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DENSE SET OF LARGE PERIODIC POINTS AND CHAOTIC GROUP ACTIONS

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Abstract. In this paper it is proved that a chaotic group action has a dense set of large periodic points. A counterexample shows that the converse doesn't hold. Furthermore, some interesting results about the topological transitivity of group actions are discussed.

1. Introduction

The study of dynamical systems describes how the state of a system transforms into another state over time. Given a space X and a continuous function $f : X \to X$, the pair (X, f) is called a dynamical system, which studies the behaviour of any point $x \in X$ under the function f. Chaos is a characteristic of complex deterministic dynamical systems that exhibit completely unpredictable behaviour. Therefore, chaos theory is a captivating aspect of the study of dynamical systems. Interestingly, many simple systems exhibit chaotic behaviour, and there are many complex systems that are not chaotic. Therefore, it is difficult to define chaos and there is no universally accepted definition for it. Chaos has been defined mathematically in various ways that are not necessarily equivalent. The term chaos was first introduced into mathematics by Li and Yorke in 1975 [12] and is known as Li-Yorke chaos. The most popular definition of chaos is that of R. L. Devaney in 1980 [5].

In this paper we focus on Devaney's definition of chaos, which states that a function on a space X is said to be Devaney chaotic if
1. it is topologically transitive, i.e. for any two non-empty sets U and V, there exists some $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$.

2. the set of periodic points is dense in X.

3. and has a sensitive dependence on the initial conditions, i.e. there exists $\delta > 0$ such that for every $x \in X$ and every $\epsilon > 0$, there exists $y \in Y$ and $n \in \mathbb{N}$ such that $d(x, y) < \epsilon$ but $d(f^n(x), f^n(y)) \geq \delta$.

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Later, in 1992, Banks et al. [2] proved that the first two conditions for an infinite system imply the sensitive dependence on the initial conditions. In the same year, it was proved in [1] that neither transitivity nor the existence of a dense set of periodic points can be derived from the other two conditions.

In 1995, as a generalized notion of chaotic functions, chaotic group actions were defined and the groups that can act chaotically on some Hausdorff (topological) spaces were characterized (see [3, 10]). Stronger versions of chaotic group actions were later discussed in [4] by replacing the transitivity condition with a stronger condition such as weak mixing, strong mixing, k-transitivity, strong transitivity and total transitivity. However, no stronger versions of chaotic group actions were defined by replacing the second condition. In 2003, a stronger version of the existence of a dense set of periodic points of a chaotic function f on an infinite topological space X was studied in [9], and the property is called the existence of *a dense set of large periodic points* (DLP). They provided two different proofs to show that if the function f is Devaney-chaotic on an infinite space X, then the system (X, f) has DLP. Later in 2015, it was proved in [7] that if f is Devaney-chaotic on a compact metric space X with no isolated points, then the system (X, f) has DLP. The DLP property is also known as the strong dense periodicity property mentioned in [6, 8].

In this paper, we prove that a chaotic group action on an infinite Hausdorff has DLP and thus introduce another stronger version of a chaotic group action. Moreover, we see that the existence of DLP need not imply the chaoticity of the action.

2. Basic results and definitions

In this section, we discuss some of the interesting findings on the transitivity of group actions and outline the necessary basic definitions.

DEFINITION 2.1. A *group action* on a topological space X is a homomorphism $\phi : G \to Sym(X)$, where $Sym(X)$ is the group of all permutations of X. The action is called *faithful (effective)* if $ker(\phi) = \{e\}$, where e is the identity element of the group G.

It is worth noting that all groups have chaotic actions on $\{x\}$, but only faithful actions are of interest. Every group has at least one faithful chaotic action. For example, given any group G, we can consider the disjoint union $G \cup \{x\}$, which is called G'. Then G can act on G' by translating elements in G and fixing x. This action is faithful and chaotic if G' is equipped with the indiscrete topology. So we will focus on faithful chaotic actions on Hausdorff spaces. Moreover, we can state that the action of any group is equivalent to the faithful action of a subgroup of the group of homeomorphisms of X.

DEFINITION 2.2. A *continuous group action* on a topological space X is a homomorphism $\phi : G \to Homeo(X)$, where $Homeo(X)$ is the group of all homomorphisms of X. For each $g \in G$, the mapping $\phi(g) : X \to X$ is simply denoted by the mapping $g : X \to X$, where $gx := \phi(g)(x)$.

Definition 2.3. The *orbit* of a point $x \in X$ under the action of G on X is defined as $Gx = \{gx : g \in G\}$.

Definition 2.4 (Topological transitivity). The action is called *topologically transitive* if for any two non-empty open sets U and V of X, there exists an element $g \in G$ such that $gU \cap V \neq \emptyset$.

Definition 2.5 (Periodic point). A point $x \in X$ is called a *periodic point* of the action if the orbit Gx is finite. The number of elements in Gx is called the period of x. The set of all periodic points is denoted by P. For each $n \in \mathbb{N}$, P_n denotes the set of all periodic points of period less than or equal to n

Definition 2.6 (DLP). We say that an action has *dense set of large periodic points (DLP)* if $P \backslash P_n$ is dense in X for every $n \in \mathbb{N}$. This property is also known as the strong dense periodicity property.

Definition 2.7 (Chaotic group action). A continuous action is called *chaotic* if it is topologically transitive and the set of periodic points forms a dense subset of X.

Definition 2.8 (Invariant subset). A subset $A \subset X$ is said to be *invariant* under the action of G or simply invariant if $gA \subset A$ for all $g \in G$.

Throughout this paper, the actions are always continuous and faithful. The relation between the transitivity of self-maps and the existence of a dense orbit was studied earlier in [13]. The properties of transitive maps were also studied before (see [11]). Most of the results are stated for continuous maps. However, many results are also true for group actions. We consider Lemma 2.9, Theorem 2.10 and Theorem 2.11 in this direction. The following lemma provides some of the equivalent definitions for topological transitivity of group actions.

Lemma 2.9. *Let G be a group acting on a topological space X. Then the following conditions are equivalent.*
(i) G acts topologically transitive on X.

(ii) Every non-empty invariant open subset is dense in X.

(iii) For any non-empty open subset U of X, $\bigcup_{g \in G} gU$ is dense in X.

Proof. (i) $\implies$ (ii): Let U be a non-empty invariant open subset of X. If $U \neq X$ then we have to prove that $\overline{U} = X$. Suppose not. Since U is invariant, we have $gU \subset U$ for all $g \in G$. Let $V = X \backslash \overline{U}$ which is non-empty and open. Then for every $g \in G$, $gU \cap V = \emptyset$ since $gU \subset U$and $U \cap V = \emptyset$. This is a contradiction.

(ii) $\implies$ (iii): Let U be a non-empty open subset of X. Then clearly $\bigcup_{g \in G} gU$ is a non-empty invariant open subset of X and therefore it is dense in X by our assumption.

(iii) $\implies$ (i): Let U, V be non-empty open subsets of X. By hypothesis, $\bigcup_{g \in G} gU$ is dense in X. Then $V \cap \bigcup_{g \in G} gU \neq \emptyset$. This implies $V \cap gU \neq \emptyset$ for some $g \in G$. This proves the topological transitivity of the action. □

Next, we have the following theorem.

THEOREM 2.10. *Let G be a group that acts on a topological space X. Suppose the action has a dense orbit at a point x. Then the action is topologically transitive.*

Proof. Let $Gx = \{gx : g \in G\}$ be dense in X and let $U \neq \emptyset$ be an open subset of X such that $gU \subset U$ for all $g \in G$. Since Gx dense in X and U is non-empty and open, there exists $g \in G$ such that $gx \in U$. Then $Gx = Ggx = \{h(gx) : h \in G\} \subset U$. This implies U contains a dense subset of X. This implies U is dense in X. Hence, by Lemma 2.9, the action is topologically transitive. □

The topological transitivity for group actions is different to topological transitivity for continuous maps. For the latter, the existence of a dense orbit does not imply topological transitivity but for group actions it does, as Theorem 2.10 shows. A topological space is called a Baire space if every countable intersection of dense open sets is dense in it. The following theorem says that topological transitivity and the existence of a dense orbit for group actions are one and the same if the underlying space is a second countable Baire space.

THEOREM 2.11. *A group G acts topologically transitive on a second countable Baire space X if and only if there exists a dense orbit in X.*

Proof. Suppose the action is topologically transitive. Let $\{V_j\}$ be a countable base for X. Then $\bigcup_{g\in G} g^{-1}V_j$ is open and intersects every non-empty open subset of X since it is dense in X. Because X is a Baire space, $\bigcap_{j=1}^{\infty} \bigcup_{g\in G} g^{-1}V_j$ is non-empty. Let $x \in \bigcap_{j=1}^{\infty} \bigcup_{g\in G} g^{-1}V_j$. This implies $x \in \bigcup_{g\in G} g^{-1}V_j$ for all j. Therefore for some $g \in G$, $gx \in V_j$ for all j. This means that $G_x = \{gx : g \in G\} \cap V_j \neq \emptyset$ for any j. Hence the orbit is dense in X. The converse is given by Theorem 2.10. □

3. Main results

In this section we prove that every chaotic action on an infinite Hausdorff space has DLP. We also provide a counterexample to show that the existence of DLP does not necessarily imply that the action is chaotic.

DEFINITION 3.1. A collection of non-empty open sets $V_1, V_2, \ldots, V_n$ shares a periodic orbit if there exists a periodic point p whose orbit meets each V_i.

In [14], Touhey proved the following lemma in the case of a single map. Here, we verify the result for an action of a group on a Hausdorff space.

LEMMA 3.2. *Let G be a group and X be a Hausdorff topological space. The action of G on X is chaotic if and only if given non-empty open sets U, V, there exists a periodic point $p \in U$ and $g \in G$ such that $gp \in V$.*

Proof. If every pair of non-empty open sets shares a periodic orbit then every open set contains a periodic point. This implies periodic points form a dense subset of X. To prove topological transitivity, let U, V be any two non-empty open sets. By assumption, there exists $g \in G$ and $p \in U$ such that $gp \in V$. Therefore $gU \cap V \neq \emptyset$. So the action is chaotic. Conversely, suppose that the action is chaotic. Let U, V be non-empty open sets. Then by topological transitivity, there exists $g \in G$ such that $gU \cap V \neq \emptyset$. Therefore there exists $x \in U$ such that $gx \in V$. Define $W = g^{-1}V \cap U \subset U$. Hence, W is open since g is continuous. We have $x \in W$ since $x \in U, x \in g^{-1}V$. Therefore W is non-empty. Also $gW \subset V$. Since the periodic points are dense in X, W must contain a periodic point, say p. Now $p \in U$ and $gp \in V$. This concludes the proof. □

LEMMA 3.3. *A group G acting on a Hausdorff space X is chaotic if and only if any finite collection of non-empty open sets shares a periodic orbit.*

Proof. Suppose the action is chaotic. Consider a finite collection of open sets $V_1, \ldots, V_n$. By topological transitivity, for all i=2, …, n, there exists $g_i \in G$ such that $g_i V_1 \cap V_i \neq \emptyset$. Set $U = V_1 \cap \bigcap_{i=2}^{n} g^{-1} V_i$. So U is a non-empty open set, $U \subset V_1$ and $g_i U \subset V_i$. As the action has a dense set of periodic points, there exists a periodic point $x \in U$ and $g_i x \in V_i$ for all $i = 2, \ldots, n$, as required. The converse is given by Lemma 3.2. □

LEMMA 3.4. *Let G be a group that acts chaotically on an infinite Hausdorff space X. Then X contains no isolated points.*

Proof. If x is an isolated point, then $\{x\}$ is open. Therefore, since the set of periodic points is dense, it must intersect $\{x\}$ also. Therefore x must be a periodic point; that is, the orbit Gx is finite. Let $y \in X \setminus Gx$. Since X Hausdorff, there exists an open set V such that $V \cap Gx = \emptyset$. But as the action is topologically transitive, there exists $g \in G$ with $V \cap g\{x\} \neq \emptyset$ which is a contradiction. □

THEOREM 3.5 (Main Theorem). *Let G be a group acting on an infinite Hausdorff space X. Then the action has DLP if it is chaotic.*

Proof. Suppose G is a group acting on an infinite Hausdorff space X. By Lemma 3.4, X contains no isolated points. We have to show that $P \setminus P_n$ is dense for each $n \in \mathbb{N}$. Let V be a non-empty open subset of X. Then V is infinite. Otherwise, V must contain an isolated point. Therefore we assume that V contains $n+1$ points. The open set V as a subspace of the Hausdorff space X, there exists disjoint non-empty open subsets $V_1, V_2, \ldots, V_{n+1}$ of V. This collection of open sets shares a periodic orbit by Lemma 3.3. Consider a point y from the orbit. Then y belongs to some $V_i, i = 1, 2, \ldots, n+1$ and the period of y is greater than n. Therefore $y \in P \setminus P_n$. Hence, $P \setminus P_n$ intersects every non-empty open set. Therefore the action has DLP. □

The following remark shows that group actions with DLP are not necessarily chaotic on infinite Hausdorff spaces.

REMARK 3.6. Consider the action of the group $\mathbb{Z}$ on $[0,1] \times \mathbb{S}^1$ generated by the transformation $(x, y) \mapsto (x, x + y \bmod 1)$. The open subsets of the form $(a, b) \times \mathbb{S}^1$ are invariant and it is not dense in $[0,1] \times \mathbb{S}^1$. Hence, the action is not topologically transitive by Lemma 2.9.

The next two remarks show that all the conditions of Theorem 3.5 are necessary.

REMARK 3.7. It can be seen that if X is a finite set and G is acting chaotically on X then it does not have DLP. Let X be a finite set with m elements. Suppose G is a finite group acting on X. Here $P = X$ but $P \backslash P_n = \emptyset$ if $n > m$. The action does not have DLP.

REMARK 3.8. Suppose G acts on X with a dense set of periodic points such that the action is not transitive. Then the action does not have DLP. For example, consider the trivial action of a group G on a space X. That is, $gx = x$ for all $x \in X$. Here $P = X$ and $P \backslash P_n = \emptyset$ for all $n \geq 2$. Therefore this action does not have DLP.

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