MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 16.06.2023

research paper
оригинални научни рад
DOI: 10.57016/MV-SPDF3739

COVERINGS, ACTIONS AND QUOTIENTS IN CAT^1-GROUPOIDS

Sedat Temel and Osman Çan

Abstract. The aim of this paper is to present the notions of actions and coverings of cat^1-groupoids and to prove the natural equivalence between their categories. Moreover, in this context, we characterize the quotient concept of cat^1-groupoids. Finally we extend these notions to cat^n-groupoids which are groupoid version of cat^n-groups.

1. Introduction

There are various 2-dimensional notions of groupoids such as 2-groupoids, double-groupoids, and crossed modules over groupoids. It is known that crossed modules over groupoids are categorically equivalent to 2-groupoids [7, 11] and to double groupoids with thin structures [18]. In this context, the notion of cat^1-groupoids as a new 2-dimensional version of groupoids was introduced in [12, 21] and it was proved that crossed modules over groupoids are equivalent to cat^1-groupoids. Since a groupoid with a single object is a group, cat^1-groupoids can be regarded as the groupoid case of cat^1-groups defined by Loday [13] (see also [6]). It is well known that the categories of cat^1-groups and of crossed modules over groups are naturally equivalent. This equivalence is useful for extending crossed modules to higher dimensions.

Studies of covering groupoids play an important role in applications of groupoids [3, 10]. The categorical equivalence between the category $\mathrm{Cov}(\mathrm{G{\scriptsize PD}})/\mathcal{G}$ of covering morphisms of a certain groupoid $\mathcal{G}$ and the category $\mathrm{Act}(\mathrm{G{\scriptsize PD}})/\mathcal{G}$ of groupoid actions of $\mathcal{G}$ on sets is well known (for the topological version, see [8]). Brown & Mucuk [4] extended this equivalence to group groupoids (i.e., 2-groups [2], $\mathcal{G}$-groupoids [5], or group objects in the category of groupoids). This result was adapted to internal groupoids in the category of groups with operations [1], to Leibniz algebras setting [19], to categorical groups [16] and rings [17].

The quotient concept of groupoids is constructed in [3, p. 420] and [10, p. 86]. Recently, normal and quotient objects in the category of crossed modules over groupoids

2020 Mathematics Subject Classification: 20L05, 18D35, 18B40, 20M50
Keywords and phrases: Cat^n-groupoid; covering; action; normality; quotient.

have been characterized and compared with the corresponding objects in 2-groupoids [20] and double groupoids with thin structures [18] using the categorical equivalences between them.

The aim of this manuscript is to introduce the notion of coverings and actions of cat^1-groupoids and to prove the categorical equivalence between their categories. In [21], the normal subcat^1-groupoids are obtained via normality in groupoids and compared with normal objects in crossed modules over groupoids by using the equivalence between these two categories. However, there is a gap in this equivalence with respect to quotient structures. In this paper, we also study the quotient concept of cat^1-groupoids. Finally, we introduce the notions of cat^n-groupoids as a groupoid case of cat^n-groups and then study coverings, actions, normality and quotient concepts in cat^n-groupoids. The results presented in Section 3 originated in the thesis [9].

2. Preliminaries

A groupoid $\mathcal{G} = (G_0, G)$ consists of the class G_0 of objects, the class $G=\bigcup_{x,y\in G_0} G(x,y)$ of morphisms, where $G(x,y)$ is the class of morphisms from x to y as follows: $x \xrightarrow{g} y$ with the source and target maps $d_0, d_1\colon G \to G_0$, respectively, such that $d_0(g) = x, d_1(g) = y$, the associative composition map $G(y,z) \times G(x,y) \to G(x,z),\ (h,g) \mapsto h\circ g$ and the identity morphism map $\varepsilon\colon G_0 \to G,\ x \mapsto 1_x \in G(x)$ (where $G(x)$ is the set of morphisms from x to x) such that $g \circ 1_x = g,\ 1_x \circ g' = g'$, where $d_1(g') = x$ and the inverse mapping $\eta\colon G_1 \to G_1, \eta(g) = g^{-1}$ is such that $g\circ g^{-1} = 1_y,\ g^{-1}\circ g = 1_x$. We write $St_{\mathcal{G}}x$ for $d_0^{-1}(x)$ and call it the star of $\mathcal{G}$ at $x \in G_0$. Briefly a groupoid is a small category in which all morphisms are invertible. For further details, see [3, 15].

A subgroupoid $\mathcal{H}$ of $\mathcal{G}$ is a subcategory $\mathcal{H}$ of $\mathcal{G}$ such that $h \in H$ implies $h^{-1} \in H$. We say $\mathcal{H}$ is wide if $H_0 = G_0$. Let $\mathcal{G}$ be a groupoid and $\mathcal{N}$ be a wide subgroupoid of $\mathcal{G}$. Then $\mathcal{N}$ is called *normal* if $g\circ N(x)\circ g^{-1} \subseteq N(y)$, i.e $g\circ N(x) = N(y)\circ g$, for each $x, y \in G_0$ and $g \in G(x,y)$ [3]. Given a normal subgroupoid $\mathcal{N}$ of $\mathcal{G}$, then $\mathcal{N}$ defines an equivalence relation on the objects of $\mathcal{G}$ by $x \sim x'$, for $x, x' \in G_0$, if and only if there is a morphism n of $\mathcal{N}$ such that $d_0(n) = x,\ d_1(n) = x'$. These equivalence classes are denoted by $[x]$ and the set of equivalence classes by G_0/N. Here $\mathcal{N}$ defines an equivalence relation on morphisms of $\mathcal{G}$ by $g \sim g'$, for $g, g' \in G$ if and only if there are morphisms m, n of $\mathcal{N}$ such that $g = m\circ g'\circ n$. Since $\mathcal{N}$ is a subgroupoid of $\mathcal{G}$, $\sim$ is an equivalence relation on $\mathcal{G}$. These equivalence classes are denoted by $[g]$, for $g \in G$ and the set of equivalence classes by G/N. Then $\mathcal{G}/\mathcal{N} = (G_0/N, G/N)$ is a groupoid, where the structure maps are defined as follows

$$d_0([g]) = [d_0(g)], \qquad d_1([g]) = [d_1(g)], \qquad 1_{[x]} = [1_x], \qquad [g]^{-1} = [g^{-1}]$$

and the composition is defined by $[g_1]\circ[g]=[g_1\circ n\circ g]$, where $d_0(g_1)\sim d_1(g)$, $d_0(n)=d_1(g)$ and $d_1(n)=d_0(g_1)$. For more details see [14, p. 9], [10, p. 86] and [3, p. 420].

Let $\mathcal{G}, \widetilde{\mathcal{G}}$ be two groupoids and $p\colon \widetilde{\mathcal{G}} \to \mathcal{G}$ be a morphism of the groupoids. If for each $\widetilde{x} \in \widetilde{G}_0$ the restriction $St_{\widetilde{\mathcal{G}}}(\widetilde{x}) \to St_{\mathcal{G}}p(\widetilde{x})$ is bijective, then p is called a covering morphism of groupoids and $\widetilde{\mathcal{G}}$ is called a covering groupoid of $\mathcal{G}$ [3].

Let $p\colon \widetilde{\mathcal{G}} \to \mathcal{G}$ and $q\colon \widetilde{\mathcal{G}}' \to \mathcal{G}$ be two covering morphisms of groupoids. A morphism $\widetilde{p}\colon \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}'$ such that $q\widetilde{p} = p$ is called a morphism of coverings of $\mathcal{G}$. Thus, coverings of $\mathcal{G}$ and their morphisms form a category which we denote by $\mathrm{Cov}(\textsc{Gpd})/\mathcal{G}$.

Corollary 2.1 ([3]). *A 1-connected covering groupoid of $\mathcal{G}$ covers every covering groupoid of $\mathcal{G}$. This covering groupoid is called a universal covering groupoid of $\mathcal{G}$.*

Let $\mathcal{G}$ be a groupoid, S a set, and $\omega\colon S \to G_0$ a mapping. An action of $\mathcal{G}$ on S via ω is a mapping $G_{d_0} \times_\omega S \to S$, $(g,s) \mapsto g \centerdot s$ where $G_{d_0} \times_\omega S = \{(g,s) \mid d_0(g) = \omega(s)\}$ satisfies the following conditions:
(AC1) $\omega(g \centerdot s) = d_1(g)$, (AC2) $1_{\omega(s)} \centerdot s = s$,

(AC3) $(h \circ g) \centerdot s = h \centerdot (g \centerdot s)$,
when $h \circ g$ and $g \centerdot s$ are defined. This action of $\mathcal{G}$ on S via ω is denoted by (S, ω). We also say that $\mathcal{G}$ acts on S via ω or S is a $\mathcal{G}$-set [3].

Given such an action, the semi-direct product $(S, S \ltimes G)$ is defined as a groupoid. Morphisms here are pairs (s,g) as $s \xrightarrow{(s,g)} g \centerdot s$, where the composition of morphisms is defined by $(g \centerdot s, h) \circ (s,g) = (s, h \circ g)$. Given such a groupoid, the projection $p\colon S \ltimes G \to G$, $(s,g) \mapsto g$ is a covering morphism, where p on objects is given by $\omega\colon S \to G_0$. For more details see [3, p. 374].

Let $\mathcal{G}$ act on S and S' via ω and ω', respectively. A morphism $f\colon (S,\omega) \to (S',\omega')$ of such actions is a mapping $f\colon S \to S'$ such that $\omega' f = \omega$ and $f(g \centerdot s) = g \centerdot f(s)$ whenever $g \centerdot s$ is defined. Therefore, actions of $\mathcal{G}$ on sets and their morphisms form a category denoted $\mathrm{Act}(\textsc{Gpd})/\mathcal{G}$.

3. Cat^1-groupoids

In this section we recall the notion of cat^1-groupoid from [21]. Then we obtain coverings, actions and quotient concepts of cat^1-groupoids.

Definition 3.1. Let $\mathcal{G} = (G_0, G)$ be a groupoid, $\sigma, \tau\colon \mathcal{G} \to \mathcal{G}$ be groupoid morphisms which are identities on objects. A cat^1-groupoid is a triple $(\mathcal{G}, \sigma, \tau)$ satisfying the following conditions:
(C1Gd1) $\sigma\tau = \tau$ and $\tau\sigma = \sigma$

(C1Gd2) $h \circ k \circ h^{-1} \circ k^{-1} = \varepsilon d_0(h)$, for all $h \in \mathrm{Ker}(\sigma), k \in \mathrm{Ker}(\tau)$ such that $d_0(h) = d_0(k)$.
Here $\mathrm{Ker}(\sigma) = \{g \in G \mid \sigma(g) = \varepsilon d_0(g)\}$ and $\mathrm{Ker}(\tau) = \{g \in G \mid \tau(g) = \varepsilon d_0(g)\}$ are totally disconnected and wide subgroupoids of $\mathcal{G}$ on the base object set G_0.

Example 3.2. Given a cat^1-group (G, s, t) and a set X, we obtain trivial cat^1-groupoid $\mathcal{G} = (X, X \times G \times X)$, where $\sigma(x,g,y) = (x, s(g), y)$ and $\tau(x,g,y) = (x, t(g), y)$.

Proposition 3.3. *Given a cat^1-groupoid $(\mathcal{G}, \sigma, \tau)$, we have*
(i) $\sigma(G) = \tau(G)$, *(ii) σ and τ are identities on $\sigma(G)$ and $\tau(G)$,*

(iii) $\sigma^2=\sigma$ *and* $\tau^2=\tau$.

Definition 3.4. Let $(\mathcal{G},\sigma,\tau)$ and $(\mathcal{G}',\sigma',\tau')$ be two cat^1-groupoids and let $f\colon \mathcal{G}\to\mathcal{G}'$ be a morphism of groupoids such that the following diagram is commutative.

$$\begin{tikzcd} \mathcal{G} \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] \arrow[d, "f"'] & \mathcal{G} \arrow[d, "f"] \\ \mathcal{G}' \arrow[r, shift left, "\sigma'"] \arrow[r, shift right, "\tau'"'] & \mathcal{G}' \end{tikzcd}$$

Then, f is called a morphism of cat^1-groupoids. Therefore, cat^1-groupoids and their morphism form a category which we denote by Cat^1-Gpd.

3.1 Coverings and actions of cat^1-groupoids

In this subsection we introduce the notions of actions and coverings of cat^1-groupoids. Then we prove the natural equivalence between their categories.

Definition 3.5. Let $(\mathcal{G},\sigma,\tau),(\widetilde{\mathcal{G}},\widetilde{\sigma},\widetilde{\tau})$ be two cat^1-groupoids and $p\colon\widetilde{\mathcal{G}}\to\mathcal{G}$ be a morphism of cat^1-groupoids. If for every $\widetilde{x}\in\widetilde{G}_0$ the restriction $St_{\widetilde{\mathcal{G}}}(\widetilde{x})\to St_{\mathcal{G}}p(\widetilde{x})$ is bijective, then p is called a covering morphism of cat^1-groupoids and $\widetilde{\mathcal{G}}$ is called a covering cat^1-groupoid of $\mathcal{G}$.

Note that the underlying groupoid $\widetilde{\mathcal{G}}$ is a covering groupoid of the underlying groupoid $\mathcal{G}$ and thus p is a covering morphism of groupoids.

Remark 3.6. Let $\widetilde{\mathcal{G}}$ be a covering cat^1-groupoid of $\mathcal{G}$. Then, we easily obtain that $p\widetilde{\sigma}=\sigma p$, $p\widetilde{\tau}=\tau p$ from the following diagram

$$\begin{tikzcd} \widetilde{\mathcal{G}} \arrow[r, shift left, "\widetilde{\sigma}"] \arrow[r, shift right, "\widetilde{\tau}"'] \arrow[d, "p"'] & \widetilde{\mathcal{G}} \arrow[d, "p"] \\ \mathcal{G} \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & \mathcal{G} \end{tikzcd}$$

Definition 3.7. Let $(\mathcal{G},\sigma,\tau)$ be a cat^1-groupoid and $p\colon\widetilde{\mathcal{G}}\to\mathcal{G}$, $q\colon\widetilde{\mathcal{G}}'\to\mathcal{G}$ be two covering morphisms of $\mathcal{G}$. A morphism $\widetilde{p}\colon\widetilde{\mathcal{G}}\to\widetilde{\mathcal{G}}'$ such that $q\widetilde{p}=p$ is called a morphism of coverings of $\mathcal{G}$. Thus, the coverings of $\mathcal{G}$ and their morphisms form a category which we denote by Cov(Cat^1-Gpd)$/\mathcal{G}$.

Definition 3.8. Let $(\mathcal{G},\sigma,\tau)$ be a cat^1-groupoid, S a set, and $\omega\colon S\to G_0$ a mapping. An action of $\mathcal{G}$ on S via ω is a mapping $G_{d_0}\times_\omega S\to S$, $(g,s)\mapsto g\centerdot s$, where $G_{d_0}\times_\omega S=\{(g,s)\mid d_0(g)=\omega(s)\}$ satisfies the following conditions:
(AC1G1) $\omega(g\centerdot s)=d_1(g)$, (AC1G2) $1_{\omega(s)}\centerdot s=s$,

(AC1G3) $(h\circ g)\centerdot s=h\centerdot(g\centerdot s)$, (AC1G4) $\sigma(g)\centerdot s=g\centerdot s$, and $\tau(g)\centerdot s=g\centerdot s$
if $h\circ g$ and $g\centerdot s$ are defined. This action of $\mathcal{G}$ on S via ω is denoted by (S,ω). We also say that $\mathcal{G}$ acts on S over ω or S is a cat^1-$\mathcal{G}$-set.

Note that the underlying groupoid $\mathcal{G}$ acts on S via ω. Under such an action, the semi-direct product cat^1-groupoid $((S, G \ltimes S), \widetilde{\sigma}, \widetilde{\tau})$ is defined as a cat^1-groupoid, whose underlying groupoid is the semi-direct product groupoid $(S, G \ltimes S)$, where $\widetilde{\sigma}(g, s) = (\sigma(g), s)$, $\widetilde{\tau}(g, s) = (\tau(g), s)$. So the projection $p : G \ltimes S \to G$ is a covering morphism of cat^1-groupoids.

Corollary 3.9. *A cat^1-groupoid whose underlying groupoid is connected has a universal covering cat^1-groupoid.*

Definition 3.10. Let S and S' be cat^1-$\mathcal{G}$-sets over ω and ω' respectively. A morphism $f\colon (S, \omega) \to (S', \omega')$ of such actions is a mapping $f\colon S \to S'$ such that $\omega' f = \omega$ and $f(g \centerdot s) = g \centerdot f(s)$ whenever $g \centerdot s$ is defined.

Therefore, the actions of a cat^1-groupoid $\mathcal{G}$ on sets and their morphisms form a category which we denote by $\mathrm{Act}(\mathrm{Cat}^1\text{-}\mathrm{Gpd})/\mathcal{G}$.

Theorem 3.11. *Let $(\mathcal{G}, \sigma, \tau)$ be a cat^1-groupoid. Then the category $\mathrm{Cov}(\mathrm{Cat}^1\text{-}\mathrm{Gpd})/\mathcal{G}$ is naturally equivalent to the category $\mathrm{Act}(\mathrm{Cat}^1\text{-}\mathrm{Gpd})/\mathcal{G}$.*

Proof. A functor $\theta\colon \mathrm{Act}(\mathrm{Cat}^1\text{-}\mathrm{Gpd})/\mathcal{G} \to \mathrm{Cov}(\mathrm{Cat}^1\text{-}\mathrm{Gpd})/\mathcal{G}$ is an equivalence of categories. Let (S, ω) be an object of $\mathrm{Act}(\mathrm{Cat}^1\text{-}\mathrm{Gpd})/\mathcal{G}$. Then $\theta(S, \omega) = (\widetilde{\mathcal{G}}, \widetilde{\sigma}, \widetilde{\tau})$ is a covering cat^1-groupoid of $\mathcal{G}$, where $\widetilde{G}_0 = S$, $\widetilde{G} = G \ltimes S = \{(g, s) \mid d_0(g) = \omega(s)\}$, $\widetilde{\sigma}(g, s) = (\sigma(g), s)$, $\widetilde{\tau}(g, s) = (\tau(g), s)$. Here the source and target maps are defined by $d_0(g, s) = s$, $d_1(g, s) = g \centerdot s$, respectively,and the composition of the morphisms is given by $(g_1, s_1) \circ (g, s) = (g_1 \circ g, s)$ with $s_1 = g \centerdot s$. The identity map is defined by $\varepsilon(s) = (1_{\omega(s)}, s)$, where the inverse of (g, s) is defined by $(g, s)^{-1} = (g^{-1}, g \centerdot s)$. The covering morphism $p = (p_0, p)$ is defined such that $p_0(s) = \omega(s)$ and $p(g, s) = g$. Since $\sigma(g) \centerdot s = g \centerdot s$ and $\tau(g) \centerdot s = g \centerdot s$ are from (AC1G4), $\widetilde{\sigma}$ and $\widetilde{\tau}$ are identities on objects. Thus, we can prove that the conditions (C1Gd1) and (C1Gd2) are satisfied.

(C1Gd1) Since $\widetilde{\sigma}\widetilde{\tau}(g, s) = \widetilde{\sigma}(\tau(g), s) = (\sigma\tau(g), s) = (\tau\sigma(g), s) = \widetilde{\tau}(\sigma(g), s) = \widetilde{\tau}\widetilde{\sigma}(g, s)$, then $\widetilde{\sigma}\widetilde{\tau} = \widetilde{\tau}$. Similarly $\widetilde{\tau}\widetilde{\sigma} = \widetilde{\sigma}$.

(C1Gd2) Since $\operatorname{Ker} \widetilde{\sigma} = \{(g, s) \mid \widetilde{\sigma}(g, s) = \varepsilon d_0(g, s)\} = \{(g, s) \mid \sigma(g) = 1_{\omega(s)}\}$ and $\operatorname{Ker} \widetilde{\tau} = \{(h, s') \mid \widetilde{\tau}(h, s') = \varepsilon d_0(h, s')\} = \{(h, s') \mid \tau(g) = 1_{\omega(s')}\}$, we get $g \centerdot s = \sigma(g) \centerdot s = 1_{\omega(s)} \centerdot s = s$, $h \centerdot s' = \sigma(h) \centerdot s' = 1_{\omega(s')} \centerdot s' = s'$ and thus $(g, s)^{-1} = (g^{-1}, s), (h, s')^{-1} = (h^{-1}, s')$. Let $s = s'$. Then $d_0(g, s) = d_0(h, s')$. So $(g, s) \circ (h, s) \circ (g, s)^{-1} \circ (h, s)^{-1} = (g \circ h \circ g^{-1} \circ h^{-1}, s) = (1_{\omega(s)}, s)$.

Let $f\colon (S, \omega) \to (S', \omega')$ be a morphism of $\mathrm{Act}(\mathrm{Cat}^1\text{-}\mathrm{Gpd})/\mathcal{G}$. Then $\theta(f) = (f, 1_G \times f)$ is a morphism of $\mathrm{Cov}(\mathrm{Cat}^1\text{-}\mathrm{Gpd})/\mathcal{G}$.

Let $p = (p_0, p)\colon (\widetilde{\mathcal{G}}, \widetilde{\sigma}, \widetilde{\tau}) \to (\mathcal{G}, \sigma, \tau)$ be a covering morphism of cat^1-groupoids. We define a functor $\psi\colon \mathrm{Cov}(\mathrm{Cat}^1\text{-}\mathrm{Gpd})/\mathcal{G} \to \mathrm{Act}(\mathrm{Cat}^1\text{-}\mathrm{Gpd})/\mathcal{G}$ as a weak inverse of θ, such that $\psi(\widetilde{\mathcal{G}}, p) = (\widetilde{G}_0, p_0)$ is an object of $\mathrm{Act}(\mathrm{Cat}^1\text{-}\mathrm{Gpd})/\mathcal{G}$. An action of $\mathcal{G}$ on $(\widetilde{G}_0, p_0)$ is given by $g \centerdot \widetilde{x} = d_1(\widetilde{g})$ where $x \xrightarrow{g} y$, $\widetilde{x} \xrightarrow{\widetilde{g}} \widetilde{y}$ and $p(\widetilde{g}) = g$.

(AC1G1) $p_0(g \centerdot \widetilde{x}) = p_0 d_1(\widetilde{g}) = p_0(\widetilde{y}) = y = d_1(g)$.

(AC1G2) $1_{p_0(\widetilde{x})} \centerdot \widetilde{x} = 1_x \centerdot \widetilde{x} = d_1(\widetilde{1_x}) = \widetilde{x}$.

(AC1G3) $(h \circ g) \centerdot \widetilde{x} = d_1(\widetilde{h} \circ \widetilde{g}) = d_1(\widetilde{h}) = h \centerdot (g \centerdot \widetilde{x})$.

(AC1G4) Since $\widetilde{\sigma}$ is identical on objects, we can write $d_1(\widetilde{g}) = d_1\widetilde{\sigma}(\widetilde{g})$. Since $p\widetilde{\sigma} = \sigma p$, we can write $p\widetilde{\sigma}(\widetilde{g}) = \sigma p(\widetilde{g}) = \sigma(g)$. Therefore, we obtain $g \centerdot \widetilde{x} = d_1(\widetilde{g}) = d_1\widetilde{\sigma}(\widetilde{g}) = \sigma(g) \centerdot \widetilde{x}$.

Let $\widetilde{p} = (\widetilde{p}_0, \widetilde{p})\colon \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}'$ be a morphism of Cov(CAT^1-GPD)$/\mathcal{G}$. Then $\psi(\widetilde{p}) = \widetilde{p}_0\colon \widetilde{G}_0 \to \widetilde{G}'_0$ is a morphism of Act(CAT^1-GPD)$/\mathcal{G}$.

It is easy to verify that $\psi\theta \cong 1$. To show $1 \cong \theta\psi$, define a natural equivalence $\xi\colon 1_{\text{cat}^1\text{-GPD}}/\mathcal{G} \to \theta\psi$ via a map $\xi_{\widetilde{G}}$ such that it is identical on objects and $\xi_{\widetilde{G}}(\widetilde{g}) = (p(\widetilde{g}), d_0(\widetilde{g}))$ for $g \in G$. Since p is bijective, $\xi_{\widetilde{G}}^{-1}$ can be defined and $\xi_{\widetilde{G}}$ preserves the composition: $\xi_{\widetilde{G}}(\widetilde{h} \circ \widetilde{g}) = (p(\widetilde{h} \circ \widetilde{g}), d_0(\widetilde{h} \circ \widetilde{g})) = (p(\widetilde{h}) \circ p(\widetilde{g}), d_0(\widetilde{g})) = (p(\widetilde{h}), d_0(\widetilde{h})) \circ (p(\widetilde{g}), d_0(\widetilde{g})) = \xi_{\widetilde{G}}(\widetilde{h}) \circ \xi_{\widetilde{G}}(\widetilde{g})$, for $\widetilde{g}, \widetilde{h} \in \widetilde{G}$. □

3.2 Quotient cat^1-groupoids

In this subsection we recall the notions of subcat^1-groupoids and normal cat^1-groupoids from [21]. We then obtain the quotient concept of cat^1-groupoids.

DEFINITION 3.12. A subcat^1-groupoid $(\mathcal{G}', \sigma', \tau')$ of a cat^1-groupoid $(\mathcal{G}, \sigma, \tau)$ is a subgroupoid $\mathcal{G}' = (G'_0, G')$ of $\mathcal{G} = (G_0, G)$ such that σ', τ' are respectively restrictions of σ, τ on $\mathcal{G}'$. We say $\mathcal{G}'$ is wide if $G'_0 = G_0$. If $\mathcal{G}'$ is a normal subgroupoid of $\mathcal{G}$, then $(\mathcal{G}', \sigma', \tau')$ is called normal subcat^1-groupoid of $(\mathcal{G}, \sigma, \tau)$.

THEOREM 3.13 ([21]). *Given a cat^1-groupoid $(\mathcal{G}, \sigma, \tau)$ and a crossed module $(\mathcal{A}, \mathcal{B}, \partial)$ over groupoids corresponding to $\mathcal{G}$, the category* NC1GD$/(\mathcal{G}, \sigma, \tau)$ *of the normal subcat^1-groupoids of $(\mathcal{G}, \sigma, \tau)$ is equivalent to the category* NCMG$/(\mathcal{A}, \mathcal{B}, \partial)$ *of the normal subcrossed modules of $(\mathcal{A}, \mathcal{B}, \partial)$.*

We now construct the quotient concept of cat^1-groupoids in the following theorem.

THEOREM 3.14. *Let $(\mathcal{G}, \sigma, \tau)$ be a cat^1-groupoid and $(\mathcal{N}, \sigma, \tau)$ be a normal subcat^1-groupoid of $(\mathcal{G}, \sigma, \tau)$. Then the quotient groupoid $\mathcal{G}/\mathcal{N}$ is a cat^1-groupoid with the following functors $\overline{\sigma}([g]) = [\sigma(g)]$, $\overline{\tau}([g]) = [\tau(g)]$, where $[g]$ is an equivalence class in $\mathcal{G}/\mathcal{N}$. This cat^1-groupoid $\big(\mathcal{G}/\mathcal{N}, \overline{\sigma}, \overline{\tau}\big)$ is called quotient cat^1-groupoid.*

Proof. Since σ and τ are identities on objects, $\overline{\sigma}(N(x)) = N(x)$ and $\overline{\tau}(N(x)) = N(x)$, for any $x \in G_0$. Then we can prove that $\overline{\sigma}$ and $\overline{\tau}$ are functors as follows:

$$\begin{aligned}\overline{\sigma}([g_1] \circ [g]) =& \overline{\sigma}([g_1 \circ n \circ g]) = [\sigma(g_1) \circ \sigma(n) \circ \sigma(g)] = [\sigma(g_1)] \circ [\sigma(g)] = \overline{\sigma}([g_1]) \circ \overline{\sigma}([g]),\\ \overline{\sigma}([1_x]) =& [\sigma(1_x)] = [1_{\sigma(x)}] = [1_x],\end{aligned}$$

and similarly, $\overline{\tau}([g_1] \circ [g]) = \overline{\tau}([g_1]) \circ \overline{\tau}([g])$, $\overline{\tau}([1_x]) = [1_x]$, where $s(g_1) \sim t(g)$ and $s(n) = t(g), t(n) = s(g_1)$.

(C1Gd1) Since $\overline{\sigma}\ \overline{\tau}([g]) = \overline{\sigma}([\tau(g)]) = [\sigma\tau(g)] = [\tau(g)] = \overline{\tau}([g])$, then $\overline{\sigma}\ \overline{\tau} = \overline{\tau}$. Similarly $\overline{\tau}\ \overline{\sigma} = \overline{\sigma}$.

(C1Gd2) Since $\operatorname{Ker}\overline{\sigma} = \{[g] \mid g \in \operatorname{Ker}\sigma\}$ and $\operatorname{Ker}\overline{\tau} = \{[g] \mid g \in \operatorname{Ker}\tau\}$, we get $[g_1] \circ [g] \circ [g_1]^{-1} \circ [g]^{-1} = [g_1 \circ n \circ g] \circ [g_1^{-1} \circ n_1 \circ g^{-1}] = [g_1 \circ n \circ g \circ n_2 \circ g_1^{-1} \circ n_1 \circ g^{-1}]$ for $g_1 \in \operatorname{Ker}\sigma$, $g \in \operatorname{Ker}\tau$, where $n, n_1, n_2 \in N(x)$ and $d_0(g_1) = d_0(g) = x$. Since $\mathcal{N}$ is normal, then $[g_1] \circ [g] \circ [g_1]^{-1} \circ [g]^{-1} = [g_1 \circ n \circ g \circ n_2 \circ g_1^{-1} \circ n_1 \circ g^{-1}] = [g_1 \circ g \circ g_1^{-1} \circ g^{-1} \circ n \circ n_2 \circ n_1] = [1_x \circ n \circ n_2 \circ n_1] = [1_x]$. □

3.3 Cat^n-groupoids

In this subsection we define cat^n-groupoids by extending the definition of cat^n-groups to the notion of groupoids. Once the notions of cat^n-groups and cat^1-groupoids are known, it is easy to obtain coverings, actions, normality and quotient concepts for cat^n-groupoids.

Definition 3.15. Let $\mathcal{G} = (G_0, G)$ be a groupoid, $\sigma_i, \tau_i \colon \mathcal{G} \to \mathcal{G}$ be 2n functors which are identities on objects. A cat^n-groupoid $(\mathcal{G}, \sigma_i, \tau_i)$ is a groupoid satisfying the following conditions for $i, j \in \{1, 2, \dots, n\}$, $i \neq j$.

(CnGd1) $\sigma_i \tau_i = \tau_i$, $\tau_i \sigma_i = \sigma_i$,

(CnGd2) $\sigma_i \sigma_j = \sigma_j \sigma_i$, $\tau_i \tau_j = \tau_j \tau_i$, $\sigma_i \tau_j = \tau_j \sigma_i$,

(CnGd3) $h_i \circ k_i \circ h_i^{-1} \circ k_i^{-1} = \varepsilon d_0(h_i)$, for all $h_i \in \mathrm{Ker}(\sigma_i), k_i \in \mathrm{Ker}(\tau_i)$ such that $d_0(h_i) = d_0(k_i)$.

Each cat^n-group can be viewed as a cat^n-groupoid with a unique object. Another example is obtained by using a trivial groupoid as in Example 3.2.

Theorem 3.16. *The category* $\mathrm{Cov}(\mathrm{C{\small AT}}^n\text{-}\mathrm{G{\small PD}})/\mathcal{G}$ *of coverings of* $\mathcal{G}$ *is naturally equivalent to the category* $\mathrm{Act}(\mathrm{C{\small AT}}^n\text{-}\mathrm{G{\small PD}})/\mathcal{G}$ *of actions of* $\mathcal{G}$*, where* $(\mathcal{G}, \sigma_i, \tau_i)$ *is a cat*^{n}*-groupoid.*

Proof. The idea of the proof is to show that the functors of Theorem 3.11 extend to an equivalence of categories. Therefore, we define the functor $\theta\colon \mathrm{Act}(\mathrm{C{\small AT}}^n\text{-}\mathrm{G{\small PD}})/\mathcal{G} \to \mathrm{Cov}(\mathrm{C{\small AT}}^n\text{-}\mathrm{G{\small PD}})/\mathcal{G}$ such that, given an object (S, ω) of $\mathrm{Act}(\mathrm{C{\small AT}}^n\text{-}\mathrm{G{\small PD}})/\mathcal{G}$, $\theta(S, \omega) = (\widetilde{\mathcal{G}}, \widetilde{\sigma}, \widetilde{\tau})$ is a covering cat^n-groupoid of $\mathcal{G}$ via the process of proving Theorem 3.11, where $\widetilde{\sigma_i}(g, s) = (\sigma_i(g), s)$, $\widetilde{\tau_i}(g, s) = (\tau_i(g), s)$. We only check if the condition (CnGd2) is satisfied. Since

$$\begin{aligned}
\widetilde{\sigma_i}\widetilde{\sigma_j}(g, s) &= (\sigma_i\sigma_j(g), s) = (\sigma_j\sigma_i(g), s) = \widetilde{\sigma_j}\widetilde{\sigma_i}(g, s),\\
\widetilde{\tau_i}\widetilde{\tau_j}(g, s) &= (\tau_i\tau_j(g), s) = (\tau_j\tau_i(g), s) = \widetilde{\tau_j}\widetilde{\tau_i}(g, s),\\
\widetilde{\sigma_i}\widetilde{\tau_j}(g, s) &= (\sigma_i\tau_j(g), s) = (\tau_j\sigma_i(g), s) = \widetilde{\tau_j}\widetilde{\sigma_i}(g, s),
\end{aligned}$$

we have $\widetilde{\sigma_i}\widetilde{\sigma_j} = \widetilde{\sigma_j}\widetilde{\sigma_i}$, $\widetilde{\tau_i}\widetilde{\tau_j} = \widetilde{\tau_j}\widetilde{\tau_i}$ and $\widetilde{\sigma_i}\widetilde{\tau_j} = \widetilde{\tau_j}\widetilde{\sigma_i}$. Other details are straightforward and therefore omitted. □

Theorem 3.17. *Let* $(\mathcal{G}, \sigma_i, \tau_i)$ *be a cat*^{n}*-groupoid and* $(\mathcal{N}, \sigma, \tau)$ *be a normal subcat*^{1}*-groupoid of* $(\mathcal{G}, \sigma_i, \tau_i)$*. Then the quotient groupoid* $\mathcal{G}/\mathcal{N}$ *is a cat*^{n}*-groupoid with the following functors* $\overline{\sigma_i}([g]) = [\sigma_i(g)]$*,* $\overline{\tau_i}([g]) = [\tau_i(g)]$*, where* $[g]$ *is an equivalence class in* $\mathcal{G}/\mathcal{N}$*. This cat*^{n}*-groupoid* $\big(\mathcal{G}/\mathcal{N}, \overline{\sigma_i}, \overline{\tau_i}\big)$ *is called quotient cat*^{n}*-groupoid.*

Proof. Following the proof of the Theorem 3.14, we show only the condition (CnGd2). Since $\overline{\sigma_i}\ \overline{\sigma_j}([g]) = \overline{\sigma_i}([\sigma_j(g)]) = [\sigma_i\sigma_j(g)] = [\sigma_j\sigma_i(g)] = \overline{\sigma_j}([\sigma_i(g)]) = \overline{\sigma_j}\ \overline{\sigma_i}([g])$, then $\overline{\sigma_i}\ \overline{\sigma_j} = \overline{\sigma_j}\ \overline{\sigma_i}$. Since $\overline{\tau_i}\ \overline{\tau_j}([g]) = \overline{\tau_i}([\tau_j(g)]) = [\tau_i\tau_j(g)] = [\tau_j\tau_i(g)] = \overline{\tau_j}([\tau_i(g)]) = \overline{\tau_j}\ \overline{\tau_i}([g])$, then $\overline{\tau_i}\ \overline{\tau_j} = \overline{\tau_j}\ \overline{\tau_i}$. Since $\overline{\sigma_i}\ \overline{\tau_j}([g]) = \overline{\sigma_i}([\tau_j(g)]) = [\sigma_i\tau_j(g)] = [\tau_j\sigma_i(g)] = \overline{\tau_j}([\sigma_i(g)]) = \overline{\tau_j}\ \overline{\sigma_i}([g])$, then $\overline{\sigma_i}\ \overline{\tau_j} = \overline{\tau_j}\ \overline{\sigma_i}$. □

Acknowledgement. We would like to thank the referee for useful remarks that helped us improve the paper; and the editors for the editorial process.

References

[1] H.F. Akız, N. Alemdar, O. Mucuk, T. Şahan, *Coverings of internal groupoids and crossed modules in the category of groups with operations*, Georgian Math. J., **20(2)** (2013), 223–238.

[2] J.C. Baez, A.D. Lauda, *Higher dimensional algebra V: 2-groups*, Theory Appl. Categ., **12** (2004), 423-–491.

[3] R. Brown, *Topology and Groupoids*, BookSurge LLC, North Carolina 2006.

[4] R. Brown, O. Mucuk, *Covering groups of non-connected topological groups revisited*, Math. Proc. Camb. Phil. Soc., **115** (1994), 97–110.

[5] R. Brown, C.B. Spencer, *$\mathcal{G}$-groupoids, crossed modules and the fundamental groupoid of a topological group*, Proc. Konn. Ned. Akad., **79** (1976), 296–302.

[6] R. Brown, P. J. Higgins, R. Sivera, *Nonabelian algebraic topology: Filtered spaces, crossed complexes, cubical homotopy groupoids*, EMS Tracts Math., **15** (2011).

[7] R. Brown, I. Icen, *Homotopies and automorphisms of crossed module over groupoids*, Appl. Categ. Structures, **11** (2003), 185–206.

[8] R. Brown, G. Danesh-Naruie, J.P.L. Hardy, *Topological groupoids: II. Covering morphisms and G-spaces*, Math. Nachr, **74** (1976), 143–156.

[9] O. Çan, *Grupoidler Üzerindeki Çaprazlanmış Modüller ve Cat^1-grupoidler* (In Turkish), M.Sc. Theses, Recep Tayyip Erdogan University, (2021).

[10] P.J. Higgins, *Categories and groupoids*, Reprints in Theory and Appl. Categ., **7** (2005), 1–195.

[11] İ. İçen, *The equivalence of 2-groupoids and crossed modules*, Commun. Fac. Sci. Univ. Ank. Series A1, **49** (2000), 39-48.

[12] K. H. Kamps, T. Porter, *A homotopy2-groupoid from a fibration*, Homology, HomotopyAppl., **1(2)** (1999), 79–93.

[13] J.-L. Loday, *Spaces with finitely many non-trivial homotopy groups*, J.Pure Appl. Algebra, **24(2)** (1982), 179–202.

[14] K. Mackenzie, *Lie groupoids and Lie algebroids in Differential Geometry*, Cambridge University Press, 1987.

[15] S. Maclane, *Categories for the Working Mathematician, Graduate Text in Mathematics*, Volume 5. Springer-Verlag, New York, 1971.

[16] O. Mucuk, T. Şahan, *Covering groupoids of categorical groups*, Hacet. J. Math. Stat., **42(4)** (2013), 419–430.

[17] O. Mucuk, S. Demir, *Covering groupoids of categorical rings*, Filomat, **29(1)** (2015), 39–49.

[18] O. Mucuk, S. Demir, *Normality and quotient in crossed modules over groupoids and double groupoids*, Turk. J. Math., **42** (2018), 2336—2347.

[19] T. Şahan,A. Erciyes, *Actions of internal groupoids in the category of Leibniz Algebras*, Commun. Fac. Sci. Univ. Ank. Series A1, **68(1)** (2019), 619–632.

[20] S. Temel, *Normality and quotient in crossed modules over groupoids and 2-groupoids*, Korean J. Math., **27(1)** (2019), 151–163.

[21] S. Temel, *Crossed squares, crossed modules over groupoids and cat$^{1-2}$-groupoids*, Categ. Gen. Algebr. Struct. Appl., **13(1)** (2020), 125–142.

(received 29.09.2022; in revised form 04.01.2023; available online 16.06.2023)

Department of Mathematics, Recep Tayyip Erdogan University, Rize 53100, Turkey

E-mail: sedat.temel@erdogan.edu.tr

Department of Mathematics, Recep Tayyip Erdogan University, Rize 53100, Turkey

E-mail: osman_can19@erdogan.edu.tr