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MULTIPLICATIVE ORDER CONTINUOUS OPERATORS ON RIESZ ALGEBRAS

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Abstract. In this paper, we investigate operators on Riesz algebras, which are continuous with respect to multiplicative modifications of order convergence and relatively uniform convergence. We also introduce and study $\mathbb{mo}$-Lebesgue, $\mathbb{mo}$-KB, and $\mathbb{mo}$-Levi operators.

1. Introduction

It is known that the order, the relatively uniform, unbounded order, and various order convergences in Riesz algebras are generally not topological in general (cf. [6, Theorem 2.2]). As far as we know, there is no sufficiently comprehensive study of operator theory on Riesz algebras. The aim of this paper is to present and study operators on Riesz algebras that are continuous, e.g. with respect to $\mathbb{mo}$- or $\mathbb{mr}$-convergences. Throughout the paper we assume that all vector spaces are real and all operators are linear. In the following, the letters X and Y stand for Riesz spaces and the letters E and F for Riesz algebras.

A net $(x_\alpha)_{\alpha\in A}$ in a Riesz space X is *order convergent* (or short $\mathbb{o}$*-convergent*) to $x \in X$ if there exists a net $(y_\beta)_{\beta\in B}$ that satisfies $y_\beta \downarrow 0$, and for any $\beta \in B$ there exists $\alpha_\beta \in A$ such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$. In this case, we write $x_\alpha \xrightarrow{\mathbb{o}} x$. A net $(x_\alpha)_{\alpha\in A}$ in X *relatively uniform converges* to $x \in X$ ($x_\alpha \xrightarrow{\mathbb{r}} x$ for short) if there exists $u \in X_+$, such that for any $n \in \mathbb{N}$, there exists α_n such that $|x_\alpha - x| \leq \frac{1}{n}u$ for all $\alpha \geq \alpha_n$. An operator T between Riesz spaces is called:

− *order-bounded*, if T transforms order-bounded sets into order-bounded sets;

− *regular*, if $T = T_1 - T_2$ with $T_1, T_2 \geq 0$;

− *order continuous*, if $Tx_\alpha \xrightarrow{\mathbb{o}} Tx$ whenever $x_\alpha \xrightarrow{\mathbb{o}} x$;

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– *relatively uniform continuous*, if $Tx_\alpha \xrightarrow{\mathbb{r}} Tx$ whenever $x_\alpha \xrightarrow{\mathbb{r}} x$.
It is known that order-continuous and relatively order-continuous operators are order bounded. The collection $\mathcal{L}_r(X,Y)$ of all regular operators between the Riesz spaces X and Y is a subspace of the vector space $\mathcal{L}_b(X,Y)$ of all bounded operators from X to Y. Let Y be Dedekind-complete, then let $\mathcal{L}_r(X,Y)$ be a Dedekind-complete Riesz space (cf. [2, Theorem 1.67]), which contains the collection $\mathcal{L}_n(X,Y)$ of all order continuous operators from X to Y as a band (cf. [2, Theorem 1.73]). We write $\mathcal{L}_r(X)$ for $\mathcal{L}_r(X,X)$; $\mathcal{L}_n(X)$ for $\mathcal{L}_n(X,X)$; etc.

Recall that a Riesz space E *Riesz algebra* if E is an associative algebra whose positive cone E_+ is closed under the algebra multiplication, i.e. $x\cdot y \in E_+$ if $x,y \in E_+$. A Riesz algebra E is called (cf. [3, 5, 7, 9]):

– *left (right) d-algebra* if $u\cdot(x\wedge y) = (u\cdot x)\wedge(u\cdot y)$ (resp., $(x\wedge y)\cdot u = (x\cdot u)\wedge(y\cdot u)$) for all $x,y\in E$ and $u\in E_+$;

– *d-algebra* if E is both left and right d-algebra;

– *left (right) f-algebra* if $x\wedge y = 0$ implies$(u\cdot x)\wedge y = 0$ (resp., $(x\cdot u)\wedge y = 0$) for all $u\in E_+$;

– *f-algebra*, if E is both left and right f-algebra;

– *semiprime* if the only nilpotent element in E is 0;

– *unital* if E has a positive multiplicative unit.

Every Riesz space E becomes a commutative f-algebra with respect to the *trivial algebraic multiplication* $x * y = 0$ for all $x,y\in E$, and $(E,*)$ is neither unital nor semiprime unless $\dim(E) = 0$.

Example 1.1 ([5, Example 5]). In the Riesz space $\mathbb{R}^D$ of all $\mathbb{R}$-valued functions on a set D, any f-algebra multiplication $*$ is uniquely determined by the function $\zeta(d) := [\mathbb{I}_{\{d\}} * \mathbb{I}_{\{d\}}](d)$, where $\mathbb{I}_A \in \mathbb{R}^D$ is the characteristic function of $A\subseteq D$. Moreover, $(\mathbb{R}^D,*)$ is unital iff it is semiprime iff ζ is a weak order unit in $\mathbb{R}^D$.

Let $\mathbb{c}$ be a linear convergence on a Riesz algebra E (see e.g. [5, Definition 1.6]).

Definition 1.2 ([5, Definition 5.3]). The algebra multiplication in E is called *left* $\mathbb{c}$*-continuous (right* $\mathbb{c}$*-continuous)* if $x_\alpha \xrightarrow{\mathbb{c}} x$ implies $y\cdot x_\alpha \xrightarrow{\mathbb{c}} y\cdot x$ (resp., $x_\alpha\cdot y \xrightarrow{\mathbb{c}} x\cdot y$) for every $y\in E$. The algebra multiplication is called $\mathbb{c}$*-continuous* if it is both left and right $\mathbb{c}$-continuous.

Definition 1.3. Let X and Y be Riesz spaces equipped with linear convergences $\mathbb{c}_1$ and $\mathbb{c}_2$ respectively. An operator $T: X\to Y$ is called $\mathbb{c}_1\mathbb{c}_2$*-continuous*, whenever $x_\alpha \xrightarrow{\mathbb{c}_1} x$ in X implies $Tx_\alpha \xrightarrow{\mathbb{c}_2} Tx$ in Y. In the case when $\mathbb{c}_1 = \mathbb{c}_2$, we say that T is $\mathbb{c}_1$*-continuous.* The collection of all $\mathbb{c}_1\mathbb{c}_2$-continuous operators from X to Y is denoted by $\mathcal{L}_{\mathbb{c}_1\mathbb{c}_2}(X,Y)$, and if $\mathbb{c}_1 = \mathbb{c}_2$, we denote $\mathcal{L}_{\mathbb{c}_1\mathbb{c}_2}(X,Y)$ by $\mathcal{L}_{\mathbb{c}_1}(X,Y)$, and $\mathcal{L}_{\mathbb{c}_1}(X,X)$ by $\mathcal{L}_{\mathbb{c}_1}(X)$.

A net x_α in a Riesz algebra E is said to be *left (right) multiplicative* $\mathbb{c}$*-convergent* to x if $u\cdot|x_\alpha - x| \xrightarrow{\mathbb{c}}$ (resp. $|x_\alpha - x|\cdot u \xrightarrow{\mathbb{c}} 0$) for all $u\in E_+$; briefly $x_\alpha \xrightarrow{\mathbb{m}_l\mathbb{c}} x$ (resp.

$x_\alpha \xrightarrow{\mathbb{m}_r\mathbb{c}} x$). If $x_\alpha \xrightarrow{\mathbb{m}_l\mathbb{c}} x$ and $x_\alpha \xrightarrow{\mathbb{m}_r\mathbb{c}} x$ simultaneously, we write $x_\alpha \xrightarrow{\mathbb{m}\mathbb{c}} x$ (cf. [5, Definition 5.4]). It is worth to notice that many multiplicative $\mathbb{c}$-convergences could be defined by specifying the collections of admissible factors, e.g. the right multiplicative $\mathbb{c}$-convergence of a net x_α in E to $x \in E$ w.r. to $A \subseteq E$:

$$x_\alpha \xrightarrow{\mathbb{m}_r\mathbb{c}(A)} x \qquad \text{whenever} \qquad |x_\alpha - x| \cdot u \xrightarrow{\mathbb{c}} 0 \qquad (\forall u \in A).$$

We postpone the study of such specified convergences to further papers.

Remark 1.4. Clearly, $\mathbb{m}_l\mathbb{c} \equiv \mathbb{m}_r\mathbb{c}$ in commutative algebras. Moreover, $\mathbb{m}_l\mathbb{c}$-convergence turns to $\mathbb{m}_r\mathbb{c}$-convergence and vice versus, if we replace the algebra multiplication " $\cdot$ " in E by " $\hat{\cdot}$ " such that $x \,\hat{\cdot}\, y := y \cdot x$.

For Riesz spaces and Riesz algebras we refer to [2, 3, 5, 9]. The structure of this paper is as follows. In Section 2 we investigate $\mathbb{mo}$-, $\mathbb{rmo}$-, and $\mathbb{omo}$-continuous operators. Section 3 is dedicated to $\mathbb{mo}$-Lebesgue, $\mathbb{mo}$-KB, and $\mathbb{mo}$-Levi operators as well as the dominance problem for such operators.

2. Basic properties of $\mathbb{mo}$-continuous operators

In this section we investigate basic properties of $\mathbb{mo}$-, $\mathbb{omo}$- and $\mathbb{rmo}$-continuous operators in Riesz algebras.

Let X be a Dedekind complete Riesz space. The space $\mathcal{L}_r(X)$ is a unital Dedekind complete Riesz algebra under the composition operation, and the space $\mathcal{L}_n(X)$ is a Riesz subalgebra of $\mathcal{L}_r(X)$. It is well known that the algebra multiplication in $\mathcal{L}_r(X)$ is right $\mathbb{o}$-continuous, while in $\mathcal{L}_n(X)$ $\mathbb{o}$- continuous (see e.g. [3, Theorem 1.56]). The following result extends this fact to $\mathbb{mo}$-convergence.

Theorem 2.1. *Let X be a Dedekind complete Riesz space. Then the algebra multiplication is*:

(i) right $\mathbb{m}_r\mathbb{o}$-continuous in $E = \mathcal{L}_r(X)$;

(ii) left and right $\mathbb{mo}$-continuous in $F = \mathcal{L}_n(X)$.

Proof. (i) Let $T_\alpha \xrightarrow{\mathbb{m}_r\mathbb{o}} T$ in E and $R \in E$, Then, for every $S \in E_+$,

$$|T_\alpha \circ R - T \circ R| \circ S \le |T_\alpha - T| \circ |R| \circ S = |T_\alpha - T| \circ (|R| \circ S) \xrightarrow{\mathbb{o}} 0. \tag{1}$$

By (1), $|T_\alpha \circ R - T \circ R| \circ S \xrightarrow{\mathbb{o}} 0$ for every $S \in E_+$ and hence $T_\alpha \circ R \xrightarrow{\mathbb{m}_r\mathbb{o}} T \circ R$. Since $R \in E$ is arbitrary, the algebra multiplication in E is right $\mathbb{m}_r\mathbb{o}$-continuous.

(ii) Let $T_\alpha \xrightarrow{\mathbb{mo}} T$ in F and $R \in F$. Since $S \circ |T_\alpha - T| \xrightarrow{\mathbb{o}} 0$ for each $S \in F_+$, it follows from

$$S \circ |T_\alpha \circ R - T \circ R| \le S \circ |T_\alpha - T| \circ |R| = (S \circ |T_\alpha - T|) \circ |R| \quad (\forall S \in F_+)$$

that $S \circ |T_\alpha \circ R - T \circ R| \xrightarrow{\mathbb{o}} 0$, because the multiplication in E is right $\mathbb{o}$-continuous by [3, Theorem 1.56]. Since $S \in F$ is arbitrary, we conclude that $T_\alpha \circ R \xrightarrow{\mathbb{m}_l\mathbb{o}} T \circ R$ and hence the multiplication in F is right $\mathbb{m}_l\mathbb{o}$-continuous. It follows from (i) that

$T_\alpha \circ R \xrightarrow{\mathbb{m}_r\mathbb{o}} T \circ R$, and hence $T_\alpha \circ R \xrightarrow{\mathbb{m}\mathbb{o}} T \circ R$. Since $R \in F$ is arbitrary, the algebra multiplication in F is right $\mathbb{m}\mathbb{o}$-continuous.

We skip similar elementary arguments which shows that the algebra multiplication in F is also left $\mathbb{m}\mathbb{o}$-continuous. □

The following example shows that, in general, the algebra multiplication is not left $\mathbb{m}_r\mathbb{o}$-continuous in $\mathcal{L}_r(X)$.

Example 2.2. Take a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Recall that any bounded real sequence x_k *converges along* $\mathcal{U}$ to some $x_{\mathcal{U}} = \lim_{\mathcal{U}} x_k \in \mathbb{R}$ in the sense that $\{k \in \mathbb{N} : |x_k - x_{\mathcal{U}}| \leq \varepsilon\} \in \mathcal{U}$ for every $\varepsilon > 0$. Define operators $L, T_n \in \mathcal{L}_r(\ell^\infty)$ by $Lx := x_{\mathcal{U}} \cdot \mathbb{1}_{\mathbb{N}}$ and $T_n x := x_{\mathcal{U}} \cdot \mathbb{1}_{\{k\in\mathbb{N}:k\geq n\}}$. It is easy to see that $T_n \downarrow 0$ and hence $T_n \xrightarrow{\mathbb{m}_r\mathbb{o}} 0$ in $\mathcal{L}_r(\ell^\infty)$. However, $L \circ T_n(x) = L(x_{\mathcal{U}} \cdot \mathbb{1}_{\{k\in\mathbb{N}:k\geq n\}}) = x_{\mathcal{U}} \cdot \mathbb{1}_{\mathbb{N}}$ implies that $L \circ T_n \circ I = L \circ T_n \equiv L \neq 0$. Thus the sequence $L \circ T_n$ does not $\mathbb{m}_r\mathbb{o}$-converge to 0, and hence the algebra multiplication in the Riesz algebra $\mathcal{L}_r(\ell^\infty)$ is neither left $\mathbb{o}$-, nor left $\mathbb{m}_r\mathbb{o}$-continuous.

The following observation is straightforward.

Observation 2.3. Let E and F be Riesz algebras.
a) The collections $\mathcal{L}_{\mathbb{m}_r\mathbb{o}}(E,F)$, $\mathcal{L}_{\mathbb{o}\mathbb{m}_r\mathbb{o}}(E,F)$, and $\mathcal{L}_{\mathbb{r}\mathbb{m}_r\mathbb{o}}(E,F)$ of all $\mathbb{m}_r\mathbb{o}$-, $\mathbb{o}\mathbb{m}_r\mathbb{o}$-, and $\mathbb{r}\mathbb{m}_r\mathbb{o}$-continuous operators from E to F are vector spaces.

b) If E is unital, then $\mathcal{L}_{\mathbb{o}\mathbb{m}_r\mathbb{o}}(E,F) \subseteq \mathcal{L}_{\mathbb{m}_r\mathbb{o}}(E,F)$.

c) If F is unital, then $\mathcal{L}_{\mathbb{o}\mathbb{m}_r\mathbb{o}}(E,F) \subseteq \mathcal{L}_n(E,F)$.

d) If E has right $\mathbb{o}$-continuous algebra multiplication then $\mathcal{L}_{\mathbb{m}_r\mathbb{o}}(E,F) \subseteq \mathcal{L}_{\mathbb{o}\mathbb{m}_r\mathbb{o}}(E,F)$ (cf. [5, Lemma 5.5]).

e) If E is Archimedean then $\mathcal{L}_{\mathbb{o}\mathbb{m}_r\mathbb{o}}(E,F) \subseteq \mathcal{L}_{\mathbb{r}\mathbb{m}_r\mathbb{o}}(E,F)$.

f) If E is Archimedean and has right $\mathbb{o}$-continuous algebra multiplication then by d) and e): $\mathcal{L}_{\mathbb{m}_r\mathbb{o}}(E,F) \subseteq \mathcal{L}_{\mathbb{r}\mathbb{m}_r\mathbb{o}}(E,F)$.

g) If F is an Archimedean f-algebra then the algebra multiplication in F is commutative and $\mathbb{o}$-continuous (cf. [7, 9]). Hence $\mathcal{L}_n(E,F) \subseteq \mathcal{L}_{\mathbb{o}\mathbb{m}\mathbb{o}}(E,F)$ (cf. [5, Lemma 5.1]).

The next result generalizes [5, Proposition 5.1] with essentially the same proof.

Proposition 2.4. *The algebra multiplication in any Riesz algebra F is both left and right $\mathbb{r}$-continuous.*

Proof. Let $x_\alpha \xrightarrow{\mathbb{r}} x$ in F and $y \in F$. Then there exists $v \in F_+$ such that, for any $k \in \mathbb{N}$, there is α_k with $|x_\alpha - x| \leq \frac{1}{k}v$ for all $\alpha \geq \alpha_k$. Since, for all $\alpha \geq \alpha_k$ we have

$$|y \cdot x_\alpha - y \cdot x| \leq |y| \cdot |x_\alpha - x| \leq \frac{1}{k}|y| \cdot v \quad \& \quad |x_\alpha \cdot y - x \cdot y| \leq |x_\alpha - x| \cdot |y| \leq \frac{1}{k}v \cdot |y|$$

for all $\alpha \geq \alpha_k$, it follows $y \cdot x_\alpha \xrightarrow{\mathbb{r}} y \cdot x$ and $x_\alpha \cdot y \xrightarrow{\mathbb{r}} x \cdot y$. The result follows from the fact that $y \in F$ was taken arbitrarily. □

Proposition 2.5. *Let X be a Riesz space and F an Archimedean Riesz algebra. Then $\mathcal{L}_{\mathbb{r}\mathbb{r}}(X,F) \subseteq \mathcal{L}_{\mathbb{r}\mathbb{m}\mathbb{o}}(X,F)$.*

Proof. Let $T \in \mathcal{L}_{\mathbb{rr}}(X,F)$. Suppose $x_\alpha \xrightarrow{\mathbb{r}} x$ in X. Then $Tx_\alpha \xrightarrow{\mathbb{r}} Tx$ in F. Take $u \in F_+$, such that, for any $n \in \mathbb{N}$, there exists α_n with $|Tx_\alpha - Tx| \leq \frac{1}{n}u$ for all $\alpha \geq \alpha_n$. Take any $w \in F_+$. Then $|Tx_\alpha - Tx| \cdot w \leq \frac{1}{n}u \cdot w$ for all $\alpha \geq \alpha_n$. Since F is Archimedean, then $\frac{1}{n}u \cdot w \downarrow 0$ and hence $|Tx_\alpha - Tx| \cdot w \xrightarrow{\mathbb{o}} 0$. Since $w \in F_+$ is arbitrary, $Tx_\alpha \xrightarrow{\mathbb{m_r o}} Tx$. Since the proof of $Tx_\alpha \xrightarrow{\mathbb{m_l o}} Tx$ is analogous, we obtain the desired result. □

Theorem 2.6. *Let X be a Riesz space and let F be an Archimedean Riesz algebra, then $\mathcal{L}_r(X,F) \subseteq \mathcal{L}_{\mathbb{rr}}(X,F) \subseteq \mathcal{L}_{\mathbb{rm}_l\mathbb{r}}(X,F) \cap \mathcal{L}_{\mathbb{rm}_r\mathbb{r}}(X,F) \subseteq \mathcal{L}_{\mathbb{rmo}}(X,F)$.*

Proof. Despite the first incision is well known, we include its proof for the convenience. Let $T \in \mathcal{L}_r(X,F)$ and $x_\alpha \xrightarrow{\mathbb{r}} x$ in X. Take $T_1, T_2 \geq 0$ with $T = T_1 - T_2$ and $u \in X_+$, so that, for each $n \in \mathbb{N}$, there exists α_n such that $|x_\alpha - x| \leq \frac{1}{n}u$ for all $\alpha \geq \alpha_n$. Then

$$|Tx_\alpha - Tx| = |(T_1 - T_2)x_\alpha - (T_1 - T_2)x| = |T_1(x_\alpha - x) - T_2(x_\alpha - x)|$$

$$\leq |T_1(x_\alpha - x)| + |T_2(x_\alpha - x)| \leq T_1|x_\alpha - x| + T_2|x_\alpha - x| \leq \frac{1}{n}(T_1 u + T_2 u),$$

and thus $Tx_\alpha \xrightarrow{\mathbb{r}} Tx$ in E. Hence $\mathcal{L}_r(X,F) \subseteq \mathcal{L}_{\mathbb{rr}}(X,F)$.

In the rest of the proof, it suffices to restrict ourselves to the "right" case only. Let $T \in \mathcal{L}_{\mathbb{rr}}(X,F)$ and $x_\alpha \xrightarrow{\mathbb{r}} x$ in X. It follows from $Tx_\alpha \xrightarrow{\mathbb{r}} Tx$ that there exist $w \in F_+$ and a sequence of indexes α_n satisfying $|Tx_\alpha - Tx| \leq \frac{1}{n}w$ for all $\alpha \geq \alpha_n$. Then, for every $f \in F_+$,

$$|Tx_\alpha - Tx| \cdot f \leq \frac{1}{n} w \cdot f \quad (\forall \alpha \geq \alpha_n). \tag{2}$$

By (2), $Tx_\alpha \xrightarrow{\mathbb{m_r r}} Tx$ and hence $\mathcal{L}_{\mathbb{rr}}(X,F) \subseteq \mathcal{L}_{\mathbb{rm}_r\mathbb{r}}(X,F)$.

The inclusions $\mathcal{L}_{\mathbb{rm}_l\mathbb{r}}(X,F) \subseteq \mathcal{L}_{\mathbb{rm}_l\mathbb{o}}(X,F)$ and $\mathcal{L}_{\mathbb{rm}_r\mathbb{r}}(X,F) \subseteq \mathcal{L}_{\mathbb{rm}_r\mathbb{o}}(X,F)$ hold true because the $\mathbb{r}$-convergence implies $\mathbb{o}$-convergence in any Archimedean Riesz space, so in F. □

Example 2.7. The set $\mathrm{Orth}(X)$ of all orthomorphisms on an Archimedean Riesz space X is an Archimedean commutative unital algebra with $\mathbb{o}$-continuous algebra multiplication [9, Theorem 8.6]. Since $\mathbb{mo}$-convergence coincides with $\mathbb{o}$-convergence in $\mathrm{Orth}(X)$, any operator from $\mathrm{Orth}(X)$ to an arbitrary Riesz algebra is $\mathbb{mo}$-continuous iff it is $\mathbb{omo}$-continuous.

Example 2.8 ([5, Example 6]). Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. We define an operation $*$ in ℓ^∞ by $x * y := \lim_{\mathcal{U}}(x_n \cdot y_n) \cdot \mathbb{1}_{\mathbb{N}}$. It is easy to see that $(\ell^\infty, *)$ is a commutative d-algebra. The identity operator I on $(\ell^\infty, *)$ is order bounded but it is neither $\mathbb{om}_l\mathbb{o}$- nor $\mathbb{om}_r\mathbb{o}$-continuous. Indeed, $f_n := \mathbb{1}_{\{k \in \mathbb{N}: k \geq n\}} \xrightarrow{\mathbb{o}} 0$ in ℓ^∞, yet the sequence $\mathbb{1} * I(f_n) = I(f_n) * \mathbb{1} = f_n * \mathbb{1} \equiv \mathbb{1}$ $\mathbb{o}$-converges to $\mathbb{1} \neq 0 = \mathbb{1} * I(0) = I(0) * \mathbb{1}$.

We continue with the following extension of [5, Theorem 12].

Proposition 2.9. *Let E be a Riesz algebra. The following conditions are equivalent:*
(i) E is a left (right) d-algebra with left (right) $\mathbb{o}$-continuous multiplication;

(ii) $\inf_E(u \cdot A) = u \cdot \inf_E A$ *(resp.,* $\inf_E(A \cdot u) = (\inf_E A) \cdot u$*) for every* $u \in E_+$ *and* $A \subseteq E$ *such that* $\inf_E A$ *exists;*

(iii) $\sup_E(u \cdot A) = u \cdot \sup_E A$ *(resp.,* $\sup_E(A \cdot u) = (\sup_E A) \cdot u$*) for every* $u \in E_+$ *and* $A \subseteq E$ *such that* $\sup_E A$ *exists.*

The proof of Proposition 2.9 for the "right" case coincides with the proof of [5, Theorem 12], and the proof for the "left" case is similar.

THEOREM 2.10. *Each* $\mathbb{m}_r\mathbb{o}$*-continuous operator from a Riesz algebra* E *satisfying* $\inf_E(A \cdot u) = (\inf_E A) \cdot u$ *for every* $u \in E_+$ *and* $A \subseteq E$*, whenever* $\inf_E A$ *exists, to a Riesz algebra* F *is* $\mathbb{o}\mathbb{m}_r\mathbb{o}$*-continuous. The same result is true with replacing "right" by "left".*

Proof. We restrict ourselves to the "right" case only. It is enough to show that $\mathbb{o}$-convergence implies $\mathbb{m}_r\mathbb{o}$-convergence in E. Let $x_\alpha \xrightarrow{\mathbb{o}} x$ in E. Then there exists a net y_β satisfying $y_\beta \downarrow 0$, and for any β, there is α_β such that $|x_\alpha - x| \le y_\beta$ for all $\alpha \ge \alpha_\beta$. Take $u \in E_+$. Then $|x_\alpha - x| \cdot u \le y_\beta \cdot u$ for all $\alpha \ge \alpha_\beta$. It follows from Proposition 2.9 $\inf\limits_{\beta \in B}(y_\beta \cdot u) = (\inf\limits_{\beta \in B} y_\beta) \cdot u = 0$. Thus $y_\beta \cdot u \downarrow 0$, and hence $|x_\alpha - x| \cdot u \xrightarrow{\mathbb{o}} 0$. Since $u \in E_+$ is arbitrary, $x_\alpha \xrightarrow{\mathbb{m}_r\mathbb{o}} x$. □

Each Archimedean f-algebra E is a commutative d-algebra (cf. [7]), and hence satisfies the conditions of the above theorem due to Proposition 2.9. The following example shows that $\mathbb{o}$-continuity of the algebra multiplication in E is essential in Theorem 2.10.

EXAMPLE 2.11. Consider the following d-algebra multiplication in the Riesz space c of convergent real sequences:

$$x * y := (\lim_{n\to\infty} x_n \cdot y_n) \cdot \mathbb{1}_{\mathbb{N}}. \tag{3}$$

The algebra multiplication $*$ is not $\mathbb{o}$-continuous since $\mathbb{1}_{\{k \ge n\}} \downarrow 0$ but $\mathbb{1}_{\mathbb{N}} * \mathbb{1}_{\{k \ge n\}} \equiv \mathbb{1}_{\mathbb{N}} \neq 0$. The identity operator I on $(c, *)$ is trivially $\mathbb{m}\mathbb{o}$-continuous. However, it is not $\mathbb{o}\mathbb{m}\mathbb{o}$-continuous: $\mathbb{1}_{\{k \ge n\}} \xrightarrow{\mathbb{o}} 0$ yet $I(\mathbb{1}_{\{k \ge n\}}) * \mathbb{1}_{\mathbb{N}} \equiv \mathbb{1}_{\mathbb{N}}$, and hence $I(\mathbb{1}_{\{k \ge n\}})$ does not $\mathbb{m}\mathbb{o}$-converge to $I(0) = 0$.

Furthermore, I is $\mathbb{m}\mathbb{o}\mathbb{m}\mathbb{r}$-continuous on $(c, *)$. Indeed, $g_\alpha \xrightarrow{\mathbb{m}\mathbb{o}} 0$ in $(c, *)$ implies $|g_\alpha| * \mathbb{1}_{\mathbb{N}} \xrightarrow{\mathbb{o}} 0$, and since by (3), the sequence $|g_\alpha| * \mathbb{1}_{\mathbb{N}}$ lies in one dimensional sublattice of c, then $|g_\alpha| * \mathbb{1}_{\mathbb{N}} \xrightarrow{\mathbb{r}} 0$, which implies that $|g_\alpha| * u \xrightarrow{\mathbb{r}} 0$ for all $u \in c_+$, and hence $g_\alpha \xrightarrow{\mathbb{m}\mathbb{r}} 0$ in $(c, *)$.

DEFINITION 2.12. A subset A of a Riesz algebra E is called:
- $\mathbb{m}_l\mathbb{o}$-*bounded* if the set $u \cdot A$ is order bounded for each $u \in E_+$;
- $\mathbb{m}_r\mathbb{o}$-*bounded* if $A \cdot u$ is order bounded for each $u \in E_+$;
- $\mathbb{m}\mathbb{o}$-*bounded* if A is both $\mathbb{m}_l\mathbb{o}$- and $\mathbb{m}_r\mathbb{o}$-bounded.

Every $\mathbb{o}$-bounded subset of E is $\mathbb{m}\mathbb{o}$-bounded; and if the algebra multiplication in E is trivial then every subset of E is $\mathbb{m}\mathbb{o}$-bounded.

Definition 2.13. An operator T from a Riesz space X to a Riesz algebra F is called $\mathbb{m}_l\mathbb{o}$-, $\mathbb{m}_r\mathbb{o}$-, or else $\mathbb{mo}$-*bounded* if T maps order bounded subsets of X into $\mathbb{m}_l\mathbb{o}$-, $\mathbb{m}_r\mathbb{o}$-, or $\mathbb{mo}$-bounded subsets of F respectively.

If the algebra multiplication in E is trivial, then every operator on E is $\mathbb{mo}$-bounded.

Remark 2.14. Let X be a Riesz space, and let F be a Riesz algebra. Then:
− every order bounded (and hence every positive) operator from X to F is $\mathbb{mo}$-bounded;

− if F is unital, then every ($\mathbb{m}_l\mathbb{o}$-) $\mathbb{m}_r\mathbb{o}$-bounded operator from X to F is order bounded.

The next result is a version of [1, Theorem 2.1] for $\mathbb{mo}$-continuous operators.

Theorem 2.15. *Let $T : E \to F$ be an operator between two Archimedean Riesz algebras.*
(i) If T is $\mathbb{mo}$-, $\mathbb{omo}$-, or $\mathbb{rmo}$-continuous then T is $\mathbb{mo}$-bounded.

(ii) If T is $\mathbb{m}_l\mathbb{o}$-, $\mathbb{om}_l\mathbb{o}$-, or $\mathbb{rm}_l\mathbb{o}$-continuous (resp., $\mathbb{m}_r\mathbb{o}$-, $\mathbb{om}_r\mathbb{o}$-, or $\mathbb{rm}_r\mathbb{o}$–continuous) then T is $\mathbb{m}_l\mathbb{o}$-bounded (resp., $\mathbb{m}_r\mathbb{o}$-bounded).

Proof. (i) Let $T : E \to F$ be an $\mathbb{mo}$-continuous (resp., $\mathbb{omo}$-continuous, $\mathbb{rmo}$-continuous) operator between two Riesz algebras. Take an arbitrary order interval $[0, b] \subseteq E$ and consider the directed set $\mathcal{I} = \mathbb{N} \times [0, b]$ with the lexicographical order [1, Theorem 2.1]. Take a net $x_{(n,y)} := \frac{1}{n} y$ indexed by $\mathcal{I}$. It follows from $0 \le x_{(n,y)} \le \frac{1}{n} b \downarrow 0$ that $x_{(n,y)} \xrightarrow{\mathbb{r}} 0$. Then $x_{(n,y)} \xrightarrow{\mathbb{o}} 0$, $x_{(n,y)} \xrightarrow{\mathbb{mr}} 0$, and $x_{(n,y)} \xrightarrow{\mathbb{mo}} 0$. If T is $\mathbb{mo}$-continuous (resp., $\mathbb{omo}$-, and $\mathbb{rmo}$-continuous) then $Tx_{(n,y)} \xrightarrow{\mathbb{mo}} 0$. Hence

$$|w \cdot T(x_{(n,y)})| \vee |T(x_{(n,y)}) \cdot w| \xrightarrow{\mathbb{o}} 0 \quad (\forall w \in F_+).$$

Take any $w \in F_+$. Then there exists a net $z_\beta \downarrow 0$ in F such that for every β there exists $(n_\beta, y_\beta) \in \mathcal{I}$ satisfying $|w \cdot T(x_{(n,y)})| \vee |T(x_{(n,y)}) \cdot w| \le z_\beta$ for all $(m, y) \ge (n_\beta, y_\beta)$. Take any z_β. Then, in particular, for all $u \in [0, b]$,

$$\frac{1}{n_\beta + 1}(|w \cdot Tu| \vee |(Tu) \cdot w|) = |w \cdot T(x_{(n_\beta+1,u)})| \vee |T(x_{(n_\beta+1,u)}) \cdot w| \le z_\beta. \tag{4}$$

By (4), $w \cdot T[0, b] \cup (T[0, b]) \cdot w \subseteq [-(n_\beta + 1)z_\beta, (n_\beta + 1)z_\beta]$ and since $w \in F_+$ is arbitrary, we conclude that T is $\mathbb{mo}$-bounded.

(ii) The modification of the proof in (i) for the "$\mathbb{m}_l\mathbb{o}$-" and "$\mathbb{m}_r\mathbb{o}$-bounded case" is trivial. □

3. $\mathbb{mo}$-Lebesgue, $\mathbb{mo}$-KB, and $\mathbb{mo}$-Levi operators

In this section, we undertake an attempt to adopt some of results of the recent paper [4] to $\mathbb{mo}$-convergence in Riesz algebras. Especially, we investigate the domination problem for related operators.

Definition 3.1. We say that an operator T
a) from a Riesz space X to a Riesz algebra F is $\mathbb{mo}$-*Lebesgue* if $Tx_\alpha \xrightarrow{\mathbb{mo}} 0$ for every net x_α in X such that $x_\alpha \downarrow 0$. In particular, every $\mathbb{omo}$-continuous operator is $\mathbb{mo}$-Lebesgue;

b) from a locally solid Riesz space $X = (X, \tau)$ to a Riesz algebra F is an $\mathbb{mo}$-*KB-operator* if, for every τ-bounded increasing net x_α in X_+, there exists $x \in X$ such that $Tx_\alpha \xrightarrow{\mathbb{mo}} Tx$;

c) from a locally solid Riesz space $X = (X, \tau)$ to a Riesz algebra F is a *quasi* $\mathbb{mo}$-*KB-operator* if, for every τ-bounded increasing net x_α in X_+, Tx_α is an $\mathbb{mo}$-Cauchy net;

d) from a Riesz algebra E to a Riesz algebra F is an $\mathbb{mo}$-*Levi operator* if, for every $\mathbb{mo}$-bounded increasing net x_α in E_+, there exists $x \in E$ such that $Tx_\alpha \xrightarrow{\mathbb{mo}} Tx$;

e) from a Riesz algebra E to a Riesz algebra F is a *quasi* $\mathbb{mo}$-*Levi operator* if, for every $\mathbb{mo}$-bounded increasing net x_α in E_+, the net Tx_α is $\mathbb{mo}$-Cauchy in F.

The $\mathbb{m}_l\mathbb{o}$- $\mathbb{m}_r\mathbb{o}$-*Lebesgue operators*, the $\mathbb{m}_l\mathbb{o}$- and $\mathbb{m}_r\mathbb{o}$-*KB-operators*, the *quasi* $\mathbb{m}_l\mathbb{o}$-*KB*- and *quasi* $\mathbb{m}_r\mathbb{o}$-*KB-operators*, the $\mathbb{m}_l\mathbb{o}$- and $\mathbb{m}_r\mathbb{o}$-*Levi operators*, the *quasi* $\mathbb{m}_l\mathbb{o}$-*Levi* and *quasi* $\mathbb{m}_r\mathbb{o}$-*Levi operators* are defined analogously.

Although it seems that the sequential versions of Definition 3.1 and the adopted for $\mathbb{r}$-convergence modifications of this definition are also interesting, we do not study them in the present paper.

Example 3.2 ([4, Example 3.1]). Let E be the f-algebra of all bounded real functions on $[0, 1]$ which differ from a constant on at most countable subset of $[0, 1]$ equipped with the poinwise algebraic multiplication. Let $T : E \to E$ be an operator that assigns to each $f \in E$ the constant function Tf on $[0, 1]$ such that the set $\{t \in [0, 1] : f(t) \neq (Tf)(t)\}$ is at most countable. Then T is a rank one operator which is continuous in the sup-norm on E. Consider the following net in E indexed by finite subsets of $[0, 1]$:

$$f_\alpha(t) = \begin{cases} 1 & \text{if } t \notin \alpha \\ 0 & \text{if } t \in \alpha \end{cases}$$

Then $f_\alpha \downarrow 0$ in E and so $f_\alpha \xrightarrow{\mathbb{mo}} 0$. However, $Tf_\alpha \equiv \mathbb{1}_{[0,1]}$ for all α, and hence $Tf_\alpha \xrightarrow{\mathbb{mo}} \mathbb{1}_{[0,1]} \neq 0$. Therefore T is neither $\mathbb{omo}$- nor $\mathbb{mo}$-Lebesgue.

It is well known that each order bounded disjointness preserving operator T between Riesz spaces X and Y has modulus $|T|$, and $|T||x| = |T|x|| = |Tx|$ for all $x \in X$; and there exist Riesz homomorphisms $R_1, R_2 : X \to Y$ such that $T = R_1 - R_2$.

The next theorem is motivated by [4, Theorem 2.5] and has the similar proof.

Theorem 3.3. *Let T be an order bounded disjointness preserving $\mathbb{mo}$-KB-operator from a locally solid Riesz space (X, τ) to a Riesz algebra F. If $|S| \leq |T|$ then S is an $\mathbb{mo}$-KB-operator. The similar result holds true for $\mathbb{m}_l\mathbb{o}$-KB and $\mathbb{m}_r\mathbb{o}$-KB operators.*

Proof. We restrict ourselves to the case of $\mathbb{mo}$-KB-operators. Take a τ-bounded increasing net x_α in X_+. Then $T(x_\alpha - x) \xrightarrow{\mathbb{mo}} 0$ for some $x \in X$. So, for every $u \in F_+$, $u \cdot |T(x_\alpha - x)| \xrightarrow{\mathbb{o}} 0$ and $|T(x_\alpha - x)| \cdot u \xrightarrow{\mathbb{o}} 0$, and hence

$$u \cdot |S(x_\alpha - x)| \leq u \cdot |S||x_\alpha - x| \leq u \cdot |T||x_\alpha - x| = u \cdot |Tx_\alpha - Tx| \xrightarrow{\mathbb{o}} 0;$$

$$|S(x_\alpha - x)| \cdot u \leq |S||x_\alpha - x| \cdot u \leq |T||x_\alpha - x| \cdot u = |Tx_\alpha - Tx| \cdot u \xrightarrow{\mathbb{o}} 0.$$

Thus $(u \cdot |Sx_\alpha - Sx|) \vee (|Sx_\alpha - Sx| \cdot u) \xrightarrow{\mathbb{o}} 0$ for every $u \in F_+$. Therefore $Sx_\alpha \xrightarrow{\mathbb{mo}} Sx$, and hence S is an $\mathbb{mo}$-KB-operator. □

Since, for a Riesz homomorphism T, $0 \leq S \leq T$ implies that S is also a Riesz homomorphism, the next result follows from Theorem 3.3.

Corollary 3.4. *Let T be an $\mathbb{mo}$-KB Riesz homomorphism from a locally solid Riesz space to a Riesz algebra. Then every operator S satisfying $0 \leq S \leq T$ is also an $\mathbb{mo}$-KB Riesz homomorphism. The similar result is true for $\mathbb{m}_l\mathbb{o}$-KB and $\mathbb{m}_r\mathbb{o}$-KB Riesz homomorphisms.*

The following two results are motivated by [4, Theorem 2.6, Theorem 2.7].

Theorem 3.5. *Let T be a positive quasi $\mathbb{mo}$-KB operator from a locally solid Riesz space (X, τ) to a Riesz algebra F. Then every operator S satisfying $0 \leq S \leq T$ is also a quasi $\mathbb{mo}$-KB operator. The similar result holds true for quasi $\mathbb{m}_l\mathbb{o}$-KB and quasi $\mathbb{m}_r\mathbb{o}$-KB operators.*

Proof. Take an increasing τ-bounded net x_α in X_+ and let $0 \leq S \leq T$. Then $Tx_\alpha \uparrow$, and since T is quasi $\mathbb{mo}$-KB, the net Tx_α is $\mathbb{mo}$-Cauchy. Pick a $u \in F_+$. Then there exists a net $z_\beta \downarrow$ in F such that, for each β, there exists α_β such that $u \cdot |Tx_{\alpha_1} - Tx_{\alpha_2}| \leq z_\beta$ for all $\alpha_1, \alpha_2 \geq \alpha_\beta$. Choosing $\alpha_1, \alpha_2 \geq \alpha_\beta$ for a fixed α_β we have

$$u \cdot |Sx_{\alpha_1} - Sx_{\alpha_2}| \leq u \cdot |S(x_{\alpha_1} - x_{\alpha_\beta})| \leq u \cdot |T(x_{\alpha_1} - x_{\alpha_\beta})| \leq z_\beta. \tag{5}$$

By (5), $u \cdot |Sx_{\alpha_1} - Sx_{\alpha_2}| \leq z_\beta$ for all $\alpha_1, \alpha_2 \geq \alpha_\beta$. Thus the net Sx_α is $\mathbb{m}_l\mathbb{o}$-Cauchy. Hence, the operator S is quasi $\mathbb{m}_l\mathbb{o}$-KB. Similar argument shows that S is quasi $\mathbb{m}_r\mathbb{o}$-KB. Therefore, S is a quasi $\mathbb{mo}$-KB-operator. □

Theorem 3.6. *Let T be a positive quasi $\mathbb{m}_l\mathbb{o}$-Levi operator from a Riesz algebra E to a Riesz algebra F. Then each operator $S : E \to F$ satisfying $0 \leq S \leq T$ is also quasi $\mathbb{m}_l\mathbb{o}$-Levi. The similar result holds true for quasi $\mathbb{m}_r\mathbb{o}$-Levi and $\mathbb{mo}$-Levi operators.*

Proof. Let x_α be an $\mathbb{m}_l\mathbb{o}$-bounded increasing net in E_+. Then the net Tx_α is $\mathbb{m}_l\mathbb{o}$-Cauchy in F. Pick some $u \in F_+$. There exists a net $z_\beta \downarrow 0$ in F such that, for each β, there exists α_β such that $u \cdot |Tx_{\alpha_1} - Tx_{\alpha_2}| \leq z_\beta$ for all $\alpha_1, \alpha_2 \geq \alpha_\beta$. Pick $\alpha_1, \alpha_2 \geq \alpha_\beta$ for a fixed α_β. Then

$$u \cdot |Sx_{\alpha_1} - Sx_{\alpha_2}| \leq u \cdot |S(x_{\alpha_1} - x_{\alpha_\beta})| \leq u \cdot |T(x_{\alpha_1} - x_{\alpha_\beta})| \leq z_\beta. \tag{6}$$

By (6), $u \cdot |Sx_{\alpha_1} - Sx_{\alpha_2}| \leq z_\beta$ for all $\alpha_1, \alpha_2 \geq \alpha_\beta$. Since $u \in F_+$ is arbitrary, the net Sx_α is $\mathbb{m}_l\mathbb{o}$-Cauchy. Hence, S is quasi $\mathbb{m}_l\mathbb{o}$-Levi. Analogously, if T is quasi $\mathbb{m}_r\mathbb{o}$-Levi or $\mathbb{mo}$-Levi then S has the same property. □

We conclude the paper with a brief discussion of the extensions of operators introduced in Definition 3.1 to the second-order duals. In what follows, we assume that the first-order dual of any Riesz space under consideration separates its points. Recall that under these conditions, any Riesz space X is embedded in its second-order dual X'' via the mapping $x \xrightarrow{i} x''$, where $x''(z) = z(x)$ for all $z \in X'$. If $(E, \cdot)$ is a Riesz algebra, then E'' is again a Riesz algebra with respect to *Arens multiplication* [8]. In all cases of an operator T considered in Definition 3.1, i.e. $T : X \to F$ and $T : E \to F$, the second dual T'' acts between spaces of the same type. Therefore, it is natural to ask whether or not, for an $\mathbb{mo}$-Lebesgue, $\mathbb{mo}$-KB-, and $\mathbb{mo}$-Levi operator T, the operator T'' is $\mathbb{mo}$-Lebesgue, $\mathbb{mo}$-KB-, and $\mathbb{mo}$-Levi respectively. The authors are not aware of any examples for T, for which the answer to the above question is negative.

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