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ON SOME POLYNOMIAL OVERRINGS OF INTEGRAL DOMAINS

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Abstract. Let D be an integral domain with quotient field K and X an indeterminate over K. A *polynomial overring of* D is a subring of $K[X]$ containing $D[X]$. The aim of this paper is to study some properties of the polynomial overrings of D, such as (faithful) flatness, locally freeness and Krull dimension.

1. Introduction

Let D be an integral domain with quotient field K and X an indeterminate over K. We recall from [12] that a *polynomial overring of* D is a subring of $K[X]$ containing $D[X]$. Notice that the known rings $\mathrm{Int}(D)$, $\mathrm{Int}(E, D)$ and $\mathbb{B}_x(D)$, which are defined in Section 3, are examples of polynomial overrings of D.

The polynomial overrings of D were studied for the first time by D.D. Anderson, D.F. Anderson and M. Zafrullah [3] at 1991, where they gave the basic properties and furthermore, they studied some very important special cases of theses rings such as $D + XD_S[X]$ and $K_1 + XK_2[X]$, where S is a multiplicatively closed subset of D and $K_1 \subseteq K_2$ are two fields. Those authors established further properties relating $A + XB[X]$ and $I(B, A) := \{f \in B[X];\ f(A) \subseteq A\}$ where $A \subseteq B$ is a pair of rings, which is a class of rings generalizing the well known ring of integer valued polynomials $I(\mathbb{Q}, \mathbb{Z})$. In 2003, Zafrullah [38] made an extensive study of these rings, especially the composite of a pair of integral domains $A \subseteq B$, that is, $R := A + XB[X]$. He gave many properties, equalities and inequalities related to the Krull dimension of R and its spectrum. In 2009, Loper and Tartarone [27] made another extensive study of the integrally closed domains D between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. More precisely, they gave classifications of the integrally closed domains D that are Prüfer, Noetherian or PvMDs; and in particular they pointed out in the introduction of their paper that the established results concerning integrally closed property run also for rings between $V[X]$ and $K[X]$, where V is a DVR with finite residue field and K its quotient field.

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Thereafter, in 2016, Chabert and Peruginelli [10] investigated polynomial overrings of $\mathbb{Z}$ containing $\mathrm{Int}(\mathbb{Z})$. Later, in 2021, the third named author [34] showed that the Krull dimension of polynomial overrings of D containing $\mathrm{Int}(D)$ is equal to the Krull dimension of $D[X]$, under the assumption that D is either locally essential or t-locally Noetherian. Moreover, in the same year, this last result was generalized by the first and third named authors in [15] for polynomial overrings of D containing $\mathrm{Int}(E, D)$ when D is an MZ-Jaffard domain and $E \subseteq D$ residually cofinite with D. Recentely, in 2022, Chang [12] studied the class group of polynomial overrings of a UFD defined by the intersection of DVRs, which allows to construct almost Dedekind polynomial overrings of $\mathbb{Z}$ with some very specific types of ideal class groups. Also, he completely characterized almost Dedekind polynomial overrings of $\mathbb{Z}$ containing $\mathrm{Int}(\mathbb{Z})$.

In the present paper, among other things, we investigate various properties and facts around the polynomial overrings of an integral domain D, such as (w-faithful) flatness and Krull dimension, and some other properties. We note that our paper extends some recent results established in [14, 15].

For the reader's convenience, we review some definitions and notation. Let D be an integral domain with quotient field K and let $\boldsymbol{F}(D)$ (resp., $\boldsymbol{f}(D)$) be the set of all nonzero fractional ideals (resp., nonzero finitely generated fractional ideals) of D. Obviously, $\boldsymbol{f}(D) \subsetneq \boldsymbol{F}(D)$. For $I \in \boldsymbol{F}(D)$, the v-operation is defined by $I_v := (I^{-1})^{-1}$, where $I^{-1} = (D : I) = \{x \in K;\ xI \subseteq D\}$; the *$t$-operation* is defined by $I_t := \cup\{J_v;\ J \in \boldsymbol{f}(D) \text{ and } J \subseteq I\}$; and the *$w$-operation* is defined by $I_w := \{x \in K;\ xJ \subseteq I \text{ for some } J \in \boldsymbol{f}(D) \text{ with } J^{-1} = D\}$. When $I = I_v$ (resp., $I_t = I$, $I_w = I$) we say that I is a *v-ideal* (resp., *t-ideal*, *w-ideal*). In general, for each nonzero fractional ideal I of D, $I \subseteq I_w \subseteq I_t \subseteq I_v$, and the inclusions may be strict as proved in [29, Proposition 1.2]. Hence v-ideals are t-ideals and t-ideals are w-ideals. If $\star$ denotes either t or w, a *$\star$-maximal ideal* is a maximal ideal among all $\star$-ideals of D and the set of all $\star$-maximal ideals of D is denoted by $\star\text{-Max}(D)$. Moreover, we have: $t\text{-Max}(D) = w\text{-Max}(D)$. The *$w$-dimension* of D, denoted by $w\text{-dim}(D)$, is defined by $w\text{-dim}(D) := \sup\{\mathrm{ht}(\mathfrak{p});\ \mathfrak{p} \in w\text{-Max}(D)\}$. We say that an integral domain D has *t-dimension one* if it is not a field and each t-maximal ideal of D has height one, i.e., $t\text{-Max}(D) = X^1(D)$, where $X^1(D)$ is the set of all height-one prime ideals of D.

The plan of this paper is as follows: In Section 2, we investigate some module-theoretic properties of extensions of integral domains, such as flatness and (w-)faithful flatness over various classes of essential domains (Proposition 2.2, Proposition 2.4 and Theorem 2.7). Moreover, we give some information on the Krull dimension of some polynomial overrings (Proposition 2.8). Then, we close the section by providing a star operation on an integral domain D issued from an overring of $D[X]$ (Proposition 2.10). In Section 3, we shall make applications of the results proved in Section 2 to the class of polynomial overrings of D contained in $\mathrm{Int}(E, D)$, namely, int polynomial overrings of D over E.

Throughout this paper D will denote an integral domain with quotient field K.

2. General results

Let us start this section by giving some module-theoretic properties of extensions of integral domains. For example the next result shows that (w-)faithful flatness is a (w-)local property, and also the only (w-)faithfully flat overring of an integral domain D is D itself.

Proposition 2.1. *Let D be an integral domain and let B be a D-module. Then the following statements are equivalent.*
(i) B is faithfully flat over D;

(ii) $B_{\mathfrak{m}}$ is faithfully flat over $D_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of D;

(iii) B is flat over D and $\mathfrak{m}B \neq B$ for each maximal ideal $\mathfrak{m}$ of D.
If, in addition, B is an overring of D, then the three statements are equivalent to $B = D$.

Proof. (i) $\Leftrightarrow$ (ii) [5, Chapitre II, §3, n^{o}4, Corollaire de la Proposition 15].
(i) $\Leftrightarrow$ (iii) [28, Theorem 7.2, page 47].

Now, assume that B is a flat overring of D such that $\mathfrak{m}B \neq B$ for every maximal ideal $\mathfrak{m}$ of D. So, let $\mathfrak{m}$ be a maximal ideal of D. Since $\mathfrak{m}B \neq B$, $\mathfrak{m}B \subseteq \mathfrak{M}$ for some maximal ideal $\mathfrak{M}$ of B and then $\mathfrak{m} \subseteq \mathfrak{M} \cap D$. So, by the maximality of $\mathfrak{m}$, necessarily $\mathfrak{m} = \mathfrak{M} \cap D$. Then by [33, Theorem 2], the flatness of B over D ensures that $B_{\mathfrak{M}} = D_{\mathfrak{M}\cap D} = D_{\mathfrak{m}}$ and hence $B \subseteq D_{\mathfrak{m}}$. Thus $B \subseteq \cap_{\mathfrak{m}\in \mathrm{Max}(D)} D_{\mathfrak{m}} = D$ and therefore $B = D$, as required. □

Notice that the additional statement of the previous proposition is known as an exercise in Matsumura's book [28].

The following result gives an answer to the question of whether an extension of integral domains $D \subseteq B$ is (faithfully) flat when D is Prüfer.

Proposition 2.2. *Let $D \subseteq B$ be an extension of integral domains with D a Prüfer domain. Then:*
(i) B is flat as a D-module.

(ii) B is faithfully flat as a D-module if and only if $D = B \cap K$.

Proof. (i) It follows from the fact that every torsion–free module over a Prüfer domain is always flat. (ii) See [22, Remark 3.4]. □

We next give the w-analog of the previous two propositions. Recall that a module M over D is said to be *w-flat* [26] if, for every short exact sequence $0 \to A \to B \to C \to 0$ of D-modules, $0 \to (M \otimes_D A)_w \to (M \otimes_D B)_w \to (M \otimes_D C)_w \to 0$ is also exact. Notice that flatness implies w-flatness. A module M over D is said to be *w-faithfully flat* if it is w-flat and $(M/\mathfrak{p}M)_w \neq 0$ for all $\mathfrak{p} \in w$-Max(D). An integral domain D is said to be a *Prüfer v-multiplication domain* (for short, PvMD) if $D_{\mathfrak{m}}$ is a valuation domain for all t-maximal ideals $\mathfrak{m}$ of D.

For the proof of the next proposition see [24, Proposition 2.5 and Corollary 2.6].

Proposition 2.3. *Let D be an integral domain and let B be a D-module. Then B is w-faithfully flat over D if and only if $B_{\mathfrak{m}}$ is faithfully flat over $D_{\mathfrak{m}}$ for each w-maximal ideal $\mathfrak{m}$ of D. If, moreover, B is an overring of D, then the two statements are equivalent to $B = D$.*

Proposition 2.4. *Let $D \subseteq B$ be an extension of integral domains with D a PvMD. Then:*

(i) B is w-flat as a D-module.

(ii) If $D = B \cap K$, then the D-module B is w-faithfully flat and hence w-$\dim(D) \leqslant w$-$\dim(B)$.

Proof. (i) See [26, Corollary 4.7]. (ii) By Proposition 2.3, we only need to prove that $B_{\mathfrak{m}}$ is faithfully flat as a $D_{\mathfrak{m}}$-module for each w-maximal ideal $\mathfrak{m}$ of D. So, let $\mathfrak{m}$ be a w-maximal ideal of D. Then, $D_{\mathfrak{m}}$ is a valuation domain and hence a Prüfer domain. So it follows from Proposition 2.2(ii) that $B_{\mathfrak{m}}$ is a faithfully flat $D_{\mathfrak{m}}$-module because $D_{\mathfrak{m}} = (B \cap K)_{\mathfrak{m}} = B_{\mathfrak{m}} \cap K$. Therefore B is w-faithfully flat D-module and so w-$\dim(D) \leqslant w$-$\dim(B)$ as asserted in [24, Theorem 2.13(2)]. □

As noticed in [7, Remark VI.3.3, page 136], if K is a field and B is a domain contained in $K[X]$, then the units of B are the units of $D := B \cap K$. On the other hand, the authors of [21] stated that if B is an overring of an integral domain R, then B is a localization of R if and only if $B = R_S$, where $S = \{r \in R;\ r \text{ is a unit in } B\}$, i.e., $B = R_{\mathcal{U}(B)\cap R}$, where $\mathcal{U}(B)$ is the multiplicative group of units of B. Therefore, from these observations, we derive the following.

Proposition 2.5. *Let D be an integral domain with quotient field K and let B be a polynomial overring of D such that $B \cap K = D$. Then B is a localization of $D[X]$ if and only if $B = D[X]$.*

For an overring R of an integral domain D, we recall that R is said to be *t-linked* over D if $I^{-1} = D$ implies that $(IR)^{-1} = R$ for each $I \in \boldsymbol{f}(D)$. Notice that any flat overring is t-linked. An integral domain D is called a *GCD domain* if the intersection of two principal ideals of D is principal. Notice that valuation domains are GCD domains and GCD domains form a subclass of PvMDs.

Corollary 2.6. *Let D be a GCD domain with quotient field K and let B be a polynomial overring of D such that $B \cap K = D$. Then B is t-linked over $D[X]$ if and only if $B = D[X]$.*

Proof. It is well known that if D is a GCD domain, then so is $D[X]$ ([19, Theorem 34.10]). So the result follows from Proposition 2.5 and the fact that GCD domains have the property that every t-linked overring is a localization [16, Corollary 3.8]. □

For a subset $\mathcal{P}$ of $\mathrm{Spec}(D)$, we say that D is an *essential domain* with defining family $\mathcal{P}$ if $D = \cap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$ and $D_{\mathfrak{p}}$ is a valuation domain for each $\mathfrak{p} \in \mathcal{P}$. We have:

Theorem 2.7. *Let D be an essential domain with quotient field K and let B be a polynomial overring of D such that $B \cap K = D$. Then B is flat over $D[X]$ if and only if $B = D[X]$.*

Proof. Denote by $\mathcal{P}$ the defining family of D. We note first that $D[X] = \cap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}[X]$ and $D_{\mathfrak{p}}[X]$ is a GCD domain, for every $\mathfrak{p} \in \mathcal{P}$ (this second statement follows from [19, Theorem 34.10] and the fact that valuation domains are GCD domains).

Assume that B is flat over $D[X]$ and let $\mathfrak{p} \in \mathcal{P}$. Then the overring $B_{\mathfrak{p}}$ of $D_{\mathfrak{p}}[X]$ is flat and hence it is also t-linked. Since $D_{\mathfrak{p}}[X]$ is a GCD domain and $B_{\mathfrak{p}} \cap K = D_{\mathfrak{p}}$, we deduce from Corollary 2.6 that $B_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$ and so $B \subseteq D_{\mathfrak{p}}[X]$. Thus $B \subseteq \cap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}[X] = D[X]$ and therefore $B = D[X]$, as desired. The converse is trivial. □

Recall that the *valuative dimension* of a domain D, denoted by $\dim_v(D)$, is defined to be the supremum of the Krull dimensions of the valuation overrings of D. Notice that $\dim(D) \leqslant \dim_v(D)$, where $\dim(D)$ denotes the Krull dimension of D, and when these two dimensions are equal and finite D is called a *Jaffard domain*. For instance, in the finite Krull dimensional case, Noetherian domains and Prüfer domains are examples of Jaffard domains. It is well known that $\dim_v(R) \leqslant \dim_v(D)$ for any overring R of D.

Proposition 2.8. *Let D be an integral domain with quotient field K and let B be a polynomial overring of D contained in $D + (X - a)K[X]$ for some element a of K. We have:*
(i) $\dim(B) = \sup\{\dim(B_{\mathfrak{m}});\ \mathfrak{m} \in \mathrm{Max}(D)\}$ *and* $\dim_v(B) = 1 + \dim_v(D)$*;*

(ii) D is Jaffard if and only if B is Jaffard and $\dim(B) = 1 + \dim(D)$*;*

(iii) if $D[X]$ is Jaffard then $1 + \dim(D) \leqslant \dim(B) \leqslant \dim(D[X])$.

Proof. (i) It follows from [34, Lemma 1.2] and [2, Theorem 2(2)].
(ii) Follows immediately from [2, Theorem 2(3)].
(iii) The first inequality follows from [2, Theorem 2(1)]. For the second inequality, we have: $\dim_v(B) = 1 + \dim_v(D) = \dim_v(D[X])$, and then $\dim_v(B) = \dim(D[X])$ because $D[X]$ is Jaffard. Thus $\dim(B) \leqslant \dim(D[X])$, as we wanted. □

The following example shows that some polynomial overrings of an integral domain do not behave in the same way for Jaffard property.

Example 2.9. Let k be a finite field, and Y and Z are indeterminates over k. Set $D = k + Zk(Y)[Z]_{(Z)}$ and let B be a polynomial overring of D contained in $D + (X - a)K[X]$ for some element a of $K := qf(D)$. It is well known that D is a one-dimensional pseudo-valuation domain that is not Jaffard but $D[X]$ is a Jaffard domain (of dimension 3). Then, it follows from Proposition 2.8 that $2 \leqslant \dim(B) \leqslant 3$ and $\dim_v(B) = 1 + \dim_v(D) = \dim_v(D[X]) = 3$. Particularly, if $B = \mathbb{B}_x(D)$, we have $\dim(B) = \dim(D[X]) = 3$ by [1, Example 5.1], and then B is a Jaffard domain. However, if $B = \mathrm{Int}(D)$, it follows from [36, Corollary 1.4] that $\dim(B) = 1 + \dim(D) = 2$ and hence B is not a Jaffard domain.

We conclude this section with some properties of a particular star-operation on D issued from an overring B of $D[X]$ such that $B \cap K = D$, where K is the quotient field of D.

Given an integral domain D with quotient field K, a star operation on D is a mapping $I \longmapsto I^*$ on $\boldsymbol{F}(D)$ that satisfies the following three properties for all $0 \neq a \in K$ and $I, J \in \boldsymbol{F}(D)$:
(i) $(a)^* = (a)$ and $(aI)^* = aI^*$,

(ii) $I \subseteq I^*$ and $I^* \subseteq J^*$ whenever $I \subseteq J$,

(iii) $(I^*)^* = I^*$.
A star operation $*$ is *of finite type* if $I^* = \cup\{J^*; J \subseteq I \text{ and } J \in \boldsymbol{f}(D)\}$, for each $I \in \boldsymbol{F}(D)$. A star operation $*$ on D is said to be *stable* if $(I \cap J)^* = I^* \cap J^*$, for all $I, J \in \mathbf{F}(D)$; *arithmetisch brauchbar* (for short a.b.) if, for all $I \in \boldsymbol{f}(D)$ and $F, G \in \boldsymbol{F}(D)$, $(IF)^* \subseteq (IG)^*$ implies $F^* \subseteq G^*$.

PROPOSITION 2.10. *Let D be an integral domain with quotient field K and let B be an overring of $D[X]$ such that $B \cap K = D$.*
(i) The mapping $\star : \mathbf{F}(D) \to \mathbf{F}(D)$, given by $I \mapsto I^\star := IB \cap K$, defines a star operation on D such that $I^\star B = IB$ for each $I \in \mathbf{F}(D)$. Moreover, $\star$ is of finite type.

(ii) For each $I \in \mathbf{F}(D)$, $IB = B$ implies that $I^\star = D$.

(iii) If B is a Prüfer domain, then $\star$ is an a.b. star operation.

(iv) Assume that B is flat as a D-module. Then:

(a) For each $I \in \mathbf{F}(D)$ and each $J \in \mathbf{f}(D)$, $(I : J)^\star = (I^\star : J^\star) = (I^\star : J)$.

(b) $\star$ is stable.

(c) B is faithfully flat as a D-module if and only if $\mathfrak{m}^\star \neq D$ for each $\mathfrak{m} \in \mathrm{Max}(D)$.

Proof. (i) It follows from [11, Lemma 1(1)] and its proof. (ii) Obvious.
(iii) Let $I \in \boldsymbol{f}(D)$ and $F, G \in \boldsymbol{F}(D)$ such that $(IF)^\star \subseteq (IG)^\star$. Then, $(IB)(FB) = (IF)B = (IF)^\star B \subseteq (IG)^\star B = (IG)B = (IB)(GB)$. Since IB is finitely generated and B is a Prüfer domain, IB is invertible and hence $FB \subseteq GB$. Thus $F^\star \subseteq G^\star$.
(iii) (a) Let $I \in \boldsymbol{F}(D)$ and $J \in \boldsymbol{f}(D)$. By flatness of B over D, $(I : J)B = (IB : JB)$. Then $(I : J)^\star = (IB : JB) \cap K = (I^\star B : J^\star B) \cap K = (I^\star : J^\star)B \cap K = (I^\star : J^\star)^\star$. Hence $(I : J)^\star = (I^\star : J^\star) = (I^\star : J)$.

(b) Let $I, J \in \boldsymbol{F}(D)$. Since B is flat as a D–module, $IB \cap JB = (I \cap J)B$ [28, Theorem 7.4(i), page 48] and then $(I \cap J)^\star = ((I \cap J)B) \cap K = (IB \cap K) \cap (JB \cap K) = I^\star \cap J^\star$. Therefore $(I \cap J)^\star = I^\star \cap J^\star$.

(c) It follows from Proposition 2.1 and statement (ii). □

We close this section by the following corollary of [19, Proposition 32.18] and Propositions 2.2 and 2.10.

COROLLARY 2.11. *Let D be a Prüfer domain. Then the star operation $\star$ is stable a.b. which is equivalent to any star operation on D and $\mathfrak{m}^\star \neq D$ for each $\mathfrak{m} \in \mathrm{Max}(D)$.*

3. Applications to polynomial overrings of D contained in $\mathrm{Int}(E, D)$

In this section, we investigate some properties of int polynomial overrings of D contained in $\mathrm{Int}(E, D)$, which we shall call later int polynomial overring of D over E, for distinguished classes of integral domains D and some special subsets E of D in order to recover some well known results.

Let D be an integral domain with quotient field K, E a subset of K and X an indeterminate over K. The ring $\mathrm{Int}(E, D) := \{f \in K[X];\ f(E) \subseteq D\}$, *of integer-valued polynomials* on E with respect to D, is known to be a D-algebra. Obviously, $\mathrm{Int}(D, D) = \mathrm{Int}(D)$, is the classical ring of integer-valued polynomials over D, $D \subseteq \mathrm{Int}(E, D) \subseteq K[X]$, and $\mathrm{Int}(E, D)$ is an overring of $D[X]$ whenever E is a subset of D.

Given a subset E of D, we call an *int polynomial overring of D over E* any domain B between $D[X]$ and $\mathrm{Int}(E, D)$. When $E = D$, we simply say int polynomial overring of D. Notice that, if B is an int polynomial overring of D over E then $B \cap K = D$.

First, we list some remarkable interesting int polynomial overrings of D over E:

— The ring $\mathrm{Int}_R(E, D)$ of D-valued R-polynomials over E [35], that is,
$$\mathrm{Int}_R(E, D) := \{f \in R[X];\ f(E) \subseteq D\},$$
where R is an overring of D and E is a subset of D.

— The Bhargava ring over E at x [2], that is,
$$\mathbb{B}_x(E, D) := \{f \in K[X];\ \forall a \in E,\ f(xX + a) \in D[X]\},$$
where x is an element of D and E is a subset of D.

— The ring of integer-valued polynomials on A with coefficients in K [17], that is,
$$\mathrm{Int}_K(A) := \{f \in K[X];\ f(A) \subseteq A\},$$
where A is a torsion-free D-algebra such that $A \cap K = D$.

Recall that an integral domain D is called *Bézout domain* if every finitely generated ideal of D is principal. Notice that PIDs and valuation domains are Bézout domains, and Bézout domains form a subclass of Prüfer domains.

It is well known that unless D is a field, the domains $\mathbb{B}_x(D)$ and $\mathrm{Int}(D)$ are never Bézout (see [2, Proposition 17] and [9, Proposition 3.1]).

We start by proving the following characterization of when some int polynomial overrings are Bézout.

PROPOSITION 3.1. *Let D be an integral domain and let B be an int polynomial overring of D over $\{0, 1\}$. Then B is a Bézout domain if and only if D is a field.*

Proof. If D is a field then $B = D[X]$ is a PID and hence Bézout. For the converse, by way of contradiction, we assume that B is a Bézout domain and that D is not a field and we argue mimicking the proof of [2, Proposition 17]. As D is not a field, there exists a nonzero element t of D which is not a unit. Then, (t, X) is a principal ideal of B and hence $(t, X) = (f)$ for some $f \in B$. Thus, $f(X) = tg_1(X) + Xg_2(X)$ for some

$g_1, g_2 \in B$. So, $d := f(0) = tg_1(0) \in D$ because $t \in D$ and $g_1(0) \in D$. Moreover, since X and t are elements of the principal ideal (f), $X = f(X)h_1(X)$ and $t = f(X)h_2(X)$ for some $h_1, h_2 \in B$. The equality $t = f(X)h_2(X)$ forces $\deg(f) = \deg(h_2) = 0$ which implies $f(X) = d \neq 0$. Now, the equality $X = f(X)h_1(X)$ implies that $\deg(h_1) = 1$ and then $h_1(X) = aX + b$. Hence, $a = h_1(1) - h_1(0) \in D$ and $1 = ad$, and thus $a = d^{-1} = (tg_1(0))^{-1}$. Therefore, since t is not a unit, $a \notin D$ which is a contradiction. □

As an immediate consequence, we have:

COROLLARY 3.2. *Let $D \subseteq R$ be an extension of integral domains and let x be an element of D. Unless D is a field, the rings $\mathrm{Int}_R(D)$ and $\mathbb{B}_x(D)$ are never Bézout.*

An *almost Dedekind domain* is defined as an integral domain D such that any localization of D at a maximal ideal is a DVR (here by a DVR we mean a rank-one discrete valuation domain). A module M over an integral domain D is said to be *locally free* if $M_{\mathfrak{m}}$ is a free $D_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m}$ of D. Note that any locally free module is (faithfully) flat.

PROPOSITION 3.3. *Let D be an integral domain with quotient field K and let B be a ring between D and $\mathrm{Int}(E, D)$ for some subset E of K.*
(i) If D is a Prüfer domain, then B is a faithfully flat D-module.

(ii) Assume that $D[X]$ is contained in B.

(a) If D is an almost Dedekind domain and E is an infinite subset of D, then B is a locally free D-module.

(b) If D is an essential domain, then B is flat over $D[X]$ if and only if $B = D[X]$.

(c) If D is a GCD domain, then B is t-linked over $D[X]$ if and only if $B = D[X]$.

Proof. (i) Since $D \subseteq B \subseteq \mathrm{Int}(E, D)$ and $\mathrm{Int}(E, D) \cap K = D$, then $B \cap K = D$ and hence the thesis follows from Proposition 2.2 because D is a Prüfer domain.
(ii) (a) Let $\mathfrak{m}$ be a maximal ideal of D. Since D is an almost Dedekind domain, $D_{\mathfrak{m}}$ is a DVR and then it follows from the inclusions $D_{\mathfrak{m}}[X] \subseteq B_{\mathfrak{m}} \subseteq \mathrm{Int}(E, D)_{\mathfrak{m}} \subseteq \mathrm{Int}(E, D_{\mathfrak{m}})$ and [7, Corollary II.1.6] that $B_{\mathfrak{m}}$ has a regular basis, and hence free as a $D_{\mathfrak{m}}$-module. Thus, B is a locally free D-module.

(b) and (c) follow from Theorem 2.7 and Corollary 2.6 because $B \cap K = D$ and $D[X] \subseteq B \subseteq K[X]$. □

EXAMPLE 3.4. Let T be an indeterminate over $\mathbb{Q}$, and set $A = \cup_{n=0}^{\infty} \mathbb{Q}[T^{\frac{1}{2^n}}]$ and $D = A_S$, where $S = \mathbb{Q}[T] \setminus (1 - T)\mathbb{Q}[T]$. Let B be a ring between D and $\mathrm{Int}(E, D)$ for some non-empty subset E of $K := qf(D)$.

As it was established in [13, Section 3] the ring $D = A_S$ is an almost Dedekind domain that is not Noetherian. Then it follows from Proposition 3.3(i) that B is a faithfully flat D-module. If, in addition, E is an infinite subset of D then B is a locally free D-module by statement (ii)(a) of Proposition 3.3.

Example 3.5. The integral domain D in [8, Example 5.1] is an almost Dedekind domain such that $\mathrm{Int}(D)$ is not a PvMD. Let B be an int polynomial overring of D over a non-empty subset E of D.
(i) If E is infinite, then by Proposition 3.3 (ii)(a) , B is locally free, and hence faithfully flat, as a D-module.

(ii) $\mathrm{Int}(E, D)$ is not flat over $D[X]$. Otherwise, it follows from Proposition 3.3 (ii)(b) that $\mathrm{Int}(E, D) = D[X]$ and hence $\mathrm{Int}(D) = D[X]$. Thus by [23, Theorem 3.7], $\mathrm{Int}(D)$ is a PvMD which is a contradiction.

The following result sheds more light on the w-analog of Proposition 3.3. For an integral domain D, a module M over D is called *w-locally free* if $M_{\mathfrak{m}}$ is a free $D_{\mathfrak{m}}$-module for each w-maximal ideal $\mathfrak{m}$ of D. Notice that any w-locally free module is w-faithfully flat. An integral domain D is said to be a *t-almost Dedekind* if $D_{\mathfrak{m}}$ is a DVR for all t-maximal ideals $\mathfrak{m}$ of D. Notice that almost Dedekind domains are t-almost Dedekind domains and these form a subclass of PvMDs.

Proposition 3.6. *Let D be an integral domain with quotient field K and let B be a ring between D and $\mathrm{Int}(E, D)$ for some subset E of K.*
(i) If D is a PvMD, then B is a w-faithfully flat D-module and w-$\dim(D) \leqslant w$-$\dim(B)$.

(ii) If D is a t-almost Dedekind domain and E is an infinite subset of D, then B is a w-locally free D-module.

Proof. (i) Follows immediately from Proposition 2.4 (ii).
(ii) The proof is similar to that of Proposition 3.3 with a slight modification. □

To avoid repetition, we fix some definitions and notation:

Following [6], a prime ideal $\mathfrak{p}$ of D is called an *associated prime* of a principal ideal aD of D if $\mathfrak{p}$ is minimal over $(aD : bD)$ for some $b \in D \setminus aD$. For brevity, we call $\mathfrak{p}$ an associated prime of D and we denote by $\mathrm{Ass}(D)$ the set of all associated prime ideals of D. Thus, for any integral domain D, we define the following partition of $\mathrm{Max}(D)$:

$$\mathcal{M}_0 := \{\mathfrak{m} \in \mathrm{Ass}(D) \cap \mathrm{Max}(D);\ D/\mathfrak{m} \text{ is finite}\} \text{ and } \mathcal{M}_1 := \mathrm{Max}(D)\backslash\mathcal{M}_0.$$

As the notion of essential domains does not carry up to localizations [20], D is said to be a *locally essential domain* if $D_{\mathfrak{q}}$ is an essential domain for each $\mathfrak{q} \in \mathrm{Spec}(D)$; or equivalently, $D_{\mathfrak{p}}$ is a valuation domain for each $\mathfrak{p} \in \mathrm{Ass}(D)$ [30]. Analogously, under the naming system of [25], an integral domain D is called an MZ-*DVR* if $D_{\mathfrak{p}}$ is a DVR for each $\mathfrak{p} \in \mathrm{Ass}(D)$.

A subset E of D is said to be *residually cofinite with* D [31] if E is non-empty and, for any prime ideal $\mathfrak{p}$ of D, E and D are simultaneously either finite or infinite modulo $\mathfrak{p}$. Notice that D is residually cofinite with itself.

Theorem 3.7. *Let D be an integral domain, $E \subseteq D$ a residually cofinite with D and B an int polynomial overring of D over E.*
(i) If D is a locally essential domain, then B is a faithfully flat D-module.

(ii) If E is infinite and $D_{\mathfrak{m}}$ is a DVR for each $\mathfrak{m} \in \mathcal{M}_0$, then B is a locally free D-module.

Proof. (i) By Proposition 2.1, we only need to prove that $B_{\mathfrak{m}}$ is a faithfully flat $D_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m}$ of D. Let $\mathfrak{m}$ be a maximal ideal of D. We then examine the following two possible cases:
Case 1. $\mathfrak{m} \in \mathrm{Ass}(D)$. Since D is locally essential, $D_{\mathfrak{m}}$ is a valuation domain and then it follows from the equality $D_{\mathfrak{m}} = B_{\mathfrak{m}} \cap K$ (because $D_{\mathfrak{m}}[X] \subseteq B_{\mathfrak{m}} \subseteq \mathrm{Int}(E,D)_{\mathfrak{m}} \subseteq \mathrm{Int}(E,D_{\mathfrak{m}})$ and $D_{\mathfrak{m}} = D_{\mathfrak{m}}[X] \cap K = \mathrm{Int}(E,D_{\mathfrak{m}}) \cap K$) and Proposition 2.2 (ii) that $B_{\mathfrak{m}}$ is a faithfully flat $D_{\mathfrak{m}}$-module.

Case 2. $\mathfrak{m} \notin \mathrm{Ass}(D)$. It follows from [35, Proposition 7] that $\mathrm{Int}(E,D)_{\mathfrak{m}} = \mathrm{Int}(E,D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$ and hence from the inclusions $D_{\mathfrak{m}}[X] \subseteq B_{\mathfrak{m}} \subseteq \mathrm{Int}(E,D_{\mathfrak{m}})$ we deduce that $B_{\mathfrak{m}} = D_{\mathfrak{m}}[X]$. Therefore $B_{\mathfrak{m}}$ is a faithfully flat $D_{\mathfrak{m}}$-module.
Therefore B is a faithfully flat D-module, as desired.
(ii) Let $\mathfrak{m}$ be a maximal ideal of D. So, we need to discuss the following cases.
Case 1. $\mathfrak{m} \in \mathcal{M}_0$. By assumption, $D_{\mathfrak{m}}$ is a DVR and so it follows from [7, Corollary II.1.6] that $B_{\mathfrak{m}}$ is free as a $D_{\mathfrak{m}}$-module because $D_{\mathfrak{m}}[X] \subseteq B_{\mathfrak{m}} \subseteq \mathrm{Int}(E,D_{\mathfrak{m}})$.

Case 2. $\mathfrak{m} \in \mathcal{M}_1$. We then examine the following possible subcases:

Case 2.1. $\mathfrak{m} \in (\mathrm{Ass}(D) \cap \mathrm{Max}(D)) \setminus \mathcal{M}_0$. We have $D/\mathfrak{m}$ is infinite and then $\mathrm{Int}(E,D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$ by [31, Lemmas 3(i) and 4(ii)]. Thus from the inclusions $D_{\mathfrak{m}}[X] \subseteq B_{\mathfrak{m}} \subseteq \mathrm{Int}(E,D_{\mathfrak{m}})$ we deduce that $B_{\mathfrak{m}} = D_{\mathfrak{m}}[X]$ is a free $D_{\mathfrak{m}}$-module.

Case 2.2. $\mathfrak{m} \notin \mathrm{Ass}(D)$. It follows from [35, Proposition 7] that $\mathrm{Int}(E,D)_{\mathfrak{m}} = \mathrm{Int}(E,D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$ and then, as in the previous subcase, $B_{\mathfrak{m}} = D_{\mathfrak{m}}[X]$. Therefore $B_{\mathfrak{m}}$ is a free $D_{\mathfrak{m}}$-module.

Consequently, B is a locally free D-module. □

Example 3.8. Let $\mathcal{E}$ be the ring of entire functions, and set $D := \mathcal{E} + T\mathcal{E}_S[T]$, where T is an indeterminate over $\mathcal{E}$ and S is the set generated by the principal primes of $\mathcal{E}$. Let $E \subseteq D$ a residually cofinite with D, and let B be an int polynomial overring of D over E.

According to [37, Example 2.6], D is a locally essential domain which is neither PvMD nor almost Krull. Then by Theorem 3.7 (i), B is a faithfully flat D-module.

From the above theorem, we derive the next corollaries.

Corollary 3.9. *Let D be an integral domain, $E \subseteq D$ an infinite residually cofinite with D and B an int polynomial overring of D over E.*
(i) If D is an MZ-*DVR, then B is locally free as a D-module.*

(ii) If D is a locally essential domain such that $\mathrm{Ass}(D) = X^1(D)$ *and $E = D$, then B is locally free as a D-module.*

Proof. (i) An immediate application of Theorem 3.7 (ii).
(ii) Let $\mathfrak{p} \in \mathrm{Ass}(D)$. We have either $B_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$ or $B_{\mathfrak{p}} \neq D_{\mathfrak{p}}[X]$. In the second case, $\mathrm{Int}(D_{\mathfrak{p}}) \neq D_{\mathfrak{p}}[X]$ and then from [7, Proposition I.3.16] we deduce that $D_{\mathfrak{p}}$ is a

valuation domain with principal maximal ideal. Thus, since $\mathfrak{p}$ is height-one, $D_{\mathfrak{p}}$ is a DVR and therefore the conclusion follows from Theorem 3.7 (ii). □

An integral domain D is called *generalized Krull* (in the sense of Gilmer [19, Section 43]) if the intersection $D = \cap_{\mathfrak{p}\in X^1(D)} D_{\mathfrak{p}}$ is locally finite and $D_{\mathfrak{p}}$ is a valuation domain for each $\mathfrak{p} \in X^1(D)$. An integral domain D is said to be *K-domain* [32] if (i) $D = \cap_{\mathfrak{p}\in X^1(D)} D_{\mathfrak{p}}$; (ii) each $D_{\mathfrak{p}}$ is a DVR; and (iii) each $\mathfrak{p}$ is divisorial. Also, an *almost Krull domain* is defined as an integral domain D such that any localization of D at a maximal ideal is a Krull domain. Notice that Krull domains form a proper subclass of generalized Krull domains, almost Krull domains and t-almost Dedekind domains. Moreover, almost Krull domains, K-domains and t-almost Dedekind domains are MZ-DVRs.

Corollary 3.10. *Let D be an integral domain, $E \subseteq D$ an infinite residually cofinite with D and B an int polynomial overring of D over E. If D is either t-almost Dedekind, almost Krull, K-domain or generalized Krull, then B is a locally free D-module.*

Notice that generalized Krull domains and t-almost Dedekind domains are PvMDs of t-dimension one, and if D has t-dimension one then $\text{Ass}(D) = X^1(D)$.

Corollary 3.11. *Let D be a PvMD of t-dimension one. Then, every int polynomial overring of D is a locally free as a D-module.*

Corollary 3.12. *Let D be an integral domain and let B be an int polynomial overring of D. If* $\text{Int}(D)$ *is a PvMD, then B is a locally free D-module.*

Proof. Let $\mathfrak{p} \in \text{Ass}(D)$. Then either $B_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$, or $B_{\mathfrak{p}} \neq D_{\mathfrak{p}}[X]$ and then it follows from [8, Corollary 1.8] that $D_{\mathfrak{p}}$ is a DVR because $\text{Int}(D_{\mathfrak{p}}) \neq D_{\mathfrak{p}}[X]$. Hence $\mathfrak{p}$ is height-one and therefore we are in the conditions of Theorem 3.7 (ii). □

Example 3.13. 1. Let $D = \mathbb{Z}\left[\{T/p_n, U/p_n\}_{n=1}^{\infty}\right]$, where T and U are indeterminates over $\mathbb{Z}$ and $\{p_n\}_{n=1}^{\infty}$ is the set of all positive prime integers, $E \subseteq D$ an infinite and residually cofinite with D and let B be an int polynomial overring of D over E. As cited in [4, Example, page 52], D is an almost Krull domain which is not PvMD. Then by Corollary 3.10, B is locally free as a D-module.

2. The integral domain D in [32, Section 3] is a K-domain that is not almost Krull. Let E be an infinite and residually cofinite with D. So, by Corollary 3.10, any int polynomial overring B of D over E is locally free as a D-module.

3. Let A be the domain of all algebraic integers and $\{p_n\}_{n=1}^{\infty}$ is the set of all positive prime integers. For each n choose a maximal ideal M_n of A lying over $p_n\mathbb{Z}$, and set $S = A \setminus \cup_{n=1}^{\infty} M_n$ and $D = A_S$.

In [18, Example 1, page 338], Gilmer proved that D is a one-dimensional Prüfer domain which is not almost Dedekind (indeed not almost Krull). Then by Corollary 3.11, any int polynomial overring B of D is locally free as a D-module.

We end this article with some results and properties of the Krull dimension of D.

THEOREM 3.14. *Let D be an integral domain, $E \subseteq D$ a residually cofinite with D and B an int polynomial overring of D over E.*
(i) If $D_{\mathfrak{m}}$ is a Jaffard domain for each $\mathfrak{m} \in \mathcal{M}_0$, then $\dim(B) = \dim(D[X])$.

(ii) If $D_{\mathfrak{m}}[X]$ is a Jaffard domain for each $\mathfrak{m} \in \mathcal{M}_0$, then $\dim(B) \leqslant \dim(D[X])$.

Proof. (i) Let $\mathfrak{m}$ be a maximal ideal of D. We then examine the following possible cases:
Case 1. $\mathfrak{m} \in \mathcal{M}_0$. By assumption $D_{\mathfrak{m}}$ is a Jaffard domain and then it follows from Proposition 2.8 (ii) that $\dim(B_{\mathfrak{m}}) = 1 + \dim(D_{\mathfrak{m}}) = \dim(D_{\mathfrak{m}}[X])$.

Case 2. $\mathfrak{m} \in \mathcal{M}_1$. We have either $\mathfrak{m} \notin \mathrm{Ass}(D)$ or $\mathfrak{m} \in (\mathrm{Ass}(D) \cap \mathrm{Max}(D)) \setminus \mathcal{M}_0$, and then it follows from the proof of Theorem 3.7 (ii) that $B_{\mathfrak{m}} = D_{\mathfrak{m}}[X]$. Thus $\dim(B_{\mathfrak{m}}) = \dim(D_{\mathfrak{m}}[X])$.
Consequently, $\dim(B_{\mathfrak{m}}) = \dim(D_{\mathfrak{m}}[X])$ for each maximal ideal $\mathfrak{m}$ of D, and hence the conclusion is settled from Proposition 2.8 (i).
(ii) For this statement, we only need to treat the previous first case. So, if $\mathfrak{m} \in \mathcal{M}_0$, then $D_{\mathfrak{m}}[X]$ is a Jaffard domain and hence by Proposition 2.8(iii), $\dim(B_{\mathfrak{m}}) \leqslant \dim(D_{\mathfrak{m}}[X])$. Thus, for each maximal ideal $\mathfrak{m}$ of D, $\dim(B_{\mathfrak{m}}) \leqslant \dim(D_{\mathfrak{m}}[X])$ and therefore by Proposition 2.8(i), $\dim(B) \leqslant \dim(D[X])$, as wanted. □

An integral domain D is called a *Mott-Zafrullah Jaffard domain* (in short, an MZ-Jaffard domain) if $D_{\mathfrak{p}}$ is Jaffard for each $\mathfrak{p} \in \mathrm{Ass}(D)$. In the finite Krull dimensional setting, it is clear that any locally essential domain is MZ-Jaffard. From the first statement of Theorem 3.14, we obtain [15, Theorem 2.11] as a corollary.

COROLLARY 3.15. *Let D be an* MZ-*Jaffard domain, $E \subseteq D$ a residually cofinite with D and B an int polynomial overring of D over E. Then* $\dim(B) = \dim(D[X])$.

Recall that an integral domain D is said to be *strong Mori* if it satisfies the ascending chain condition (a.c.c.) on integral w-ideals. Thus, the class of strong Mori domains includes that of Noetherian domains and Krull domains. We also recall that an integral domain D is a *t-locally Noetherian domain* if any localization of D at a t-maximal ideal is a Noetherian domain. It is well known that strong Mori domains are t-locally Noetherian. Moreover, as mentioned in [25], t-locally Noetherian domains are MZ-Jaffard.

COROLLARY 3.16. *Let D be an integral domain, $E \subseteq D$ a residually cofinite with D and B an int polynomial overring of D over E. If D is a t-locally Noetherian domain (in particular, a strong Mori domain), then* $\dim(B) = \dim(D[X])$.

EXAMPLE 3.17. Let T be a non-Noetherian Krull domain with a maximal ideal $\mathfrak{m}$ such that $T_{\mathfrak{m}}$ is Noetherian. Assume that $T/\mathfrak{m}$ contains properly a finite field k.

Let D be defined by the following pullback diagram:

$$\begin{array}{ccc} D & \longrightarrow & k \cong D/\mathfrak{m} \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/\mathfrak{m}, \end{array}$$

$E \subseteq D$ a residually cofinite with D and let B be an int polynomial overring of D of E.

It follows from [29, Example 3.15(3)] that D is a strong Mori domain which is neither Noetherian nor Krull. Then by Corollary 3.16, $\dim(B) = \dim(D[X])$.

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