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PROPERTIES OF f-RECTIFYING CURVES IN GALILEAN 3-SPACE

Joydeep Sengupta, Zafar Iqbal and Sarani Chakraborty

Abstract. The purpose of this paper is to introduce a new class of admissible curves, referred to as f-rectifying curves, and study their geometric properties in Galilean 3-space $\mathbb{G}_3$. For some non-vanishing real-valued smooth function f, an f-rectifying curve in $\mathbb{G}_3$ is introduced as an admissible curve γ of class at least C^4 such that its f-position vector field, given by $\gamma_f = \int f d\gamma$, lies on its rectifying planes (i.e., the planes generated by its tangent and binormal vectors). Some geometric characterizations of such curves are explored in $\mathbb{G}_3$. Moreover, they are investigated in the equiform geometry of $\mathbb{G}_3$.

1. Introduction

Let $\mathbb{E}^3$ be the 3D Euclidean space (i.e. the real vector space $\mathbb{R}^3$ endowed with the *standard inner product* $\langle \cdot , \cdot \rangle$). Let $\gamma : I \to \mathbb{E}^3$ be a curve of class at least C^4 parameterized by *arc length* s (and thus unit-speed). As usual, here I is a non-trivial interval in $\mathbb{R}$, i.e. a connected subset of $\mathbb{R}$ that contains at least two points. Let us consider the *Frenet-Serret apparatus* $\{T_\gamma, N_\gamma, B_\gamma, \kappa_\gamma, \tau_\gamma\}$ for γ, defined as: $T_\gamma = \gamma'$ is the *tangent field* along γ; N_γ is the *principal normal field* along γ, derived by normalizing the acceleration field T'_γ; $B_\gamma = T_\gamma \times N_\gamma$ is the *binormal field* along γ and it is the unique vector field along γ that is orthogonal to both T_γ and N_γ, so that the *dynamic Frenet frame* $\{T_\gamma, N_\gamma, B_\gamma\}$ is positive definite along the curve γ with the right-handed standard orientation of $\mathbb{E}^3$; κ_γ is the *curvature* and τ_γ is the *torsion* of γ [2]. At every point $\gamma(s)$ on γ, the planes generated by $\{T_\gamma(s), B_\gamma(s)\}$, $\{T_\gamma(s), N_\gamma(s)\}$ and $\{N_\gamma(s), B_\gamma(s)\}$ are referred to as the *rectifying plane*, *osculating plane* and *normal plane* of γ respectively. From elementary *Differential Geometry* we know that a space curve γ lies in a *plane* in $\mathbb{E}^3$ iff its position vector field always lies in its osculating planes, and it lies on a *sphere* in $\mathbb{E}^3$ iff its position vector field always lies in its normal planes (cf. [2]). From this point of view, it is natural to ask the geometric question: *Does there exist a space curve whose position vector field always*

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remains in its rectifying planes? The existence of such space curves was established by B.Y. Chen in his paper [3] and they were called *rectifying curves*. For a rectifying curve $\gamma : I \to \mathbb{E}^3$ parameterized by the arc length s, its position vector field satisfies

$$\gamma(s) = \lambda(s)T_\gamma(s) + \mu(s)B_\gamma(s)$$

for smooth functions $\lambda, \mu : I \to \mathbb{R}$. In [3], B.Y. Chen studied some characterizations of rectifying curves in $\mathbb{E}^3$ in terms of distance functions as well as tangential, normal and binormal components of the position vector field and also in terms of the ratios of their curvature and torsion. He also endeavored to classify such curves in $\mathbb{E}^3$ on the basis of a kind of dilation applied to curves with unit-speed curves lying on $\mathbb{S}^2(1)$, the unit sphere in $\mathbb{E}^3$.

In [5], B.Y. Chen and F. Dillen observed that rectifying curves can be viewed as *centrodes* and *extremal curves* in $\mathbb{E}^3$. They also found a relation between rectifying curves and centrodes, which plays an important role in the definition of curves with constant procession in *Differential Geometry* and *Kinematics* (in general, *Mechanics*). After that, many characterizations of rectifying curves in $\mathbb{E}^3$ were developed in [4, 7]. In the meantime, the study of rectifying curves has been extended to several ambient spaces; we mention the Galilean 3-space $\mathbb{G}_3$ [6,15,18]. Moreover, a new type of curves in $\mathbb{E}^3$ was studied, which generalizes rectifying curves and helices [8].

In the previous works, we have given some characterizations of null and non-null f-rectifying curves in *Minkowski 3-space* $\mathbb{E}^3_1$ [9, 10], *Minkowski spacetime* $\mathbb{E}^4_1$ [11], Euclidean 4-space $\mathbb{E}^4$ [12] and Euclidean n-space $\mathbb{E}^n$ [13]. In the present paper we extend the study to the Galilean 3-space $\mathbb{G}_3$. We organize the paper as follows:

- In Section 2 we discuss some fundamental notions of Galilean 3-space $\mathbb{G}_3$ and the Frenet system for curves in it.
- In Section 3 the notion of f-rectifying curves in $\mathbb{G}_3$ is introduced.
- In Section 4 we study some simple geometric characterizations of f-rectifying curves in $\mathbb{G}_3$. We also consider how they generalize general helices and rectifying curves in $\mathbb{G}_3$ with respect to their conical curvature.
- In Section 5 we investigate some characterizations of f-rectifying curves in the equiform geometry of $\mathbb{G}_3$. First and foremost, we establish a relation between their equiform curvature and equiform torsion.

2. Galilean 3-space and Frenet system

The *Galilean 3-space*, denoted by $\mathbb{G}_3$, is a three-dimensional space modelled by a *real Cayley-Klein space* endowed with a metric with *projective signature* $(0, 0, +, +)$ [14]. It can be described in the three-dimensional *real projective space* $\mathbb{RP}^3$ and its *absolute figure* is defined as a triplet $(\Pi, L, \mathfrak{I})$, consisting of a plane Π (referred to as *ideal plane* or *absolute plane*), a line L (referred to as *absolute line*) in Π and the (fixed) *elliptic involution* $\mathfrak{I}$ of points on L [17]. In $\mathbb{G}_3$ one can introduce homogeneous coordinates so that the absolute plane Π is obtained by the equation

$x_0 = 0$, the absolute line L by $x_0 = 0 = x_1$ and finally the elliptic involution $\mathfrak{I}$ by $(0, 0, x_2, x_3) \mapsto (0, 0, x_3, -x_2)$. In fact, the projective transformations for which the absolute form of the eight-parameter *similarity group* $\boldsymbol{H_8}$ of $\mathbb{G}_3$ remains invariant can be written in terms of non-homogeneous affine coordinates as follows (cf. [16, 17]):

$$\begin{cases} \bar{x} = a_1 + a_2 x, \\ \bar{y} = b_1 + b_2 x + \xi(y \cos\varphi + z \sin\varphi), \\ \bar{z} = c_1 + c_2 x - \xi(y \sin\varphi - z \cos\varphi), \end{cases} \tag{1}$$

where $a_1, a_2, b_1, b_2, c_1, c_2, \xi, \varphi \in \mathbb{R}$. As a special case, if $a_2 = \xi = 1$, we find the following transformations (called the *Galilean transformations*), which describe the *Galilean motions*:

$$\begin{cases} \bar{x} = a_1 + x, \\ \bar{y} = b_1 + b_2 x + y \cos\varphi + z \sin\varphi, \\ \bar{z} = c_1 + c_2 x - y \sin\varphi + z \cos\varphi. \end{cases} \tag{2}$$

In this case, we obtain the six-parameter group $\boldsymbol{B_6} \subset \boldsymbol{H_8}$ of the isometries of $\mathbb{G}_3$, which is called the *group of Galilean motions* of $\mathbb{G}_3$ [16, 17].

In $\mathbb{G}_3$ there are the following two categories of planes:

1. *Euclidean planes*: the planes that contain the absolute line L;

2. *Isotropic planes*: the planes that do not contain the absolute line L.

Obviously, the planes obtained by $x =$ constant are Euclidean and all others are isotropic. In particular, the absolute plane Π is Euclidean. On the other hand, the lines in $\mathbb{G}_3$ belong to the following four categories:

1. *Absolute line* L;

2. *Proper isotropic lines*: the lines that are not contained in the absolute plane Π but intersect the absolute line L;

3. *Proper non-isotropic lines*: the lines that do not intersect the absolute line L;

4. *Improper non-isotropic lines*: the lines contained in the absolute plane Π, except for the absolute line L.

In addition, the vectors in $\mathbb{G}_3$ belong to the following two categories:

1. *Isotropic vectors*: the vectors whose first component vanishes identically, i.e. vectors of the form $(0, v_2, v_3)$;

2. *Non-isotropic vectors*: the vectors whose first component does not vanish.

The metric on $\mathbb{G}_3$ induces a scalar product $\langle \cdot\,, \cdot \rangle_{\mathbb{G}_3} : \mathbb{G}_3 \times \mathbb{G}_3 \to \mathbb{R}$ (referred to as the *Galilean scalar product*), defined by

$$\langle v, w \rangle_{\mathbb{G}_3} := \begin{cases} v_1 w_1 & \text{if both } v \text{ and } w \text{ are non-isotropic}, \\ 0 & \text{if either } v \text{ or } w \text{ is non-isotropic}, \\ v_2 w_2 + v_3 w_3 & \text{if both } v \text{ and } w \text{ are isotropic} \end{cases}$$

for all vectors $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3)$ in $\mathbb{G}_3$. It is trivial to mention that two vectors in $\mathbb{G}_3$ are *orthogonal* if and only if their Galilean scalar product vanishes. In $\mathbb{G}_3$, *two non-isotropic vectors cannot be orthogonal, while a non-isotropic vector* *and an isotropic vector are always orthogonal.* Now the Galilean scalar product on $\mathbb{G}_3$ induces a norm $\|\cdot\|_{\mathbb{G}_3} : \mathbb{G}_3 \to \mathbb{R}$ (known as the *Galilean norm*), defined by

$$\|v\|_{\mathbb{G}_3} := \sqrt{\langle v, v\rangle_{\mathbb{G}_3}} = \begin{cases} |v_1| & \text{if } v \text{ is non-isotropic,} \\ \sqrt{v_2^2 + v_3^2} & \text{if } v \text{ is isotropic} \end{cases}$$

for all vectors $v = (v_1, v_2, v_3)$ in $\mathbb{G}_3$. As usual, a vector in $\mathbb{G}_3$ is *unit* if and only if its Galilean norm is one. It is therefore clear that vectors of the form $(1, v_2, v_3)$ are non-isotropic unit vectors in $\mathbb{G}_3$. Moreover, for any two vectors $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3)$ in $\mathbb{G}_3$ their *Galilean cross product*, denoted by $v \times_{\mathbb{G}_3} w$, is defined as follows [1]:

$$v \times_{\mathbb{G}_3} w = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} & \text{if } v \text{ or } w \text{ is non-isotropic,} \\ \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} & \text{if both } v \text{ and } w \text{ are isotropic,} \end{cases}$$

where $e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$ for each $i = 1, 2, 3$; $\delta_{ij} = 1$ iff $i = j$. It is evident that $v \times_{\mathbb{G}_3} v = 0$ and $v \times_{\mathbb{G}_3} w$ is orthogonal to both v and w.

The Galilean analogues of the regular curves in Euclidean space are now *admissible curves* [16]. In $\mathbb{G}_3$ an admissible curve is a curve $\gamma : I \to \mathbb{G}_3$ of class at least C^3 with the coordinate expression $\gamma(t) = (x(t), y(t), z(t))$, $t \in I$, such that

1. it has no point of inflection, i.e. $\gamma'(t) \times_{\mathbb{G}_3} \gamma''(t) \neq 0$ for all $t \in I$;

2. all its velocities $\gamma'(t)$ are non-isotropic, i.e. $x'(t) \neq 0$ for all $t \in I$.

Let $\gamma : I \to \mathbb{G}_3$ be an admissible curve of class at least C^4 with arc length parameter s (i.e. γ has the coordinate expression $\gamma(s) = (x(s), y(s), z(s))$, $s \in I$). Then the (non-isotropic) velocity vectors $\gamma'(s)$ are normalized by s, which implies $s = x$ and is an invariant of the group $\boldsymbol{B_6}$. Thus γ takes the coordinate expression $\gamma(s) = (s, y(s), z(s))$, $s \in I$, and is called unit-speed in $\mathbb{G}_3$. With T_γ we now denote the unit *tangent* γ' of γ. Then we have

$$T_\gamma(s) = \gamma'(s) = (1, y'(s), z'(s)), \quad T'_\gamma(s) = \gamma''(s) = (0, y''(s), z''(s)).$$

It is obvious that the *acceleration* $T'_\gamma = \gamma''$ is isotropic. Furthermore, we find $\langle T_\gamma(s), T'_\gamma(s)\rangle_{\mathbb{G}_3} = 0$, which means that T'_γ is orthogonal to T_γ, i.e. orthogonal to γ. We define the function $\kappa_\gamma : I \to (0, \infty)$, called *curvature* of γ, by

$$\kappa_\gamma(s) := \|T'_\gamma(s)\|_{\mathbb{G}_3} = \sqrt{[y''(s)]^2 + [z''(s)]^2}.$$

We normalize T'_γ to define the unit *principal normal* of γ, denoted by N_γ, as follows:

$$N_\gamma(s) := \frac{1}{\kappa_\gamma(s)} T'_\gamma(s) = \frac{1}{\kappa_\gamma(s)} (0, y''(s), z''(s)).$$

We also define the unit *binormal* of γ, which is denoted by B_γ, as:

$$B_\gamma(s) := T_\gamma(s) \times_{\mathbb{G}_3} N_\gamma(s) = \frac{1}{\kappa_\gamma(s)}\,(0, -z''(s), y''(s))\,.$$

Thus, B_γ is the unique vector field along γ that is orthogonal to both T_γ and N_γ, so that $\{T_\gamma, N_\gamma, B_\gamma\}$ forms the *dynamic Frenet frame* along γ. Note that T_γ is non-isotropic, while both N_γ and B_γ are isotropic vector fields along γ. Moreover, when studying the motion of the dynamic Frenet frame $\{T_\gamma, N_\gamma, B_\gamma\}$ along γ, we need to consider another function $\tau_\gamma : I \to \mathbb{R}$, called *torsion* of γ, defined as follows:

$$\tau_\gamma(s) = \frac{\langle \gamma'(s) \times_{\mathbb{G}_3} \gamma''(s), \gamma'''(s)\rangle_{\mathbb{G}_3}}{\|\gamma'(s) \times_{\mathbb{G}_3} \gamma''(s)\|^2_{\mathbb{G}_3}} = \frac{y''(s)z'''(s) - y'''(s)z''(s)}{\kappa^2_\gamma(s)}.$$

Thus, the *Frenet apparatus* $\{T_\gamma, N_\gamma, B_\gamma, \kappa_\gamma > 0, \tau_\gamma\}$ is obtained along γ and the motion of the dynamic Frenet frame $\{T_\gamma, N_\gamma, B_\gamma\}$ along γ is described by the following *Frenet formulae* (cf. also [16]):

$$\begin{pmatrix} T'_\gamma \\ N'_\gamma \\ B'_\gamma \end{pmatrix} = \begin{pmatrix} 0 & \kappa_\gamma & 0 \\ 0 & 0 & \tau_\gamma \\ 0 & -\tau_\gamma & 0 \end{pmatrix} \begin{pmatrix} T_\gamma \\ N_\gamma \\ B_\gamma \end{pmatrix}. \tag{3}$$

3. f-rectifying curves in $\mathbb{G}_3$

Let $\gamma : I \to \mathbb{G}_3$ be a unit-speed admissible curve (parameterized by the Galilean invariant parameter s) with the Frenet apparatus $\{T_\gamma, N_\gamma, B_\gamma, \kappa_\gamma > 0, \tau_\gamma\}$. According to the general notion, γ is called a *rectifying curve* in $\mathbb{G}_3$ iff its position vector field lies in the rectifying planes, i.e. in the planes generated by $\{T_\gamma(s), B_\gamma(s)\}$. In this case, we have $\gamma(s) = \lambda(s)T_\gamma(s) + \mu(s)B_\gamma(s)$ for smooth functions $\lambda, \mu : I \to \mathbb{R}$. Now for any non-vanishing smooth mapping $f : I \to \mathbb{R}$ the f-position vector field along γ, denoted by γ_f, is defined by

$$\gamma_f(s) := \int f(s)\, d\gamma,$$

where the integral sign is used in the sense that γ_f is an integral curve of the vector field fT_γ along γ such that $\gamma'_f(s) = f(s)T_\gamma(s)$. Motivated by this, we define an *f-rectifying curve* as an admissible curve in $\mathbb{G}_3$ as:

Definition 3.1. Let $\gamma : I \to \mathbb{G}_3$ be a unit-speed admissible curve with Frenet apparatus $\{T_\gamma, N_\gamma, B_\gamma, \kappa_\gamma > 0, \tau_\gamma\}$ and let $f : I \to \mathbb{R}$ be non-vanishing and smooth. Then γ is called a *f-rectifying curve* in $\mathbb{G}_3$ if the f-position vector field γ_f along γ remains in its rectifying planes, i.e. γ_f satisfies

$$\gamma_f(s) = \lambda(s)T_\gamma(s) + \mu(s)B_\gamma(s) \tag{4}$$

for some smooth functions $\lambda, \mu : I \to \mathbb{R}$.

In coordinate form, the f-position vector field γ_f along an f-rectifying curve γ :

$I \to \mathbb{G}_3$ with the expression $\gamma(s) = (s, y(s), z(s))$ is another curve in $\mathbb{G}_3$ given by

$$\gamma_f(s) = \left(\lambda(s), \lambda(s)y'(s) - \frac{\mu(s)}{\kappa_\gamma(s)}z''(s), \lambda(s)z'(s) + \frac{\mu(s)}{\kappa_\gamma(s)}y''(s)\right)$$

for some smooth functions $\lambda, \mu : I \to \mathbb{R}$.

Remark 3.2. From the definition it follows that if $f \equiv 1$, then γ reduces to a rectifying curve in $\mathbb{G}_3$. From this point of view, f-rectifying curves generalize the rectifying curves in $\mathbb{G}_3$.

4. Characterizations of f-rectifying curves in $\mathbb{G}_3$

First, let us obtain several simple characterizations of an f-rectifying curve in $\mathbb{G}_3$ with respect to its f-position vector field.

Theorem 4.1. *Let $\gamma : I \to \mathbb{G}_3$ be a unit-speed admissible curve with Frenet apparatus $\{T_\gamma, N_\gamma, B_\gamma, \kappa_\gamma > 0, \tau_\gamma\}$ and $f : I \to \mathbb{R}$ be a non-vanishing smooth function with a non-vanishing primitive F. Then γ is an f-rectifying curve in $\mathbb{G}_3$ iff any of the following holds:*
(i) The function $\rho = \|\gamma_f\|_{\mathbb{G}_3}$ satisfies $\rho^2(s) = F^2(s) + a$ for some constant $a > 0$.
(ii) The tangential component $\langle \gamma_f, T_\gamma \rangle_{\mathbb{G}_3}$ is nothing but the primitive F.
(iii) The normal part $\gamma_f^{N_\gamma}$ is of non-zero constant length and ρ is non-constant.
(iv) τ_γ is non-vanishing and the binormal component $\langle \gamma_f, B_\gamma \rangle_{\mathbb{G}_3}$ is a non-zero constant.

Proof. First, let $\gamma : I \to \mathbb{G}_3$ be an f-rectifying curve. Then γ_f satisfies the (4) for smooth functions $\lambda, \mu : I \to \mathbb{R}$. A simple calculation on differentiation of (4) and then the application of (3) results in

$$\lambda'(s) = f(s), \quad \lambda(s)\kappa_\gamma(s) - \mu(s)\tau_\gamma(s) = 0, \quad \mu'(s) = 0$$

which imply

$$\lambda(s) = \int f(s)ds = F(s), \quad F(s)\kappa_\gamma(s) = c\tau_\gamma(s), \quad \mu(s) = c \tag{5}$$

for a constant $c \neq 0$ (otherwise κ_γ vanishes). Using (4) and (5) we get

$$\rho^2(s) = \langle \gamma_f(s), \gamma_f(s) \rangle_{\mathbb{G}_3} = \lambda^2(s) + \mu^2(s) = F^2(s) + c^2$$

which proves the statement (i). Again using (4) and (5), we find $\langle \gamma_f(s), T_\gamma(s) \rangle_{\mathbb{G}_3} = \lambda(s) = F(s)$ which proves the statement (ii). Now γ_f can be decomposed as

$$\gamma_f(s) = \langle \gamma_f(s), T_\gamma(s) \rangle_{\mathbb{G}_3} T_\gamma(s) + \gamma_f^{N_\gamma}(s), \tag{6}$$

where $\gamma_f^{N_\gamma}$ denotes the normal part of γ_f. Then (4) and (5) suggest that $\gamma_f^{N_\gamma}(s) = cB_\gamma(s)$, and therefore we have

$$\left\langle \gamma_f^{N_\gamma}(s), \gamma_f^{N_\gamma}(s) \right\rangle_{\mathbb{G}_3} = c^2, \quad \rho^2(s) = \langle \gamma_f(s), T_\gamma(s) \rangle^2_{\mathbb{G}_3} + c^2 = F^2(s) + c^2.$$

Consequently, the statement (iii) is true. Also, the non-vanishing nature of F and (5) ensures that τ_γ is non-vanishing. Moreover, with (4) and (5) we obtain $\langle \gamma_f(s), B_\gamma(s)\rangle_{\mathbb{G}_3} = \mu(s) = c$. This proves the statement (iv).

Conversely, let $\gamma : I \to \mathbb{G}_3$ be a unit-speed admissible curve and $f : I \to \mathbb{R}$ be non-vanishing and smooth with a primitive F, so that the statement (i) or (ii) holds. In both cases we must have $\langle \gamma_f(s), T_\gamma(s)\rangle_{\mathbb{G}_3} = F(s)$. By differentiating and then applying (3) and the non-vanishing nature of κ_γ we obtain $\langle \gamma_f(s), N_\gamma(s)\rangle_{\mathbb{G}_3} = 0$. This shows that γ_f lies in the rectifying planes of γ and therefore γ is an f-rectifying curve.

Next, let (iii) be true. Then we can find a constant $a > 0$ so that

$$\left\langle \gamma_f^{N_\gamma}(s), \gamma_f^{N_\gamma}(s) \right\rangle_{\mathbb{G}_3} = a.$$

Since γ_f can be decomposed as (6), it follows together with the previous equation that $\langle \gamma_f(s), \gamma_f(s)\rangle_{\mathbb{G}_3} = \langle \gamma_f(s), T_\gamma(s)\rangle^2_{\mathbb{G}_3} + a$. If we differentiate and then apply (3) and the non-vanishing nature of κ_γ, we find $\langle \gamma_f(s), N_\gamma(s)\rangle_{\mathbb{G}_3} = 0$ and therefore γ is an f-rectifying curve.

Finally, let the statement (iv) hold. Then let $\tau_\gamma \neq 0$ and $\exists$ a constant $c \neq 0$ such that $\langle \gamma_f(s), B_\gamma(s)\rangle_{\mathbb{G}_3} = c$. By differentiating and then applying (3) and the non-vanishing nature of τ_γ we obtain $\langle \gamma_f(s), N_\gamma(s)\rangle_{\mathbb{G}_3} = 0$ which indicates that γ is an f-rectifying curve. This proves the result. □

Coincidentally, general helices and rectifying curves in $\mathbb{G}_3$ can be characterized by their *conical curvature* (i.e. the ratio of torsion and curvature), which are similar to those in Euclidean 3-space. That is, up to Galilean motions from $\boldsymbol{B_6}$, an admissible curve γ in $\mathbb{G}_3$ parameterized by invariant parameter is
(i) a *general helix* iff its conical curvature is a non-zero constant (cf. [17]);

(ii) a *rectifying curve* iff its conical curvature is a non-constant linear function in the invariant parameter $s = x$ (cf. [6,15]).
An analogous characterization can be obtained for any f-rectifying curve γ in $\mathbb{G}_3$ with respect to the ratio $\frac{\tau_\gamma}{\kappa_\gamma}$ as follows.

THEOREM 4.2. *Let $\gamma : I \to \mathbb{G}_3$ be a unit-speed admissible curve with curvature $\kappa_\gamma > 0$ and torsion $\tau_\gamma \neq 0$, and let $f : I \to \mathbb{R}$ be non-vanishing and smooth with a primitive F. Then, up to Galilean motions from $\boldsymbol{B_6}$, γ is congruent to an f-rectifying curve in $\mathbb{G}_3$ iff its conical curvature is a non-zero constant multiple of the primitive F.*

Proof. First, let $f : I \to \mathbb{R}$ be a non-vanishing smooth function with primitive F and $\gamma : I \to \mathbb{G}_3$ be an f-rectifying curve. Then, by Theorem 4.1, we obtain (5) in which third equation gives

$$\frac{\tau_\gamma(s)}{\kappa_\gamma(s)} = \frac{1}{c} F(s), \tag{7}$$

where $c \neq 0$ is a constant. Conversely, let $\gamma : I \to \mathbb{G}_3$ be a unit-speed admissible curve such that its conical curvature satisfies (7). Then a straightforward calculation using (3) and (7) gives $\gamma_f'(s) = \frac{d}{ds}\left[F(s)T_\gamma(s) + cB_\gamma(s)\right]$ which produces $\gamma_f(s) =$

$p+F(s)T_\gamma(s)+cB_\gamma(s)$ for some arbitrary point $p \in \mathbb{G}_3$. Consequently, up to Galilean motions from $\boldsymbol{B_6}$, γ is an f-rectifying curve in $\mathbb{G}_3$. □

REMARK 4.3. Relaxing the conditions laid down, if we allow f to vanish on I, then its primitive F is a constant and hence Theorem 4.2 gives $\frac{\tau_\gamma}{\kappa_\gamma}$ = a constant. Consequently, γ reduces to either a *plane curve* or a *general helix* in $\mathbb{G}_3$. To avoid these circumstances, we have assumed f as non-vanishing. On the other hand, if f is a non-zero constant, then its primitive F is given by $F(s) = as + b$ for some constants $a \neq 0$ and b, and hence Theorem 4.2 yields

$$\frac{\tau_\gamma(s)}{\kappa_\gamma(s)} = \frac{a}{c}s + \frac{b}{c},$$

where $c \neq 0$, i.e., $\frac{\tau_\gamma}{\kappa_\gamma}$ is non-constant and linear in s. Consequently, up to Galilean motions from $\boldsymbol{B_6}$, γ becomes a rectifying curve in $\mathbb{G}_3$.

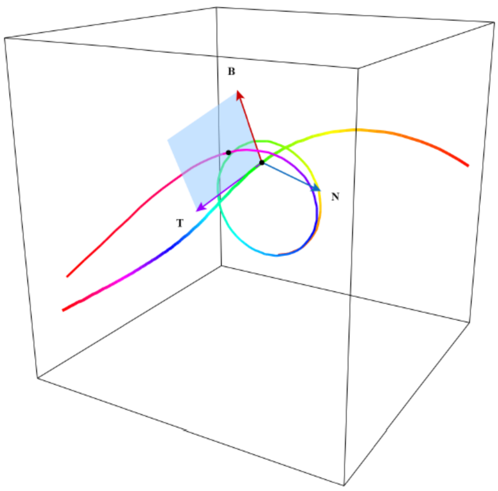

Figure 1: The thick curve with TNB-frame and rectifying plane represents γ whereas the thinner one represents γ_f.

EXAMPLE 4.4. Let γ be a unit-speed admissible curve (parametrized by Galilean invariant parameter s) in $\mathbb{G}_3$, and f be a non-vanishing and smooth function defined by $f(s) := e^s$. Then f has primitive F given by $F(s) = e^s + k$, where k is an arbitrary constant. We assume $k = 0$ and the f-position vector field γ_f given by

$$\gamma_f(s) = \left(e^s, -\frac{1}{2}(\sin s + \cos s), -\frac{1}{2}(\sin s - \cos s)\right).$$

Then γ is an f-rectifying curve in $\mathbb{G}_3$ with curvature $\kappa_\gamma = e^{-s}$ and torsion $\tau_\gamma = 1$, and it has coordinate expression

$$\gamma(s) = \left(s, -\frac{1}{2}e^{-s}\sin s, \frac{1}{2}e^{-s}\cos s\right).$$

One can easily draw the curves γ and γ_f (see Figure 1).

5. Equiform geometry of f-rectifying curves in $\mathbb{G}_3$

To begin with, let us recall the idea of *equiform geometry of curves* in $\mathbb{G}_3$, incorporated in [17]. In general, the projective transformations given by (1) from the similarity group $\boldsymbol{H_8}$ of $\mathbb{G}_3$ do not preserve angles between lines and planes in $\mathbb{G}_3$. If we restrict the condition $a_2 = \xi \neq 1$ in (1), then we obtain the following transformations:

$$\begin{cases} \bar{x} = a_1 + \xi x, \\ \bar{y} = b_1 + b_2 x + \xi(y\cos\varphi + z\sin\varphi), \\ \bar{z} = c_1 + c_2 x - \xi(y\sin\varphi - z\cos\varphi), \end{cases} \tag{8}$$

where $a_1, b_1, b_2, c_1, c_2, \xi, \varphi \in \mathbb{R}$, and these transformations clearly preserve angles between lines and planes in $\mathbb{G}_3$. Such transformations are called *equiform transformations* of $\mathbb{G}_3$ and give rise to a seven-parameter subgroup $\boldsymbol{H_7} \subset \boldsymbol{H_8}$, referred to as the *group of equiform transformations* of $\mathbb{G}_3$. It is found in [17] that an equiform transformation of $\mathbb{G}_3$ is a *homothety* followed by a *Galilean motion* of $\mathbb{G}_3$. This is how the equiform group $\boldsymbol{H_7}$ is structured and its geometry is known as the *equiform geometry* of $\mathbb{G}_3$.

Let $\gamma : I \to \mathbb{G}_3$ be an admissible curve (parametrized by invariant parameter s) of class at least C^4 with Frenet apparatus $\{T_\gamma, N_\gamma, B_\gamma, \kappa_\gamma > 0, \tau_\gamma\}$. We introduce the *equiform parameter* σ defined by

$$\sigma := \int \kappa_\gamma(s)\, ds$$

so that $\frac{ds}{d\sigma} = \frac{1}{\kappa_\gamma}$. Then, in equiform geometry, the tangent, normal and binormal vector fields of γ, respectively denoted by T^σ_γ, N^σ_γ and B^σ_γ, are related to the Frenet frame $\{T_\gamma, N_\gamma, B_\gamma\}$ along γ as follows:

$$T^\sigma_\gamma = \frac{d\gamma}{d\sigma} = \frac{d\gamma}{ds}\frac{ds}{d\sigma} = \frac{1}{\kappa_\gamma}\, T_\gamma, \quad N^\sigma_\gamma = \frac{1}{\kappa_\gamma}\, N_\gamma, \quad B^\sigma_\gamma = \frac{1}{\kappa_\gamma}\, B_\gamma.$$

It is clear that the trihedron $\{T^\sigma_\gamma, N^\sigma_\gamma, B^\sigma_\gamma\}$ forms a dynamic non-orthonormal frame, known as *equiform frame*, along γ. Now, in describing the motion of the equiform frame $\{T^\sigma_\gamma, N^\sigma_\gamma, B^\sigma_\gamma\}$ along γ, two functions $\kappa^\sigma_\gamma, \tau^\sigma_\gamma : I \to \mathbb{R}$ will come up which are defined by

$$\kappa^\sigma_\gamma := \left(\frac{1}{\kappa_\gamma}\right)', \quad \tau^\sigma_\gamma := \frac{\tau_\gamma}{\kappa_\gamma},$$

where "prime" stands for the derivative with respect to s as earlier. In equiform geometry, the functions κ^σ_γ and τ^σ_γ are called the *equiform curvature* and *equiform torsion* of γ, respectively, so that $\{T^\sigma_\gamma, N^\sigma_\gamma, B^\sigma_\gamma, \kappa^\sigma_\gamma, \tau^\sigma_\gamma\}$ forms an *equiform invariant apparatus* for γ and the motion of the equiform frame $\{T^\sigma_\gamma, N^\sigma_\gamma, B^\sigma_\gamma\}$ along γ is described by the following *Frenet-type formulae* [17]:

$$\begin{pmatrix} \dot{T}^\sigma_\gamma \\ \dot{N}^\sigma_\gamma \\ \dot{B}^\sigma_\gamma \end{pmatrix} = \begin{pmatrix} \kappa^\sigma_\gamma & 1 & 0 \\ 0 & \kappa^\sigma_\gamma & \tau^\sigma_\gamma \\ 0 & -\tau^\sigma_\gamma & \kappa^\sigma_\gamma \end{pmatrix} \begin{pmatrix} T^\sigma_\gamma \\ N^\sigma_\gamma \\ B^\sigma_\gamma \end{pmatrix}, \tag{9}$$

where "dot" denotes the derivative with respect to equiform parameter σ.

In equiform geometry, f-rectifying curves (more precise to say *equiform f-rectifying curves*) may be described as follows.

Definition 5.1. Let $\gamma : I \to \mathbb{G}_3$ be an admissible curve parametrized by equiform parameter σ, and with equiform invariant apparatus $\{T^\sigma_\gamma, N^\sigma_\gamma, B^\sigma_\gamma, \kappa^\sigma_\gamma, \tau^\sigma_\gamma\}$. Let $f : I \to \mathbb{R}$ be a non-vanishing smooth function in σ. Then γ is called an *equiform f-rectifying curve* in $\mathbb{G}_3$ if the equiform f-position vector field γ_f along γ, defined by

$$\gamma_f(\sigma) := \int f(\sigma)\, d\gamma,$$

lies in its rectifying planes (spanned by $\{T^\sigma_\gamma(\sigma), B^\sigma_\gamma(\sigma)\}$), i.e., γ_f satisfies

$$\gamma_f(\sigma) = \xi(\sigma) T^\sigma_\gamma(\sigma) + \zeta(\sigma) B^\sigma_\gamma(\sigma) \tag{10}$$

for some smooth functions $\xi, \zeta : I \to \mathbb{R}$ in parameter σ.

Accordingly, equiform f-rectifying curves in $\mathbb{G}_3$ are characterized by means of their equiform curvature and equiform torsion as:

Theorem 5.2. *Let $\gamma : I \to \mathbb{G}_3$ be an admissible curve parametrized by equiform parameter σ, and having equiform invariant apparatus $\{T^\sigma_\gamma, N^\sigma_\gamma, B^\sigma_\gamma, \kappa^\sigma_\gamma, \tau^\sigma_\gamma\}$. Also let $f : I \to \mathbb{R}$ be non-vanishing and smooth. Then, up to equiform transformations from $\boldsymbol{H_7}$, γ is an equiform f-rectifying curve iff both of its equiform curvature κ^σ_γ and equiform torsion τ^σ_γ are non-vanishing and satisfy*

$$\dot{\tau}^\sigma_\gamma(\sigma) = c\, f(\sigma) \exp\left(\int \kappa^\sigma_\gamma(\sigma)\, d\sigma\right) \tag{11}$$

for some constant $c \neq 0$.

Proof. First, let $\gamma : I \to \mathbb{G}_3$ be an equiform f-rectifying curve. Then γ_f satisfies (10) for some smooth functions $\xi, \zeta : I \to \mathbb{R}$. Differentiating (4) and then using (9), we find

$$\begin{cases} \dot{\xi}(\sigma) + \xi(\sigma)\kappa^\sigma_\gamma(\sigma) & = f(\sigma), \\ \xi(\sigma) - \zeta(\sigma)\tau^\sigma_\gamma(\sigma) & = 0, \\ \dot{\zeta}(\sigma) + \zeta(\sigma)\kappa^\sigma_\gamma(\sigma) & = 0. \end{cases} \tag{12}$$

Using non-vanishing nature of f, first equation in (12) imply that both of ξ and κ^σ_γ are non-vanishing and consequently second equation in (12) assures that both of ζ and τ^σ_γ are non-vanishing. Now, last equation in (12) yields

$$\zeta(\sigma) = c_1 \exp\left(-\int \kappa^\sigma_\gamma(\sigma)\, d\sigma\right), \tag{13}$$

for some constant $c_1 \neq 0$. On the other hand, applying last two equations in the first one in (12), we get

$$\zeta(\sigma)\,\dot{\tau}^{\sigma}_{\gamma}(\sigma) = f(\sigma). \tag{14}$$

Substituting (13) in (14) and letting $c = \frac{1}{c_1}\,(\neq 0)$, we obtain (11).

Conversely, let $\gamma : I \to \mathbb{G}_3$ be an admissible curve such that both of κ^{σ}_{γ} and τ^{σ}_{γ} are non-vanishing and satisfy (11). We define a new vector field V along γ by

$$V(\sigma) := \gamma_f(\sigma) - \omega(\sigma)T^{\sigma}_{\gamma}(\sigma) - \eta(\sigma)B^{\sigma}_{\gamma}(\sigma), \tag{15}$$

where $\omega, \eta : I \to \mathbb{R}$ are smooth functions defined by

$$\begin{cases} \omega(\sigma) &= \frac{1}{c}\,\tau^{\sigma}_{\gamma}(\sigma)\exp\left(-\int \kappa^{\sigma}_{\gamma}(\sigma)\,d\sigma\right), \\ \eta(\sigma) &= \frac{1}{c}\exp\left(-\int \kappa^{\sigma}_{\gamma}(\sigma)\,d\sigma\right). \end{cases} \tag{16}$$

Differentiating (16) and then applying (9), we obtain

$$\begin{aligned} \dot{V}(\sigma) = \big[f(\sigma) - \dot{\omega}(\sigma) - \omega(\sigma)\kappa^{\sigma}_{\gamma}(\sigma)\big]\,T^{\sigma}_{\gamma}(\sigma) + \big[-\omega(\sigma) + \eta(\sigma)\tau^{\sigma}_{\gamma}(\sigma)\big]\,N^{\sigma}_{\gamma}(\sigma) \\ + \big[-\dot{\eta}(\sigma) - \eta(\sigma)\kappa^{\sigma}_{\gamma}(\sigma)\big]\,B^{\sigma}_{\gamma}(\sigma). \end{aligned}$$

Using (16) and (11), previous equation reduces to $\dot{V}(\sigma) = 0$ which implies V is a constant vector field along γ. Hence, up to equiform transformations, γ is an equiform f-rectifying curve in $\mathbb{G}_3$. □

In particular, equiform rectifying curves in $\mathbb{G}_3$ are characterized with respect to their equiform curvature and equiform torsion as:

Corollary 5.3. *Let $\gamma : I \to \mathbb{G}_3$ be an admissible curve parametrized by equiform parameter σ, and with equiform invariant apparatus $\{T^{\sigma}_{\gamma}, N^{\sigma}_{\gamma}, B^{\sigma}_{\gamma}, \kappa^{\sigma}_{\gamma}, \tau^{\sigma}_{\gamma}\}$. Then, up to equiform transformations from $\boldsymbol{H_7}$, γ is an equiform rectifying curve iff both of its equiform curvature κ^{σ}_{γ} and equiform torsion τ^{σ}_{γ} are non-vanishing and satisfy*

$$\dot{\tau}^{\sigma}_{\gamma}(\sigma) = c\,\exp\left(\int \kappa^{\sigma}_{\gamma}(\sigma)\,d\sigma\right) \tag{17}$$

for some constant $c \neq 0$.

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Department of Mathematics and Statistics, Aliah University, Kolkata - 700160, West Bengal, India
E-mail: joydeep1972@yahoo.com
ORCID iD: https://orcid.org/0000-0002-1609-0798

Department of Mathematics, Kaliyaganj College, Uttar Dinajpur - 733 129, West Bengal, India
E-mail: zafariqbal_math@yahoo.com
ORCID iD: https://orcid.org/0000-0003-4405-1160

Department of Mathematics, Sukanta Mahavidyalaya, Jalpaiguri - 735210, West Bengal, India
E-mail: sarani121179@gmail.com
ORCID iD: https://orcid.org/0000-0003-0359-3713