MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 14.08.2024

research paper
оригинални научни рад
DOI: 10.57016/MV-9K9832wr

AUTOMATIC CONTINUITY OF ALMOST DERIVATIONS ON LMC Q-ALGEBRAS

G. Siva

Abstract. In this article, almost derivation on LMC algebras is introduced. Also, it is proved that every almost derivation(or, surjective almost derivation) T on semisimple LMC Q-algebras Γ with an additional condition on Γ has a closed graph. Moreover, it is derived that every almost derivation(or, surjective almost derivation) T on semisimple commutative(or, non commutative) Fréchet Q-algebra Γ with an additional condition on Γ is continuous. To further illustrate our primary results, an example is provided.

1. Introduction

In this part we give a brief overview of concepts and known results. For more information, see [2,9]. All vector spaces considered here are over the complex number field $\mathbb{C}$. A complete normed algebra is called a Banach algebra Γ, and a normed algebra Γ is an algebra with a norm $||.||$ that also satisfies the condition $||\varpi_1.\varpi_2|| \leq ||\varpi_1||.||\varpi_2||$, $\forall$ $\varpi_1, \varpi_2 \in \Gamma$. If all algebraic operations are jointly continuous, an algebra with a Hausdorff topology is called a topological algebra. The intersection of all maximal right (or left) regular ideals is the Jacobson radical $rad(\Gamma)$ of the algebra Γ. If $rad(\Gamma) = \{0\}$, we say that the algebra Γ is semisimple.

Definition 1.1. [9] Let Γ be an algebra. Let an element $\varpi_1 \in \Gamma$ be quasi-invertible if there exists $\varpi_2 \in \Gamma$ such that $\varpi_1 + \varpi_2 - \varpi_1\varpi_2 = 0 = \varpi_2 + \varpi_1 - \varpi_2\varpi_1$.

Definition 1.2 ([2]). Let Γ be a unital algebra and $p \in \Gamma$. The set of all complex numbers c such that $c.e - p$ is not invertible in Γ is called the spectrum $\sigma_\Gamma(p)$ of the element p. The quantity $r_\Gamma(p) = \sup\{|c| : c \in \sigma_\Gamma(p)\}$ is called the spectral radius of p.

Furthermore, for every Banach algebra Γ: $rad(\Gamma) = \{\varpi_1 \in \Gamma : r_\Gamma(\varpi_1\varpi_2) = 0$, for every $\varpi_2 \in \Gamma\}$ (see [16, Lemma 1]).

2020 Mathematics Subject Classification: 46H40, 46H05
Keywords and phrases: LMC-algebra; almost derivation; automatic continuity.

LEMMA 1.3 ([16, Lemma 2]). *Let Γ be a Banach algebra and $Q(\zeta)$ a polynomial with coefficients in Γ and $R > 0$. Then*

$$r_\Gamma^2(Q(1)) \leq \sup_{|\zeta|=R} r_\Gamma(Q(\zeta)). \sup_{|\zeta|=\frac{1}{R}} r_\Gamma(Q(\zeta)).$$

DEFINITION 1.4 ([9]). A topological algebra Γ is called advertibly complete if every Cauchy net $(\upsilon_\alpha)_{\alpha\in\Lambda}$ converges in Γ, provided that there is some $\nu \in \Gamma$ such that $\upsilon_\alpha + \nu - \upsilon_\alpha.\nu$ converges to 0.

A complete metrizable topological algebra is called an F-algebra. A topological algebra Γ is called an LMC algebra if its topology is defined by a separating family of submultiplicative seminorms $(p_\alpha)_{\alpha\in J}$. A Fréchet algebra is an LMC algebra which is also an F-algebra. A Q-algebra is a topological algebra in which the set of all quasi-invertible elements is open.

REMARK 1.5. If Γ is a topological Q-algebra, then Γ is advertibly complete (see [9, page 45]).

REMARK 1.6. Let Γ be an LMC algebra with a family of seminorms $(p_\alpha)_{\alpha\in J}$, and let Γ_α be the completion of the quotient algebra $\Gamma/\ker\ p_\alpha$, with respect to the norm $p_\alpha{}'(y + \ker\ p_\alpha) = p_\alpha(y), y \in \Gamma$, then Γ_α is a Banach algebra. And if Γ is advertibly complete, then $r_\Gamma(b) = \sup_\alpha r_{\Gamma_\alpha}(b + \ker\ p_\alpha) = \sup_\alpha(\lim_{n\to\infty}(p_\alpha(b^n)^{\frac{1}{n}}))$ (see [9, Chapter III, Theorem 6.1]).

DEFINITION 1.7 ([8]). Let Γ be an algebra. A linear mapping $T : \Gamma \to \Gamma$ is called derivation, if $T(\varpi_1.\varpi_2) = \varpi_1.T(\varpi_2) + T(\varpi_1).\varpi_2, \forall\varpi_1, \varpi_2 \in \Gamma$.

Next, we introduce almost derivations on LMC algebras.

DEFINITION 1.8. Let Γ be an LMC algebra with a family of seminorms $(p_\alpha)_{\alpha\in J}$. A linear mapping $T : \Gamma \to \Gamma$ is called almost derivation, if there exists $\epsilon_\alpha \geq 0$ such that $p_\alpha(T(\varpi_1.\varpi_2) - \varpi_1.T(\varpi_2) - T(\varpi_1).\varpi_2) \leq \epsilon_\alpha p_\alpha(\varpi_1)\ p_\alpha(\varpi_2);\ \forall\alpha \in J, \forall\varpi_1, \varpi_2 \in \Gamma$.

REMARK 1.9. If $\epsilon_\alpha = 0$, for every α, then almost derivations on Γ turn out to be derivations on Γ, because (p_α) is a separating family of seminorms on Γ. Moreover, every derivation is an almost derivation, for every $\epsilon_\alpha \geq 0$. Let Γ be an LMC algebra. If $T : \Gamma \to \Gamma$ is defined by $T(k) = \beta k$, $k \in \Gamma$ and for any $(\epsilon_\alpha =)\beta \in (0, \infty)$, then T is an almost derivation, but not a derivation on Γ.

DEFINITION 1.10 ([9]). Let Γ be a topological algebra and $a \in \Gamma$. If for every open set $G \supseteq \sigma_\Gamma(a)$ there exists a neighbourhood H of a such that $\sigma_\Gamma(x) \subseteq G$ whenever $x \in H$, then the spectrum function $x \mapsto \sigma_\Gamma(x)$ is called upper semicontinuous at a.

THEOREM 1.11 ([6]). *Let Γ be a topological algebra, then Γ is a Q-algebra if the spectral radius function (spectrum function) $x \mapsto r_\Gamma(x)(x \mapsto \sigma_\Gamma(x))$ is upper semicontinuous on A.*

LEMMA 1.12 ([6]). *Let W be a topological space and $L \subseteq W$ a compact set. If $g : W \to \mathbb{R}$ is upper semicontinuous, then g takes its maximum on L.*

A conjecture of Kaplansky [8] can be formulated in the following question form. Is every derivation on a semisimple Banach algebra continuous? The Kaplansky conjecture was proved by Johnson and Sinclair [7] in 1968. Every derivation on a semisimple commutative Fréchet algebra with identity is continuous, as R. L. Carpenter [1] showed in 1971. More recent publications [11–15] on topological algebras deal with the automatic continuity of derivations. Similar results as in this article were obtained in 1993 by M. Fragoulopoulou [3] for homomorphisms.

In this paper we prove that every almost-derivation (or surjective almost-derivation) T on a semisimple LMC Q-algebra Γ satisfying the condition $r_\Gamma(Ta) \leq r_\Gamma(a), \forall a \in \Gamma$ has a closed graph, and we deduce that every almost-derivation (or surjective almost-derivation) T on a semisimple commutative (or non-commutative) Fréchet Q-algebra Γ satisfying $r_\Gamma(Ta) \leq r_\Gamma(a), a \in \Gamma$ is continuous.

2. Main results for surjective maps

In this section we only deal with unital algebras.

THEOREM 2.1. *Let Γ be a semisimple LMC-algebra with a family of seminorms $(p_\alpha)_{\alpha\in J}$, which is also a Q-algebra. If $T : \Gamma \to \Gamma$ is such a surjective almost derivation such that $r_\Gamma(Ta) \leq r_\Gamma(a), \forall a \in \Gamma$, then T has a closed graph.*

Proof. Let$(\varrho_i)_{i\in\Lambda}$ be a net in Γ such that $\varrho_i \to 0$, and $T(\varrho_i) \to b$. Since T is onto, there exists $a \in \Gamma$ such that $Ta = b$.

We define $Q_i(\zeta) = \zeta T\varrho_i + T(a - \varrho_i)$, for each $i \in \Lambda$, and $\zeta \in \mathbb{C}$. Let $g_i(\zeta) = (\zeta - 1)\varrho_i + a$, for $\zeta \in \mathbb{C}$. Since Γ is a Q-algebra, the function $x \mapsto r_\Gamma(x)$ is upper semicontinuous due to Theorem 1.11. Moreover, the composite function $f_i = r_\Gamma \circ g_i : \mathbb{C} \to \mathbb{R}$ is upper semicontinuous function because g_i is continuous.

According to Lemma 1.12, for every $R > 0$ there is $\zeta_i \in \mathbb{C}$ such that $|\zeta_i| = R$ and $\sup_{|\zeta|=R} f_i(\zeta) = f_i(\zeta_i)$. Since $(\zeta_i - 1)\varrho_i + a \to a$ and the r_Γ is upper semicontinuous on Γ, for every $\epsilon > 0$ there exists $\mu \in \Lambda$ such that $r_\Gamma((\zeta_i - 1)\varrho_i + a)\mu$.

Assuming that Γ_α is the completion of the quotient algebra $\Gamma/\ker\ p_\alpha$, with respect to the norm ${p_\alpha}'$, then for each $i \in \Lambda$, we have $r_{\Gamma_\alpha}(Q_i(\zeta) + \ker\ p_\alpha) \leq r_\Gamma(Q_i(\zeta)) \leq r_\Gamma((\zeta - 1)\varrho_i + a)$, because of the hypothesis.

Since for each $i \in \Lambda$, we also have

$$\begin{aligned} r_{\Gamma_\alpha}(Q_i(\zeta) + \ker\ p_\alpha) &\leq {p_\alpha}'(Q_i(\zeta) + \ker\ p_\alpha) = p_\alpha(Q_i(\zeta)) \\ &= p_\alpha(\zeta T\varrho_i + T(a - \varrho_i)) \leq |\zeta| p_\alpha(T\varrho_i) + p_\alpha(T(a - \varrho_i)). \end{aligned}$$

By Lemma 1.3 we have for each $i > \mu$

$$\begin{aligned} r^2_{\Gamma_\alpha}(b + \ker\ p_\alpha) &= r^2_{\Gamma_\alpha}(Q_i(1) + \ker\ p_\alpha) \\ &\leq \sup_{|\zeta|=R} r_{\Gamma_\alpha}(Q_i(\zeta) + \ker\ p_\alpha). \sup_{|\zeta|=\frac{1}{R}} r_{\Gamma_\alpha}(Q_i(\zeta) + \ker\ p_\alpha) \\ &\leq \sup_{|\zeta|=R} r_\Gamma((\zeta - 1)\varrho_i + a). \sup_{|\zeta|=\frac{1}{R}} (|\zeta| p_\alpha(T\varrho_i) + p_\alpha(T(a - \varrho_i))) \end{aligned}$$

$$\leq r_\Gamma((\zeta_i - 1)\varrho_i + a).(\frac{1}{R}p_\alpha(T\varrho_i) + p_\alpha(b - T\varrho_i))$$
$$\leq (r_\Gamma(a) + \epsilon)(\frac{1}{R}p_\alpha(T\varrho_i) + p_\alpha(b - T\varrho_i)).$$

Now, passing to the limit on i, we get $r^2_{\Gamma_\alpha}(b + \ker\ p_\alpha) \leq (r_\Gamma(a) + \epsilon).(\frac{1}{R}p_\alpha(b))$.

Now let $R \to \infty$ to get $r_{\Gamma_\alpha}(b + \ker\ p_\alpha) = 0$, for each $\alpha \in J$. Since Γ is a Q-algebra, Γ is advertibly complete, and $r_\Gamma(b) = \sup_{\alpha\in J} r_{\Gamma_\alpha}(b + \ker\ p_\alpha) = 0$, according to Remark 1.6.

Let $c \in \Gamma$. Since $\varrho_i \to 0$, $p_\alpha(c.\varrho_i) \to 0$ for every α. Let $w = T(c)$. Since T is an almost derivation, we have

$$\begin{aligned} p_\alpha(T(c.\varrho_i) - c.b) &\leq p_\alpha(T(c.\varrho_i) - c.T(\varrho_i) - T(c).\varrho_i) + p_\alpha(c.T(\varrho_i) + w.\varrho_i - c.b) \\ &\leq p_\alpha(T(c.\varrho_i) - c.T(\varrho_i) - T(c).\varrho_i) + p_\alpha(c.T(\varrho_i) - c.b) + p_\alpha(w.\varrho_i) \\ &\leq \epsilon_\alpha p_\alpha(c)\, p_\alpha(\varrho_i) + p_\alpha(c)\, p_\alpha(T(\varrho_i) - b) + p_\alpha(w.\varrho_i). \end{aligned}$$

Since $p_\alpha(T(\varrho_i) - b) \to 0$, $p_\alpha(\varrho_i) \to 0$ and $p_\alpha(w.\varrho_i) \leq p_\alpha(w).p_\alpha(\varrho_i) \to 0$, $\forall\alpha$, we have $p_\alpha(T(c.\varrho_i) - c.b) \to 0$, for every α, and therefore $T(c.\varrho_i) \to c.b$, if $c.\varrho_i \to 0$. By the argument used above, we have $r_\Gamma(c.b) = 0$. Since $c \in \Gamma$ is arbitrary, we conclude that $b \in rad(\Gamma) = \{0\}$, and this proves the theorem. □

Corollary 2.2. *Let $(\Gamma, (p_\alpha))$ be a semisimple Fréchet Q-algebra. If $T : \Gamma \to \Gamma$ is a surjective almost derivation satisfying $r_\Gamma(Ta) \leq r_\Gamma(a)$, $a \in \Gamma$, then T is continuous.*

3. Main results for commutative algebras

Definition 3.1 ([5]). Let Γ and Ω be two LMC algebras. Let $T : \Gamma \to \Omega$ be a linear map. The separating space of T is defined by $G(T) = \{q \in \Omega$: there exists a net $(q_i)_{i\in\Lambda}$ in Γ such that $q_i \to 0$ and $Tq_i \to q\}$.

With the same proof as [14, Theorem 2.2], one gets the following theorem, but for a net instead of a sequence.

Theorem 3.2. *Let Γ be a semisimple LMC algebra with family of seminorms $(p_\alpha)_{\alpha\in J}$, which is also a Q-algebra. If $T : \Gamma \to \Gamma$ is an almost derivation, then the separating space $G(T)$ is a closed two sided ideal in $(\Gamma, (p_\alpha))$.*

Theorem 3.3. *Let Γ be a semisimple LMC Q-algebra with family $(p_\alpha)_{\alpha\in J}$ of seminorms. If r_Γ is continuous and $T : \Gamma \to \Gamma$ is an almost derivation with $r_\Gamma(Ta) \leq r_\Gamma(a)$, $a \in \Gamma$, then T has a closed graph.*

Proof. For $b \in G(T)$, there exists $(\varrho_i)_{i\in\Lambda}$ in Γ such that $\varrho_i \to 0$ and $T\varrho_i \to b$. By the inequality $r_\Gamma(Ta) \leq r_\Gamma(a)$ and $r_\Gamma(\varrho_i) \to 0$, we have $r_\Gamma(T\varrho_i) \to 0$. On the other hand, $r_\Gamma(T\varrho_i) \to r_\Gamma(b)$, because of the continuity of r_Γ on Γ. Therefore $r_\Gamma(b) = 0$. Also, $G(T)$ is an ideal in Γ, because of Theorem 3.2. So $b.c \in G(T)$, for every $c \in \Gamma$. By the above process, $r_\Gamma(b.c) = 0$. It is known that $rad(\Gamma) = \{\varpi_1 \in \Gamma : r_\Gamma(\varpi_1.\varpi_2) = 0, \forall\varpi_2 \in \Gamma\}$, and so $b \in rad(\Gamma)$. Hence, $G(T) \subseteq rad(\Gamma)$. Therefore

$G(T) = \{0\}$, because Γ is semisimple. Now, we conclude that T has a closed graph. $\square$

COROLLARY 3.4. *Let $(\Gamma, (p_\alpha))$ be a commutative Fréchet Q-algebra such that Γ is semisimple. If $T : \Gamma \to \Gamma$ is an almost derivation with $r_\Gamma(Ta) \leq r_\Gamma(a), \forall a \in \Gamma$, then T is continuous.*

Proof. If Γ is a commutative Fréchet Q-algebra, then the spectral radius function r_Γ is uniformly continuous (see, e.g. [4, Theorem 6.18]). By previous theorem and the Closed Graph theorem, T is continuous. $\square$

4. An example

EXAMPLE 4.1. Let $(\Gamma, (p_\alpha))$ be a semisimple commutative Fréchet Q-algebra. A linear mapping $T : \Gamma \to \Gamma$ is defined by $T(a) = \beta a$, $\forall a \in \Gamma$, where $\beta \in (0, \infty)$. Since

$$\begin{aligned} p_\alpha(T(\varpi_1.\varpi_2) - \varpi_1.T(\varpi_2) - T(\varpi_1).\varpi_2) &= p_\alpha(\beta\varpi_1.\varpi_2 - \varpi_1.\beta\varpi_2 - \beta\varpi_1.\varpi_2) \\ &= p_\alpha(-\beta\varpi_1.\varpi_2) \leq |-\beta| p_\alpha(\varpi_1).p_\alpha(\varpi_2), \forall \varpi_1, \varpi_2 \in \Gamma, \end{aligned}$$

T is an almost derivation but not a derivation on $(\Gamma, (p_\alpha))$. Since Γ is a Q-algebra, there exists $k \in N$ such that $r_\Gamma(a) = \lim_{n\to\infty}(p_k(a^n))^{\frac{1}{n}}$, $\forall a \in \Gamma$ (see e.g. [4, Theorem 6.18]). Thus

$$r_\Gamma(Ta) = r_\Gamma(\beta a) = \lim_{n\to\infty}(p_k((\beta a)^n))^{\frac{1}{n}} = |\beta| \lim_{n\to\infty}(p_k(a^n))^{\frac{1}{n}} \leq r_\Gamma(a).$$

All hypotheses of Corollary 2.2 and Corollary 3.4 are satisfied, so T is continuous.

5. Conclusion

We can ask the open question whether a almost derivation on a semisimple Fréchet algebra is continuous or not by extending the Kaplansky conjecture from Banach algebras to Fréchet algebras. In Corollaries 2.2 and 3.4 we have found partial answers to this open problem.

ACKNOWLEDGEMENT. The author's study is being funded by a fellowship from the Council of Scientific and Industrial Research (CSIR), India (File No. 09/688(0031)/ 2018-EMR-I).

REFERENCES

[1] R. L. Carpenter, *Continuity of derivations in F-algebras*, Amer. J. Math., **93** (1971), 500–502.

[2] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Mathematical Society Monographs **24**, Clarendon Press, Oxford, 2000.

[3] M. Fragoulopoulou, *Uniqueness of topology for semisimple LFQ-algebras*, Proc. Amer. Math. Soc., **117** (1993), 963–969.

[4] M. Fragoulopoulou, *Topological Algebras with Involution*, Elsevier Science, Amsterdam, 2005.

[5] T. G. Honary, M. Omidi, A. H. Sanatpour, *Automatic continuity of almost multiplicative maps between Fréchet algebras*, Bull. Korean Math. Soc., **41** (2015), 1497–1509.

[6] T. G. Honary, M. N. Tavani, *Upper semicontinuity of the spectrum functtion and automatic continuity in topological Q-algebra*, Note Mat., **28** (2008), 57–62.

[7] B. E. Johnson, A. M. Sinclair, *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math., **90** (1968), 1067–1073.

[8] I. Kaplansky, *Derivations, Banach algebras seminar on analytic functions*, Vol. II, Institute for Advanced Study, Princeton, 1958.

[9] A. Mallios, *Topological Algebras, Selected Topics*, North-Holland Publishing Co., Amsterdam, 1986.

[10] E. A. Michael, *Locally multiplicatively convex topological algebras*, Mem. Amer. Math. Soc. 11, 1952.

[11] A. A. Mohammed, *On automatic continuity of closable derivations*, Int. Journal of Math. Analysis., 8 (2014), 1161-1164.

[12] A. A. Mohammed, S. M. Ali, *On the nonassociative Jewell Sinclair theorem*, J. Pure Appl. Math., Adv. Appl., **11** (2014), 137–146.

[13] C. G. Moorthy, G. Siva, *Automatic continuity of almost Jordan derivations on special Jordan Banach algebras*, MACA, Math. Anal. Contemp. Appl., **4** (2022), 11–16.

[14] C. G. Moorthy, G. Siva, *Automatic continuity of almost derivations on Fréchet Q-algebras*, CAMS, Commun. Adv. Math. Sci., **5** (2022), 88–91.

[15] C. G. Moorthy, G. Siva, *Automatic continuity of surjective almost derivations on Fréchet Q-algebras*, MACO, Math. Anal. Convex Optim., **3** (2022), 1–6.

[16] T. J. Ransford, *A short proof of Johnson's uniqueness-of-norm theorem*, Bull. London Math. Soc., **21** (1989), 487–488.

(received 02.03.2023; in revised form 14.08.2023; available online 14.08.2024)

Department of Mathematics, Rajiv Gandhi National Institute of Youth Development, Sriperumbudur-602105, Tamil Nadu, India

E-mail: gsivamaths2012@gmail.com

ORCID iD: https://orcid.org/0000-0002-4676-6883