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FIXED POINTS OF ENRICHED ρ-NONEXPANSIVE MAPPINGS IN MODULAR FUNCTION SPACES

Safeer Hussain Khan

Abstract. In this paper, we initiate the study of enriched ρ-nonexpansive mappings in modular function spaces. First we show that in modular function spaces, every ρ-nonexpansive mapping is enriched ρ-nonexpansive mapping but not conversely and that their sets of fixed point are same. Next, we prove a ρ-convergence result on approximation of fixed points of enriched ρ-nonexpansive mappings in modular function spaces. We verify the validity of the result by an example. We construct a table to show our findings. Finally, we give one more ρ-convergence result under different conditions. Our results are new for ρ-nonexpansive mappings in modular function spaces.

1. Introduction and preliminaries

Nakano [4] began the study of modular spaces in connection with the theory of ordered spaces. Musielak and Orlicz [5] presented their generalization. As part of nonlinear functional analysis, fixed point theory for nonlinear mappings has found many applications in nonlinear integral equations and differential equations. The study of this theory in the context of modular function spaces was initiated by Khamsi [6]. Kozlowski [10] has made remarkable contributions in this field.

First, of course, the results on the existence of fixed points were proved. The approximation of fixed points in modular function spaces continued to seek attention until Dehaish and Kozlowski [3] used Mann iterative methods to approximate fixed points of asymptotically point-wise nonexpansive mappings. The class of asymptotically pointwise non-expansive mappings is more general than that of non-expansive mappings. Kumam [8] has obtained some fixed point results for nonexpansive mappings in arbitrary modular spaces. However, his results are more restrictive, while those of Dehaish and Kozlowski [3] hold for a more general class and under less restrictive conditions.

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The author [9] introduced a hybrid iterative Picard-Mann process and proved analytically and supported by numerical examples that it is faster than many existing iterative processes, including Mann's and thus Krasnoselskii's iterative processes. On the other hand, the idea of enriched mappings in a metric or a normed space is a relatively new idea introduced by Berinde [1]. His work on enriched non-expansive mappings in uniformly convex Banach spaces can be seen, for example, in [2], where he introduced the Krasnoselskii's iterative process to obtain convergence to fixed points.

In this paper, we start by investigating the approximation of fixed points of enriched ρ-nonexpansive mappings in modular function spaces using Picard-Mann hybrid iterative process. We substantiate our main convergence theorem by a numerical example, the results of which are shown in Table 1.

Some basic facts and notations needed in this paper are recalled below.

Let Ω be a non-empty set and Σ a non-trivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for every $E \in \mathcal{P}$ and $A \in \Sigma$. Suppose that there is an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \cup K_n$. With 1_A, we denote the characteristic function of the set A in Ω. With $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. With $\mathcal{M}_\infty$ we denote the space of all extended measurable functions.

A set $A \in \Sigma$ is called ρ-null if $\rho(g1_A) = 0$ for every $g \in \mathcal{E}$. A property $p(\omega)$ means that ρ holds -almost everywhere (ρ-a.e.) if the set $\{\omega \in \Omega : p(\omega)$ does not hold$\}$ is ρ-null. We define $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty : |f(\omega)| < \infty \ \rho\text{-a.e.}\}$, where $f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal to ρ-a.e. and not a single function. Where there is no confusion, we write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

Definition 1.1. Let ρ be a regular function pseudomodular. We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies$f = 0$ ρ-a.e.

It is known (see [10]) that ρ satisfies the following properties:

(1) $\rho(0) = 0$ iff $f = 0$ ρ-a.e.

(2) $\rho(\alpha f) = \rho(f)$ for every scalarα with $|\alpha| = 1$ and $f \in \mathcal{M}$.

(3) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ and $f, g \in \mathcal{M}$.

ρ is called a convex modular if the following property is also fulfilled:

(3') $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ and $f, g \in \mathcal{M}$.

Definition 1.2. The convex function modular ρ defines the modular function space L_ρ as $L_\rho = \{f \in \mathcal{M};\ \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}$.

In general, the modular ρ is not sub-additive and therefore does not behave like a norm or a distance. However, the modular space L_ρ can be equipped with an F-norm, which is defined by

$$\|f\|_\rho = \inf\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \leq \alpha\}.$$

The class of all nonzero regular convex function modulars defined on Ω is denoted by $\Re$.

The following properties of the uniform convexity type of ρ can be found in [3].

DEFINITION 1.3. Let ρ be a regular non-zero convex function modular defined on Ω. Let $t \in (0,1)$, $r > 0$, $\varepsilon > 0$. Define

$$D(r_1,\epsilon) = \{(f,g) : f,g \in L_\rho, \rho(f) \le r, \rho(g) \le r, \rho(f-g) \ge \varepsilon r\}.$$

Let $$\delta_1^t(r,\epsilon) = \inf\left\{1 - \frac{1}{r}\rho(tf + (1-t)g) : (f,g) \in D(r_1,\epsilon)\right\} \text{ if } \backslash D(r_1,\epsilon) \ne \phi,$$

and $\delta_1(r,\epsilon) = 1$ if $D(r_1,\epsilon) = \phi$. The usual notation is $\delta_1 = \delta_1^{\frac{1}{2}}$.

DEFINITION 1.4. A regular non-zero convex function which is modular ρ satisfies $(UUC1)$ if for every $s \ge 0$, $\epsilon > 0$, there exists an $\eta_1(s,\epsilon) > 0$ that depends only on s and ϵ such that $\delta_1(r,\epsilon) > \eta_1(s,\epsilon) > 0$ for each $r > s$.

DEFINITION 1.5. Let L_ρ be a modular space. The sequence $\{f_n\} \subset L_\rho$ is called:
(i) ρ-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$.

(ii) ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as n and $m \to \infty$.

DEFINITION 1.6. A subset $D \subset L_\rho$ is called
(i) ρ-closed if the ρ-limit of a ρ-convergent sequence of D always belongs to D.

(ii) ρ-compact if every sequence in D has a ρ-convergent subsequence in D.

(iii) ρ-bounded if $\mathrm{diam}_\rho(D) = \sup\{\rho(f-g) : f,g \in D\} < \infty$.

The following lemma can be seen as an analogy to a famous lemma of Schu [12] in Banach spaces.

LEMMA 1.7. *Let $\rho \in \Re$ satisfy $(UUC1)$ and let $\{t_k\} \subset (0,1)$ be bounded away from* 0 *and* 1. *If there exists $R > 0$ such that* $\limsup_{n\to\infty} \rho(u_n) \le R$, $\limsup_{n\to\infty} \rho(v_n) \le R$ *and* $\lim_{n\to\infty} \rho(t_n u_n + (1-t_n)v_n) = R$, *then* $\lim_{n\to\infty} \rho(u_n - v_n) = 0$.

A function $f \in L_\rho$ is called a fixed point of $T : L_\rho \to L_\rho$ if $f = Tf$. The set of all fixed points of T is denoted by $F_\rho(T)$.

DEFINITION 1.8. A mapping $T : D \to D$ is called ρ-nonexpansive if

$$\rho(Tf - Tg) \le \rho(f-g) \text{ for all } f,g \in D.$$

The following theorem about the existence of fixed point of the so-called pointwise ρ-nonexpansive mappings, a class that is broader than ρ-nonexpansive mappings, can be found in Khamsi and Kozlowski [7].

THEOREM 1.9. *Let $\rho \in \Re$ satisfy $(UUC1)$. Let D be a ρ-closed ρ-bounded convex non-empty subset. Then any asymptotically pointwise ρ-nonexpansive mapping $T : D \to D$ has a fixed point. Moreover, the set of all fixed points $F_\rho(T)$ is ρ-closed.*

Considering that the class of asymptotically pointwise ρ-nonexpansive mappings contains the class of ρ-nonexpansive mappings, the above theorem yields the following theorem about the existence of fixed points of ρ-nonexpansive mappings.

THEOREM 1.10. *Let $\rho \in \Re$ satisfy $(UUC1)$. Let D be a ρ-closed ρ-bounded convex non-empty subset. Then any ρ-nonexpansive mapping $T : D \to D$ has a fixed point. Moreover, the set of all fixed points $F_\rho(T)$ is ρ-closed.*

2. Fixed-point approximation of enriched ρ-nonexpansive mappings

In this section, we prove a ρ-convergence result for the approximation of fixed points of enriched ρ-nonexpansive mappings in modular function spaces using a hybrid Picard-Mann iteration method. We define enriched ρ-nonexpansive mappings in modular function spaces as follows.

DEFINITION 2.1. Let D be a subset of L_ρ. We say that a mapping $T : D \to D$ is called enriched ρ-nonexpansive if there exist $\alpha \in (0,1)$ such that

$$\rho((1-\alpha)(f-g)+\alpha(Tf-Tg)) \leq \rho(f-g) \text{ for all } f,g \in D. \tag{1}$$

The following important proposition gives a relationship between enriched ρ-nonexpansiveness and ρ-nonexpansive and the sets of their fixed points. We support later this Proposition by Example 2.6.

PROPOSITION 2.2. *Suppose that T is an enriched ρ-nonexpansive mapping as defined in* (1). *Define*

$$T_\alpha f = (1-\alpha)f + \alpha Tf. \tag{2}$$

Then
(i) T_α is ρ-nonexpansive.
(ii) The set of fixed pints of T is the same as that of T_α. That is, $F_\rho(T) = F_\rho(T_\alpha)$.

Proof. (i) By (1) and (2) , we have $\rho(T_\alpha f - T_\alpha g) \leq \rho(f-g)$ for all $f,g \in D$.
Consequently, T_α is a ρ-nonexpansive.
(ii) Let $f \in F_\rho(T)$. Then $Tf = f$ implies that $T_\alpha f = (1-\alpha)f + \alpha f = f$. Conversely, if $T_\alpha f = f$ then $(1-\alpha)f + \alpha Tf = f$ implies $\alpha Tf = \alpha f$ and therefore $Tf = f$. □

Before proving our main convergence theorem, we need the following key result for enriched ρ-nonexpansive mappings in modular function spaces using the hybrid Picard-Mann iteration method.

THEOREM 2.3. *Let $\rho \in \Re$ satisfy $(UUC1)$ and D be a non-empty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : D \to D$ be an enriched ρ-nonexpansive mapping as defined in* (1). *Let $\{f_n\} \subset D$ be defined by the iterative process as follows.*

$$f_{n+1} = T_\alpha g_n, \quad g_n = (1-\beta)f_n + \beta T_\alpha f_n, \tag{3}$$

where T_α is defined as in (2). *Then $\lim_{n\to\infty} \rho(f_n - c)$ exists for all $c \in F_\rho(T_\alpha)$, and $\lim_{n\to\infty} \rho(f_n - T_\alpha f_n) = 0$.*

Proof. Since $T : D \to D$ is enriched ρ-nonexpansive, by Proposition 2.2, T_α is ρ-nonexpansive and $F_\rho(T) = F_\rho(T_\alpha)$. Then, according to the Theorem 1.10, $F_\rho(T_\alpha) \neq \varnothing$. Let $c \in F_\rho(T_\alpha)$. To prove that $\lim_{n\to\infty} \rho(f_n - c)$ exists for all $c \in F_\rho(T_\alpha)$, we consider $\rho(f_{n+1} - c) = \rho(T_\alpha g_n - T_\alpha c) \leq \rho(g_n - c)$.
Also because T_α is a ρ-nonexpansive, $\rho(T_\alpha f_n - T_\alpha c) \leq \rho(f_n - c)$, so

$$\rho(f_{n+1} - c) \leq \rho(g_n - c) = \rho[(1-\beta)(f_n - c) + \beta\rho(T_\alpha f_n - T_\alpha c)]$$

$$\leq (1-\beta)\rho\left(f_n - c\right) + \beta\rho\left(f_n - c\right) = \rho\left(f_n - c\right).$$

Thus, $\lim_{n\to\infty}\rho(f_n - c)$ exists for every $c \in F_\rho(T_\alpha)$.

To prove the rest of the result, we assume that

$$\lim_{n\to\infty}\rho(f_n - c) \ = m \tag{4}$$

where $m \geq 0$.

Note that the above calculations also result in the following inequality:

$$\rho\left(g_n - c\right) \leq \rho\left(f_n - c\right).$$

Next, we prove that $\lim_{n\to\infty}\rho(f_n - T_\alpha f_n) = 0$. Now

$$m = \lim_{n\to\infty}\rho(f_{n+1} - c) \leq \lim_{n\to\infty}\rho(\ g_n - c) \leq \lim_{n\to\infty}\rho\left(f_n - c\right) = m.$$

This results in $\lim_{n\to\infty}\rho(g_n - c) = m$. In addition,

$$\limsup_{n\to\infty}\rho(\ T_\alpha f_n - c) \leq \lim_{n\to\infty}\rho(\ f_n - c) = \ m. \tag{5}$$

But then $\rho(f_{n+1} - c) \leq \rho(g_n - c)$ means that

$$\begin{aligned}\lim_{n\to\infty}\rho\left[(1-\beta)(f_n - c) + \beta(T_\alpha f_n - c)\right] &= \lim_{n\to\infty}\rho\left[(1-\beta)f_n + \beta T_\alpha f_n) - c\right] \\ &= \lim_{n\to\infty}\rho(g_n - c) = m. \end{aligned} \tag{6}$$

Now through (4), (5), (6) and Lemma 1.7, we have $\lim_{n\to\infty}\rho(f_n - T_\alpha f_n\ ) = 0$ as required. □

Using the above result, we now prove the following convergence theorem by our iterative process (3) for approximating fixed points of enriched ρ-nonexpansive mappings in modular function spaces as follows.

THEOREM 2.4. *Let $\rho \in \Re$ satisfy $(UUC1)$. Let D be a non-empty ρ-compact and convex subset of L_ρ. Let $T : D \to D$ be an enriched ρ-nonexpansive mapping. Let $\{f_n\}$ be as defined by* (3). *Then let $\{f_n\}$ ρ-converges to a fixed point of T.*

Proof. Since D isρ- compact, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\lim_{k\to\infty}(f_{n_k} - w) = 0$ for some $w \in D$. Since T_α is a ρ-nonexpansive, using the convexity of ρ, we have

$$\begin{aligned}\rho\left(\frac{w - T_\alpha w}{3}\right) &= \rho\left(\frac{w - f_{n_k}}{3} + \frac{f_{n_k} - T_\alpha f_{n_k}}{3} + \frac{T_\alpha f_{n_k} - T_\alpha w}{3}\right) \\ &\leq \frac{1}{3}\rho(w - f_{n_k}) + \frac{1}{3}\rho(f_{n_k} - T_\alpha f_{n_k}) + \frac{1}{3}\rho(T_\alpha f_{n_k} - T_\alpha w) \\ &\leq \rho(w - f_{n_k}) + \rho(f_{n_k} - T_\alpha f_{n_k}) + \rho(f_{n_k} - w) \\ &\leq 2\rho\left(w - f_{n_k}\right) + \rho(f_{n_k} - T_\alpha f_{n_k}).\end{aligned}$$

Applying the Theorem 2.3, we therefore have $\rho(\frac{w - T_\alpha w}{3}) = 0$. Hence w is a fixed point of T_α and by Proposition 2.2, w is also a fixed point of T. This means that $\{f_n\}$ ρ-converges to a fixed point of T. □

COROLLARY 2.5. *Let $\rho \in \Re$ satisfy $(UUC1)$. Let D be a non-empty ρ-compact and convex subset of L_ρ. Let $T : D \to D$ be a ρ-nonexpansive mapping. Let $\{f_n\}$ be as defined by* (3)*. Then $\{f_n\}$ ρ-converges to a fixed point of T.*

Proof. As already mentioned, every ρ-nonexpansive mapping is an enriched ρ-nonexpansive mapping, the proof is complete. □

We now give an example to support our main result Theorem 2.4 above.

EXAMPLE 2.6. Let us consider the modular space $L_\rho = \mathbb{R}$ equipped with the norm $\|.\|$, that is, $\rho(f) = |f|$ and $D = \left\{f \in L_\rho : \frac{1}{3} \leq f \leq 3\right\}$. Obviously, D is a non-empty ρ-compact and ρ-convex subset of L_ρ. Define $T : D \to D$ by $Tf = \frac{1}{f}$ for all $f \in D$. Then $F_\rho(T) = \{1\}$. We show that T is not ρ-nonexpansive, but it is enriched ρ-nonexpansive. Now, for all $f, g \in D$

$$\rho(Tf - Tg) = \left|\frac{1}{f} - \frac{1}{g}\right| = \left|\frac{g-f}{fg}\right| \nleq |f - g| = \rho(f - g).$$

For example, let us take $f = 0.6$ and $g = 0.9$. Then $\rho(Tf - Tg) = 0.556 > 0.3 = \rho(f - g)$. T is therefore not ρ-nonexpansive.

However, T is $\frac{1}{5}$-enriched ρ-nonexpansive as follows.

$$\begin{aligned}\rho((1-\alpha)(f-g) + \alpha(Tf - Tg)) &= \left|\frac{4}{5}(f-g) + \frac{1}{5}(\frac{1}{f} - \frac{1}{g})\right| = \left|\frac{4}{5}(f-g) + \frac{1}{5}(\frac{g-f}{fg})\right| \\ &= |f-g|\left|\frac{4}{5} - \frac{1}{5fg}\right| \leq \rho\,(f-g)\end{aligned}$$

for all $f, g \in D$ because $\left|\frac{4}{5} - \frac{1}{5fg}\right| \leq 1$ iff $-1 \leq \frac{4}{5} - \frac{1}{5fg} \leq 1$ iff $-\frac{9}{5} \leq -\frac{1}{5fg} \leq \frac{1}{5}$. That is, $\frac{1}{5} \geq -\frac{1}{5fg}$ and $\frac{1}{5fg} \leq \frac{9}{5}$ or $1 \leq \frac{-1}{fg}$ and $\frac{1}{9} \leq fg$ or $fg \leq -1$ and $\frac{1}{9} \leq fg$, which are true for all $f, g \in D$.

Iteration#	f_n	g_n	f_{n+1}
1	0.6000000000	0.8133333333	1.0266666667
2	1.0266666667	1.0161385281	1.0056103896
3	1.0056103896	1.0033724939	1.0011345983
6	1.0000455075	1.0000273049	1.0000091023
8	1.0000018205	1.0000010923	1.0000003641
10	1.0000000728	1.0000000437	1.0000000146
12	1.0000000029	1.0000000017	1.0000000006
14	1.0000000001	1.0000000001	1.0000000000
15	1.0000000000	1.0000000000	1.0000000000

Table 1: Convergence of our process to the fixed point

Next, we construct a sequence as in (3) and show that it converges to 1, the unique fixed point of T_α that is identical to the fixed point of T. Take $f_1 = 0.6$, $\alpha = \frac{2}{5}$, $\beta = \frac{1}{2}$. With these assumptions, the Table 1 shows that $\{f_n\}$ given by (3) converges to 1 with an accuracy of 7 decimal places at the 10th iteration and 10 decimal places at the 15th iteration. The speed of convergence naturally depends on the choice of the parameters α, β and the initial estimate f_1. For example, if α, β

remain the same, the convergence speed increases the closer we move from the left to $\frac{2}{3}$, e.g. $0.66, 0.666$ etc. But as soon as we move above or further below $\frac{2}{3}$, we take for example $f_1 = 0.6700000000$ or $f_1 = 0.57$, the convergence slows down until we get $f_1 = 1$. Of course, if we start with $f_1 = 1$, there is nothing to show and $\{f_n\}$ given by (3) converges to 1 as a constant sequence.

Finally, we give another ρ-convergence theorem under different conditions. Consistent with Kilmer et al. [11], we define the ρ-distance from a $f \in L_\rho$ to a set $D \subset L_\rho$ as $dist_\rho(f, D) = \inf\{\rho(f-h) : h \in D\}$. The following definition is a function-modular space version of Senter and Dotson's condition (I) [13]. Let $D \subset L_\rho$. A mapping $T : D \to D$ is said to satisfy the condition (I) if there is a non-decreasing function $\ell : [0, \infty) \to [0, \infty)$ with $\ell(0) = 0$, $\ell(r) > 0$ for all $r \in (0, \infty)$ such that

$$\rho(f - Tf) \geq \ell(dist_\rho(f, F_\rho(T)) \text{ for all } f \in D.$$

Theorem 2.7. *Let $\rho \in \Re$ satisfy $(UUC1)$ and Δ_2-condition. Let D be a non-empty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : D \to D$ be an enriched ρ-nonexpansive satisfying the condition (I). Let $\{f_n\}$ be as defined by (3). Then $\{f_n\}$ ρ-converges to a fixed point of T.*

Proof. According to Theorem 2.3, $\lim_{n\to\infty} \rho(f_n - w)$ exists for all $w \in F_\rho(T_\alpha)$. Suppose that $\lim_{n\to\infty} \rho(f_n - w) = m > 0$ because otherwise $\lim_{n\to\infty} \rho(f_n - w) = 0$ means that there is nothing left to prove. Again by the same theorem, we have $\rho(f_{n+1} - w) \leq \rho(f_n - w)$ so that $dist_\rho(f_{n+1}, F_\rho(T_\alpha)) \leq dist_\rho(f_n, F_\rho(T_\alpha))$. This means that $\lim_{n\to\infty} dist_\rho(f_n, F_\rho(T_\alpha))$ exists. Applying the condition (I) and the Theorem 2.3 results in

$$\lim_{n\to\infty} \ell(dist_\rho(f_n, F_\rho(T_\alpha))) \leq \lim_{n\to\infty} \rho(f_n - T_\alpha f_n) = 0.$$

Since ℓ is a non-decreasing function and $\ell(0) = 0$, therefore

$$\lim_{n\to\infty} dist_\rho(f_n, F_\rho(T_\alpha)) = 0. \tag{7}$$

To prove that $\{f_n\}$ is a ρ-Cauchy sequence in D, let $\varepsilon > 0$. By (7), there exists a constant n_0 such that for all $n \geq n_0$, $dist_\rho(f_n, F_\rho(T_\alpha)) < \frac{\varepsilon}{2}$. There therefore exists a $y \in F_\rho(T_\alpha)$ such that $\rho(f_{n_0} - y) < \varepsilon$. Now for $m, n \geq n_0$,

$$\rho\left(\frac{f_{n+m} - f_n}{2}\right) \leq \frac{1}{2}\rho(f_{n+m} - y) + \frac{1}{2}\rho(f_n - y) \leq \rho(f_{n_0} - y) < \varepsilon.$$

This implies by the Δ_2-condition that $\rho(f_{n+m} - f_n) < \varepsilon$ for $m, n \geq n_0$. Hence, $\{f_n\}$ is a ρ-Cauchy sequence in a ρ-closed subset D of the ρ-complete space L_ρ, and thus ρ-converges in D. Let $\lim\limits_{n\to\infty} f_n = w$. Then $dist_\rho(w, F_\rho(T_\alpha)) = \lim\limits_{n\to\infty} dist_\rho(f_n, F_\rho(T_\alpha)) = 0$ by (7). Since $F_\rho(T_\alpha)$ is closed, $w \in F_\rho(T_\alpha)$. This means that $\{f_n\}$ ρ-converges to a fixed point T_α and thus from T. □

Corollary 2.8. *Let $\rho \in \Re$ satisfy $(UUC1)$ and Δ_2 the condition. Let D be a non-empty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : D \to D$ be a ρ-nonexpansive satisfying the condition (I). Let $\{f_n\}$ be as defined by (3). Then $\{f_n\}$ ρ-converges to a fixed point of T.*

Proof. The proof follows from the fact that every ρ-nonexpansive is enriched ρ-nonexpansive. □

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Department of Mathematics and Statistics, North Carolina Agricultural and Technical State University, Greensboro NC 27411, USA

E-mail: safeerhussain5@yahoo.com, shkhan@ncat.edu