantity H, which is derived from the mean Berwald curvature through covariant horizontal differentiation along geodesics. The quantity H is defined as $H = H_{ij}dx^i dx^j$, where $H_{ij} := E_{ijs}y^s$. The main result of the paper states that for a Finsler manifold with scalar flag curvature K and dimension $n \geq 3$, if the flag curvature $\mathbf{K}$ is constant, then the non-Riemannian quantity H must be equal to zero. This result establishes a relationship between the constancy of flag curvature and the vanishing of the H-curvature.

The concept of almost vanishing H-curvature was initially introduced by Shen et al. [12]. They investigated the close connection between the non-Riemannian quantity H and flag curvature.

A Finsler metric F on an n-dimensional manifold M is said to be of almost vanishing H-curvature if $H_{ij} = \dfrac{(n+1)}{2F}\theta h_{ij}$, for some 1-form θ on M, where h_{ij} is the angular metric.

THEOREM 9.1. *Suppose $\bar{F}$ is an exponentially transformed m-th root Finsler metric on a manifold, and $\bar{F}$ has almost vanishing H-curvature. Then, $H = 0$.*

Proof. Let $\bar{F} = Fe^{\beta/F}$ be an exponentially transformed m-th root Finsler metric. Suppose that $\bar{F}$ has almost vanishing $\bar{H}$-curvature, that is,

$$\bar{H}_{ij} = \frac{(n+1)}{2\bar{F}}\theta\bar{h}_{ij}, \tag{32}$$

for some 1-form θ on M, where $\bar{h}_{ij}$ is the angular metric tensor. By putting the angular metric $\bar{h}_{ij}$ given by equation (5) in (32), we have

$$\bar{H}_{ij} = \frac{(n+1)c\bar{F}}{2}\left[\frac{(m-1)(F-\beta)}{F^{m+1}}A_{ij} + \left\{\frac{(m-1)(\beta-F)}{F^{2m+1}} + \frac{\beta^2}{F^{2m+2}}\right\}A_iA_j - \frac{\beta}{F^{m+2}}(A_ib_j + A_jb_i) + \frac{1}{F^2}b_ib_j\right]. \tag{33}$$

The left side of equation (33) is a rational function in y while its right side is an irrational function in y. Thus, equation (33) implies either $\theta = 0$ or

$$\left[\frac{(m-1)(F-\beta)}{F^{m+1}}A_{ij} + \left\{\frac{(m-1)(\beta-F)}{F^{2m+1}} + \frac{\beta^2}{F^{2m+2}}\right\}A_iA_j - \frac{\beta}{F^{m+2}}(A_ib_j + A_jb_i) + \frac{1}{F^2}b_ib_j\right] = 0. \tag{34}$$

If equation (34) holds, then $\bar{h}_{ij} = 0$, which is not possible. Hence, $\theta = 0$ and $\bar{H}_{ij} = 0$. Hence the proof is complete. □

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